8.3.28
\[ a_n = \frac{1 + \sin n}{10^n}. \]
We have \( 0 \leq 1 + \sin n \leq 2 \)
for all \( n \). Thus, \( 0 \leq a_n \leq \frac{2}{10^n} \).
The series \( \sum_{n=1}^{\infty} \frac{2}{10^n} \) is the geometric series,
\[ r = \frac{1}{10}, \quad |r| < 1 \rightarrow \text{it is convergent}. \]
By the Comparison Test, \( \sum_{n=1}^{\infty} \frac{1 + \sin n}{10^n} \) is convergent.

8.3.29
When \( n \to \infty \) sequence \( \frac{1}{n} \) converges to 0.
Near 0, \( \sin x \) behaves like \( x \). More precisely,
\[ \lim_{x \to 0} \frac{\sin x}{x} = 1. \]
Therefore,
\[ \lim_{n \to \infty} \frac{\sin \left( \frac{1}{n} \right)}{\frac{1}{n}} = 1, \quad \text{since} \quad \sin \frac{1}{n} > 0 \quad \text{and} \quad \frac{1}{n} > 0 \]
for all \( n \), we can use the Limit Comparison Test. We have: \( \sum_{n=1}^{\infty} \frac{1}{n} \) is divergent. Therefore,
\( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) is also divergent.

8.3.34
Let \( S_n = \sum_{k=1}^{n} \frac{1}{k} \) be the \( n \)-th partial sum and \( R_n = \sum_{k=n+1}^{\infty} \frac{1}{k} \) be the \( n \)-th remainder of the series. Let \( f(x) = \frac{1}{x^s} \) so that \( a_n = \frac{1}{n^s} = f(n) \).
By the Remainder Estimate for the Integral Test,
\[ \int_{n}^{\infty} \frac{1}{x^s} \, dx \leq R_n \leq \int_{n+1}^{\infty} \frac{1}{x^s} \, dx \]
(since \( \frac{1}{x^s} \) is positive, continuous and decreasing on \([1, \infty)\))
Thus, \( R_n \leq -\frac{1}{4 n^4} \int_1^{\infty} = NW^{-4} \) \(1\)

\[ 0 - \left( -\frac{1}{4 n^4} \right) = \frac{1}{4 n^4} \] and \( R_n \geq -\frac{1}{4 n^4} \int_1^{\infty} = \frac{1}{4(n+1)^4} \).

To estimate \( S \) correct to three digits, let's take \( n \) so that \( R_n \leq 10^{-3} \). Solving \( \frac{1}{4 n^4} < 10^{-3} \), \( n^4 > \frac{10^3}{4} = 250 \) we see that \( n = 4 \) is sufficient. We have:

\[ S_4 = 1 + \frac{1}{32} + \frac{1}{243} + \frac{1}{1024} \approx 1.03634 \text{ and } \]

\[ S_4 + \frac{1}{4.5^4} \leq S = S_4 + R_4 \leq S_4 + \frac{1}{4.9^4} \]

\[ 1.03624 \leq S \leq 1.03732. \]

Therefore, round of \( S \) to three digits is \( 1.037 \).

\[ \text{Answer: } 1.037. \]

\[ \text{No. 3.38} \]

\[ S_{10} = \sum_{n=1}^{10} \frac{\sin^2(n)}{n^3} = 0.83253 \]

\[ R_{10} = \sum_{n=11}^{\infty} \frac{\sin^2(n)}{n^3} \] we cannot use the integral test directly to \( f(x) = \frac{\sin^2(x)}{x^3} \) since there is no formula for \( \int f(x) \, dx \).

Instead, first use the comparison test.
we have:
\[ 0 \leq \frac{\sin^2 n}{n^3} \leq \frac{1}{n^3}, \quad \sum_{n=11}^{\infty} \frac{1}{n^3} \text{ is convergent (p-series, p = 3 > 1)}. \]
Therefore,
\[ \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \leq \sum_{n=11}^{\infty} \frac{1}{n^3}. \]
For the latter sum, use the integral test to estimate the remainder. Set \( f(x) = \frac{1}{x^3} \). Then \( f(x) \) is decreasing, positive, continuous. Therefore,
\[ \sum_{n=11}^{\infty} \frac{1}{n^3} \leq \int_{10}^{\infty} \frac{1}{x^3} \, dx = -\frac{1}{2x^2} \bigg|_{10}^{\infty} = 0 - \left(-\frac{1}{2 \cdot 10^2}\right) \]
\[ = \frac{1}{200} = 0.005. \]
Thus, \( 0 \leq R_n \leq 0.005 \).

Answer: The error is less or equal to 0.005.

\( S \approx S_{10} \approx 0.83253 \).

Remark: Here we could not use improved estimate, because
\[ R_{10} = \sum_{n=11}^{\infty} \frac{\sin^2 n}{n^3} \leq \sum_{n=11}^{\infty} \frac{1}{n^3}. \]
and saying that
\[ \sum_{n=11}^{\infty} \frac{1}{n^3} \geq \int_{11}^{\infty} \frac{1}{x^3} \, dx \]
would not give any information about \( R_{10} \).
N8.4.3
\[ a_n = \frac{4 \cdot (-1)^{n-1}}{n+6} \]

\( \sum a_n \) is an alternating series.

\[ |a_n| = \frac{4}{n+6} \] is decreasing and convergent to 0. By the Alternating series test, \( \sum a_n \) is convergent.

N8.4.10
\[ a_n = (-1)^n \cos \left( \frac{\pi}{n} \right) \]
\[ 1|a_n| = \left| \cos \left( \frac{\pi}{n} \right) \right| \]

From the picture we can see that \( 1|a_n| \) approaches 1. Let's prove this. For \( n \geq 2 \)
\[ 1|a_n| = \cos \frac{\pi}{n} \]
since \( \cos \frac{\pi}{n} \geq 0 \) for \( n \geq 2 \).

Let \( f(x) = \cos x \), \( b_n = \frac{\pi}{n} \).

Then \( b_n \to 0 \) when \( n \to \infty \). Since \( \cos x \) is continuous everywhere (in particular at 0) we have: \( \lim_{n \to \infty} |a_n| = \cos 0 = 1 \).

Thus, \( 1|a_n| \) does not converge to 0 \( \Rightarrow \) \( a_n \) does not converge to 0. By the Divergence Test, \( \sum a_n \) is divergent.
\[ a_n = (-1)^n b_n \text{ where } b_n = \frac{1}{n \cdot 5^n} \]

\( \{b_n\} \) is decreasing, since \( n \cdot 5^n \) is increasing.

\[ \lim_{n \to \infty} b_n = 0. \]

By the Alternating Series Test, \( \sum_{n=1}^\infty (-1)^n b_n \) is convergent. Moreover, the remainder \( R_n = S - S_n \) satisfies

\[ |R_n| \leq b_{n+1} = \frac{1}{(n+1)5^{n+1}}. \]

To make the error \( < 0.0001 \), choose \( n \) so that \( b_{n+1} < 0.0001 \). We have

\[ \frac{1}{(n+1)5^{n+1}} < 0.0001 \iff (n+1)5^{n+1} > 10^4 \]

By trial and error we find that \( n = 4 \) is sufficient.

Thus, \( S_4 = \sum_{n=1}^{4} (-1)^n b_n = -0.19226 \) approximates \( S \) with an error \( < 0.0001 \).

\[ a_n = \frac{(-1)^n}{n \cdot 5^n} \]

Use the Ratio Test.

\[ \left( \frac{a_{n+1}}{a_n} \right) = \frac{(n+1)!}{100^{n+1}} \cdot \frac{n!}{100^n} = \frac{n+1}{100} \]

since \( n+1 \) grows without a bound. By the Ratio Test, \( \sum_{n=1}^\infty |a_n| \) is divergent \( \implies \) not absolutely convergent.

Remark: This problem can be also solved by the Divergence Test by showing that \( \lim_{n \to \infty} a_n = \infty \).
N8.4.30

\[ a_n = \frac{\sin 4n}{4^n}, \quad |a_n| = \frac{|\sin 4n|}{4^n} \]

We have:

\[ 0 \leq \frac{18 \sin 4n}{4^n} \leq \frac{1}{4^n}. \]

The series \( \sum_{n=1}^{\infty} \frac{1}{4^n} \) is the geometric series with \( r = \frac{1}{4} \). Hence, \( |r| < 1 \implies \) convergent. By the Comparison Test, \( \sum_{n=1}^{\infty} \frac{18 \sin 4n}{4^n} \) is convergent.

Thus, \( \sum_{n=1}^{\infty} \frac{\sin 4n}{4^n} \) is absolutely convergent.

N8.4.35

Clearly, \( a_n > 0 \) for all \( n \). Use the Ratio Test.

We have:

\[ \left| \frac{a_{n+1}}{a_n} \right| = \frac{5n+1}{4n+3} = \frac{5 + \frac{1}{n}}{4 + \frac{3}{n}} \rightarrow \frac{5}{4} \]

when \( n \to \infty \). Therefore, \( \sum_{n=1}^{\infty} a_n \) is divergent.