To Hand in

HW 5

1

\[ a_n = \frac{(x-2)^n}{n^2+1}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{\left(\frac{(x-2)^n}{(n+1)^{2}+1}\right)}{\left(\frac{(x-2)^n}{n^2+1}\right)} = \]

\[ = |x-2| \cdot \frac{n^2+1}{n^2+2n+2} = |x-2| \cdot \frac{1 + \frac{1}{n^2}}{1 + \frac{2}{n} + \frac{n}{n^2}} \rightarrow |x-2|. \]

By the Ratio Test, if \( |x-2| < 1 \) the series is convergent, if \( |x-2| > 1 \) the series is divergent.

Thus, \( k = 1 \).

Let's check the boundary points: \( |x-2| = 1 \),

\( x = 1 \) or \( x = 3 \). Then \( |a_1| = \frac{1}{n^2+1} < \frac{1}{n^2} \).

By the Comparison Test, \( \Sigma |a_n| \) is convergent since \( \Sigma \frac{1}{n^2} \) is convergent. Thus, \( \Sigma a_n \) is absolutely convergent \( \Rightarrow \) convergent.

Answer: \( k = 1 \), \( [1, 3] \)

\[ a_n = \frac{n}{4^n} \cdot (x+1)^n. \]

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{n+1}{4^{n+1}} \cdot |x+1|^{n+1}}{\frac{n}{4^n} \cdot |x+1|^n} = \frac{n+1}{n} \cdot \frac{|x+1|}{4} \rightarrow \frac{|x+1|}{4}. \]

By the Ratio Test, if \( \frac{|x+1|}{4} < 1 \) the series is convergent, if \( \frac{|x+1|}{4} > 1 \) it is divergent. \( \frac{|x+1|}{4} < 1 \Leftrightarrow \)

\( |x+1| < 4 \Rightarrow k = 4 \).

When \( |x+1| = 4 \), \( x = -5 \) or \( x = 3 \).

We have: \( |a_n| = \frac{n \cdot |x+1|^n}{4^n} = \frac{n \cdot 4^n}{4^n} = n \rightarrow \infty \Rightarrow \)

\( a_n \) does not converge to 0. By the divergence test, \( \sum a_n \) is divergent. Answer: \( k = 4 \), \( (-5, 3) \).
\[ a_n = n! \cdot (2x-1)^n, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)! \cdot (2x-1)^{n+1}}{n! \cdot (2x-1)^n} = \]
\[ = (n+1) \cdot |2x-1| \to \infty \text{ when } 2x-1 \neq 0. \]

Thus, \( R = \infty , \quad \sum_{n=1}^{\infty} n! \cdot (2x-1)^n \text{ converges only for } x = \frac{1}{2}. \)

**Answer:** \( R = \infty , \quad \left[ \frac{1}{2}, \frac{3}{2} \right). \)

**8.5.20** \( a_n = \frac{(3x-2)^n}{n \cdot 3^n}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{(3x-2)^{n+1}}{(n+1) \cdot 3^{n+1}} = \]
\[ = |3x-2| \cdot \frac{n \cdot \frac{1}{3}}{n+1} \to \frac{13x-21}{3} = |x - \frac{2}{3}|. \text{ Ratio Test} \]

\( \Rightarrow \) If \( |x - \frac{2}{3}| < 1 \) the series is convergent, if \( |x - \frac{2}{3}| > 1 \) the series is divergent. Thus, \( R = 1 \).

Let's check the boundary points: \( |x - \frac{2}{3}| = 1, \)

\( x = \frac{5}{3} \) and \( x = -\frac{1}{3}. \)

a) \( x = \frac{5}{3} \). Then \( a_n = \frac{(3 \cdot \frac{5}{3} - 2)^n}{n \cdot 3^n} = \frac{1}{n}, \)

\( \sum \) is the harmonic series, divergent.

b) \( x = -\frac{1}{3} \), then \( a_n = \frac{(3 \cdot \frac{-1}{3} - 2)^n}{n \cdot 3^n} = \frac{(-1)^n}{n}. \)

\( \sum a_n = \sum \frac{(-1)^n}{n} \) is alternating series.

\( 1a_n = \frac{1}{n} \) is decreasing, convergent to 0. By the alternating series test, \( \sum a_n \) is convergent.

**Answer:** \( R = 1 , \quad \left[ -\frac{1}{3}, \frac{5}{3} \right). \)
N 8.5.8

\[ a_n = \frac{10^n x^n}{n^3}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{10^{n+1} x^{n+1}}{(n+1)^3}}{\frac{10^n x^n}{n^3}} = 1 \]

\[ = 101x1 \cdot \frac{n^3}{(n+1)^3} = 101x1 \cdot \left(1 + \frac{1}{n} \right)^3 \rightarrow 101x1. \]

By the ratio test, if \( 101x1 < 1 \) then the series is convergent, if \( 101x1 > 1 \) then the series is divergent. \( 101x1 < 1 \iff |x| < \frac{1}{10}. \) Therefore, \( R = \frac{1}{10}. \) Boundary points: \( x = -\frac{1}{10} \) and \( x = \frac{1}{10}. \)

In both cases we have \( \sum a_n = \sum \frac{1}{n^3} \) is a \( p \)-series with \( p = 3 > 1 \implies \) is convergent. By the absolute convergence test, \( \sum |a_n| \) is convergent as well.

Cluster \( \bar{R} = \frac{1}{10}, \quad [-\frac{1}{10}, \frac{1}{10}]. \)

N 8.5.22.

\[ a_n = \frac{x^{2n}}{n(e^n h)^2}, \quad \left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{x^{2(n+1)}}{(n+1)(e^{(n+1)h})^2}}{\frac{x^{2n}}{n(e^n h)^2}} = \frac{n(e^n h)^2}{(n+1)(e^{(n+1)h})^2} = \frac{n(e^n h)^2}{e^{2n+2}h^2(n+1)} \]

\[ = \frac{1x^{2n}}{n+1 \cdot \left(\frac{e^n h}{e^{(n+1)h}}\right)^2}. \]

We have \( \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \rightarrow 1 \)

When \( n \rightarrow \infty. \) To find \( \lim_{n \rightarrow \infty} \frac{e^n h}{e^{(n+1)h}} \) use the L'Hospital's rule. Set \( f(x) = e^n h, \quad g(x) = e^{(n+1)h} \)

Then \( f'(x) \rightarrow \infty \) and \( g(x) \rightarrow \infty \) when \( x \rightarrow \infty \Rightarrow \)
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0
\]

\[
= \lim_{x \to \infty} \left(1 + \frac{1}{x}\right) = 1.
\]
Thus, by the limit laws,

\[
\lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right| = 1\cdot \frac{1}{x^2}.
\]

By the Ratio Test, when \(1x^2 < 1\) the series is convergent,

when \(1x^2 > 1\) the series is divergent.

\[
1x^2 < 1 \iff 1x < 1 \implies R = 1.
\]

Boundary points: \(1x = 1, \quad x = -1\) and \(x = 1\).

In both cases we have \(\frac{a_{n+1}}{a_n} = 1\).

The series \(\sum_{n=2}^{\infty} 1\cdot(n\cdot1)^n\) is convergent (see example 9 on the course web page). Thus,

\[
\sum_{n=2}^{\infty} \frac{1}{n(\text{en})^n}\] is convergent.

By the Absolute Convergence Test, \(\sum_{n=2}^{\infty} a_n\) is convergent.

Answer: \(R = 1, \quad [-1, 1]\).

N 8.5.26 Let \(R\) be the radius of convergence.

Then the series is convergent when \(1x < R\) and divergent when \(1x > R\). Convergence for \(x = -9\) implies that \(R \geq 1x = 9\). Divergence for \(x = 6\) implies that \(R \leq 1x = 6\). Thus, \(4 \leq R \leq 6\).

(a) \(\sum_{n=0}^{\infty} C_n = \sum_{n=0}^{\infty} C_n (1)^n\) is convergent since

\(1 < 4 \leq R \implies 1 < R\)

(b) \(\sum_{n=0}^{\infty} C_n (8)^n\) is divergent since \(8 > 6 \geq R \implies 8 > R\).
(c) \( \sum_{n=0}^{\infty} C_n(-3)^n \) is convergent since 
\[ |-3| = 3 \leq R \]

(d) \( \sum_{n=0}^{\infty} (-1)^nC_n(9)^n \) is divergent

since \( |9| = 9 > 6 > R \).

8.5.31

\[ f(x) = (1 + x^2 + x^4 + x^6 + \ldots) + (2x + 2x^3 + 2x^5 + \ldots) = \sum_{n=0}^{\infty} x^{2n} + 2\sum_{n=0}^{\infty} x^{2n+1} \]

Using the Ratio test we obtain that the radius of convergence for both series is 1. \( \Rightarrow \) by the Sum Law for the sum does not imply that \( f(x) \) is divergent for \( |x| > 1 \). Notice, however, if \( |x| \geq 1 \) the terms of \( f(x) \) do not converge to 0 \( \Rightarrow \) by the Divergence Test \( f(x) \) is divergent.

Thus, the interval of convergence is \( |x| < 1 \): \( (-1, 1) \), and so \( R = 1 \).

We have: \( \sum_{n=0}^{\infty} x^{2n} = \frac{1}{1-x^2} \), \( \sum_{n=0}^{\infty} x^{2n+1} = \frac{x}{1-x^2} \),

by geometric series. Thus, \( f(x) = \frac{1}{1-x^2} + \frac{2x}{1-x^2} = \frac{2x + 1}{1-x^2} \) for \( |x| < 1 \).

\[ \text{Answer} \quad R = 1, (-1, 1), \quad f(x) = \frac{2x+1}{1-x^2} \]