sequence does not approach one number ⇒ has no limit. Let us prove this by contradiction. Assume that this sequence has a limit. Let \( \lim_{n \to \infty} \cos(\pi n/3) = L \). Take a small \( \varepsilon > 0 \), say, \( \varepsilon = \frac{1}{2} \). By definition of the limit, for all indexes \( n \) starting from some index \( N \) we have: \( |\cos(\pi n/3) - L| < \varepsilon = \frac{1}{2} \). Take \( n > N \) of the form \( 6k \). Then \( \cos(\pi n/3) = \cos(2\pi k) = 1 \). Thus, \( 1 - L \mid < \frac{1}{2} \). Take \( n > N \) of the form \( 6k + 3 \). Then \( \cos(\pi n/3) = \cos(2\pi k + \pi) = -1 \). Thus, \( 1 - L \mid < \frac{1}{2} \). But then \( |(1 - L) - (-1 - L)| \leq 1 - L \mid + 1 - L \mid \) and \( 2 < \frac{1}{2} + \frac{1}{2} = 1 \). This is a contradiction. It shows that the sequence \( \{\cos(\pi n/3)\} \) has no limit.

N8.1.8. \( A_n = \frac{(-1)^n n}{(n+1)^2} \)
An = \frac{3^{n+2}}{5^n} = 9 \cdot \left(\frac{3}{5}\right)^n.\) Observe that \((\frac{3}{5})^n\) converges to 0, since it is of the form \(r^n\) with \(|r| < 1\). By the product rule,
\[
\lim_{n \to \infty} An = \lim_{n \to \infty} 9 \cdot \left(\frac{3}{5}\right)^n = 9 \cdot \lim_{n \to \infty} \left(\frac{3}{5}\right)^n = 9 \cdot 0 = 0.
\]

Answer: \(\lim_{n \to \infty} \frac{3^{n+2}}{5^n} = 0\).

N 8.1.30

\(-\frac{1}{\sqrt{n}} \leq \frac{-1}{1 + \sqrt{n}} \leq \frac{\sin 2n}{1 + \sqrt{n}} \leq \frac{1}{1 + \sqrt{n}} \leq \frac{1}{\sqrt{n}}.

Thus, \(-\frac{1}{\sqrt{n}} \leq \frac{\sin 2n}{1 + \sqrt{n}} \leq \frac{1}{\sqrt{n}}\).

Observe that \\{\frac{1}{\sqrt{n}}\} converges to 0 since it is of the form \(\{n^r\}\) with \(r = -\frac{1}{2} < 0\). Then also
\[
\lim_{n \to \infty} \frac{-1}{\sqrt{n}} = -1, \lim_{n \to \infty} \frac{1}{\sqrt{n}} = -1, 0 = 0.
\]

By squeeze theorem, \(\lim_{n \to \infty} \frac{\sin 2n}{1 + \sqrt{n}} = 0\).
The sequence does not approach one number, so it does not have a limit. Prove it using contradiction. Assume that it has limit \( L \). Then for all \( \epsilon > 0 \) starting we can find index \( N \) starting from which
\[ |a_n - L| < \epsilon. \]
Take \( \epsilon = \frac{1}{10} \). There are infinitely many terms an equal to 0, so there is an index \( n > N \) with \( a_n = 0 \). Thus, \( |0 - L| < \epsilon \), that is \( |L| < \frac{1}{10} \). But also there are infinitely many indexes \( n \) for which \( a_n = 1 \). Therefore, we can find \( n > N \) with \( a_n = 1 \). We obtain
\[ |1 - L| < \epsilon = \frac{1}{10} \]
But \( |1 - L| \geq |L| \geq 1 - \frac{1}{10} = \frac{9}{10} > \frac{1}{10} \). This contradiction shows that this sequence cannot have a limit.

Answer: No limit.
To find the formula observe that 5 and 1 are on the same distance from their mean 
\[ \frac{5+1}{2} = 3 \]: 5 = 3 + 2, 1 = 3 - 2. Thus,
\[ a_n = 3 + (-1)^{n-1} \cdot 2 \]

We have \[ a_n = \frac{n^3}{n^3 + 1} = \frac{1}{1 + \frac{1}{n^3}} \]. By the Sum and the Quotient Laws,
\[ \lim_{n \to \infty} a_n = \frac{\lim_{n \to \infty} 1}{\lim_{n \to \infty} 1 + \lim_{n \to \infty} \frac{1}{n^3}} = \frac{1}{1 + 0} = 1 \]

\[ a_n = \frac{n^3}{n+1} = \frac{n^2}{1 + \frac{1}{n}} \]. Observe that \[ \frac{1}{n} \in [1, 2] \Rightarrow 1 + \frac{1}{n} \leq 2 \Rightarrow \frac{n^2}{1 + \frac{1}{n}} \geq \frac{n^2}{2} \].

In particular, \( a_n \) becomes larger than any number when \( n \) grows. Therefore, \( a_n \) diverges to \( \infty \).
8.1.25

Since \( 0 < \cos^2 n < 1 \), we have:

\[
0 \leq a_n \leq \frac{1}{2^n}.
\]

The sequence \( \frac{1}{2^n} \) is the geometric sequence with \( r = \frac{1}{2} \) (since \( |r| < 1 \)).

Therefore, \( \lim_{n \to \infty} \frac{1}{2^n} = 0 \). By the Squeeze Theorem, \( \lim_{n \to \infty} a_n = 0 \).

8.1.38

The main term under the root sign is \( 5^n \) (\( 3^n \) is comparably small: \( \lim_{n \to \infty} \frac{3^n}{5^n} = \lim_{n \to \infty} \left( \frac{3}{5} \right)^n = 0 \)).

It is convenient to rewrite \( a_n \) as follows:

\[
a_n = \sqrt{5^n \cdot \left( 1 + \left( \frac{3}{5} \right)^n \right)} = 5 \cdot \sqrt{1 + \left( \frac{3}{5} \right)^n}.
\]

Observe that \( 1 \leq \sqrt{x} \leq x \) for all \( x \geq 1 \).

Therefore, \( 5 \leq a_n \leq 5 \cdot \left( 1 + \left( \frac{3}{5} \right)^n \right) \).

We have:

\[
\lim_{n \to \infty} 5 \left( 1 + \left( \frac{3}{5} \right)^n \right) = 5 + 5 \cdot \lim_{n \to \infty} \left( \frac{3}{5} \right)^n = 5 + 5 \cdot 0 = 5.
\]

By the Squeeze Theorem, \( a_n \) is convergent to 5:

\[
\lim_{n \to \infty} a_n = 5.
\]