MAT 126 Calculus B Spring 2007
Practice Midterm II Solutions

Answer each question in the space provided and on the reverse side of the sheets. Show your work whenever possible. Unless otherwise indicated, **answers without justification will get little or no partial credit!** Cross out anything that grader should ignore and circle or box the final answer. The actual exam will contain 5 problems. This practice test contains more problems to give you more practice.

1. Evaluate the following definite integrals
   
   (a) \( \int_{0}^{13} \frac{2}{(2x + 1)^{\frac{3}{2}}} dx \)

   **Solution.** Substitution \( u = 2x + 1 \) gives \( du = 2dx \) and \( x = 0 \) corresponds to \( u = 1 \) and \( x = 13 \) — to \( u = 27 \). Thus by the substitution rule,
   \[
   \int_{0}^{13} \frac{2}{(2x + 1)^{\frac{3}{2}}} dx = \int_{1}^{27} \frac{1}{u^{\frac{3}{2}}} du = 3u^{\frac{1}{3}} \bigg|_{1}^{27} = 3(27)^{\frac{1}{3}} - 3 = 3 \cdot 3 - 3 = 6.
   \]

   (b) \( \int_{0}^{\pi/2} e^{\sin x} \cos x dx \)

   **Solution.** Substitution \( u = \sin x \) gives \( du = \cos x dx \) and \( x = 0 \) corresponds to \( u = 0 \) and \( x = \pi/2 \) — to \( u = 1 \). Thus by the substitution rule,
   \[
   \int_{0}^{\pi/2} e^{\sin x} \cos x dx = \int_{0}^{1} e^{u} du = e^{u} \bigg|_{0}^{1} = e - 1.
   \]

   (c) \( \int_{0}^{1} x^4(1 + x^5)^{20} dx \)

   **Solution.** Substitution \( u = 1 + x^5 \) gives \( du = 5x^4 dx \) and \( x = 0 \) corresponds to \( u = 1 \) and \( x = 1 \) — to \( u = 2 \). Thus by the substitution rule,
   \[
   \int_{0}^{1} x^4(1 + x^5)^{20} dx = \frac{1}{5} \int_{1}^{2} u^{20} du = \frac{u^{21}}{5 \cdot 21} \bigg|_{1}^{2} = \frac{2^{21} - 1}{105}.
   \]
(d) \[ \int_0^1 \tan^{-1} x \, dx \]

Solution. We use integration by parts with \( u = \tan^{-1} x \) and \( dv = dx \). We have \( du = \frac{dx}{1 + x^2} \) and \( v = x \), so that using \( \tan^{-1}(1) = \frac{\pi}{4} \), we get

\[ \int_0^1 \tan^{-1} x \, dx = \int_0^1 udv = uv\big|_0^1 - \int_0^1 vdu \]
\[ = \frac{\pi}{4} - \int_0^1 \frac{x}{1 + x^2} \, dx. \]

To evaluate the remaining integral, we use the substitution \( u = 1 + x^2 \), so that \( du = 2x \, dx \) and \( x = 0 \) corresponds to \( u = 1 \) and \( x = 1 \) to \( u = 2 \). Thus

\[ \int_0^1 \frac{x}{1 + x^2} \, dx = \frac{1}{2} \int_1^2 \frac{du}{u} = \ln u\big|_1^2 = \ln 2. \]

Therefore, we get

\[ \int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \ln 2. \]

(e) \[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sin t \right| \, dt. \]

Solution. This is the integral over the symmetric interval \([ -\frac{\pi}{2}, \frac{\pi}{2} ] \) of the even function

\[ f(t) = |\sin t|. \]

Since \( \sin t \) is non-negative on \([ 0, \frac{\pi}{2} ] \), we get

\[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin t| \, dt = 2 \int_0^{\frac{\pi}{2}} |\sin t| \, dt = 2 \int_0^{\frac{\pi}{2}} \sin t \, dt \]
\[ = -2 \cos t\big|_0^{\frac{\pi}{2}} = -2(\cos \frac{\pi}{2} - \cos 0) = 2. \]

(f) \[ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + t^2)^2 \sin^5 t \, dt. \]
Solution. It is an integral over the symmetric interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the odd function 

$$f(t) = (1 + t^2)^2 \sin^5 t.$$ 

(Verify that $f(-t) = -f(t)$!). Therefore,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 + t^2)^2 \sin^5 t \, dt = 0.$$

2. Evaluate the following indefinite integrals

(a) 

$$\int x^3 e^{x^4} \, dx$$

Solution. Setting $u = x^4$, we get $du = 4x^3 \, dx$, so by the substitution rule,

$$\int x^3 e^{x^4} \, dx = \frac{1}{4} \int e^u \, du = \frac{1}{4} e^u + C = \frac{1}{4} e^{x^4} + C.$$

(b) 

$$\int te^t \, dt$$

Solution. We use integration by parts with $u = t$ and $dv = e^t \, dt$. We have $du = dt$ and $v = e^t$, so that

$$\int te^t \, dt = \int udv = uv - \int vdu = te^t - \int e^t \, dt = te^t - e^t + C.$$

(c) 

$$\int x^2 \cos x \, dx$$

Solution. We use integration by parts with $u = x^2$ and and $dv = \cos x \, dx$. We have $du = 2x \, dx$ and $v = \sin x$, so that

$$\int x^2 \cos x \, dx = \int udv = uv - \int vdu = x^2 \sin x - 2 \int x \sin x \, dx.$$

For the remaining integral we again use integration by parts with $u = 2x$ and $dv = \sin x \, dx$, so that $du = 2 \, dx$ and $v = -\cos x$. We have

$$2 \int x \sin x \, dx = \int udv = uv - \int vdu = -2x \cos x + 2 \int \cos x \, dx$$

$$= -2x \cos x + 2 \sin x + C,$$

so that

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$
(Double-check the answer by differentiating!)

(d) \[ \int \cos(\sqrt{x})dx \]

Solution. First, we use the substitution rule with \( t = \sqrt{x} \), so that \( dt = \frac{1}{2\sqrt{x}} dx \), or \( dx = 2\sqrt{x} dt = 2t \, dt \). We get

\[ \int \cos(\sqrt{x})dx = 2 \int t \cos t \, dt. \]

To evaluate this integral, we use integration by parts with \( u = 2t \) and \( dv = \cos t \, dt \). We have \( du = 2 \, dt \) and \( v = \sin t \), so that

\[ 2 \int t \cos t \, dt = \int uv - \int vdu = 2t \sin t - 2 \int \sin t \, dt = 2t \sin t + 2 \cos t + C. \]

Finally, remembering that \( t = \sqrt{x} \), we get

\[ \int \cos(\sqrt{x})dx = 2\sqrt{x} \sin \sqrt{x} + 2 \cos \sqrt{x} + C. \]

3. Evaluate the following indefinite integrals

(a) \[ \int \frac{1}{x^2} \ln x \, dx \]

Solution. Here we use integration by parts with \( u = \ln x \) and \( dv = \frac{1}{x^2} \, dx \), so that

\[ du = \frac{1}{x} \, dx \quad \text{and} \quad v = -\frac{1}{x}. \]

(Note that substitution rule with \( u = \ln x \) does not simplify the integral since in the denominator we have \( x^2 \); if it was \( x \), then the substitution rule would work.) Thus we have

\[ \int \frac{1}{x^2} \ln x \, dx = \int uv - \int vdu = -\frac{\ln x}{x} + \int \frac{1}{x^2} \, dx = -\frac{\ln x}{x} - \frac{1}{x} + C. \]

(b) \[ \int \frac{1}{x} (\ln x)^2 \, dx \]
**Solution.** Here we use the substitution rule with \( u = \ln x \) and \( du = \frac{1}{x} \, dx \) (since we have \( x \) in the denominator). Therefore,

\[
\int \frac{1}{x} (\ln x)^2 \, dx = \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} (\ln x)^3 + C.
\]

(c) \[
\int x^7 \ln x \, dx
\]

**Solution.** Here we use integration by parts with \( u = \ln x \) and \( dv = x^7 \, dx \), so that

\[
du = \frac{1}{x} \, dx \quad \text{and} \quad v = \frac{1}{8} x^8.
\]

Thus we have

\[
\int x^7 \ln x \, dx = \int udv = uv - \int vdu = \frac{1}{8} x^8 \ln x - \frac{1}{8} \int x^8 \frac{dx}{x} = \frac{1}{8} x^8 \ln x - \frac{1}{64} x^8 + C.
\]

4. Evaluate the following indefinite integrals

(a) \[
\int \frac{2x^2}{x^2 + 1} \, dx
\]

**Solution.** We have

\[
\frac{2x^2}{x^2 + 1} = 2 - \frac{2}{x^2 + 1}
\]

(either by doing the long division, or by writing \( 2x^2 = 2x^2 + 2 - 2 = 2(x^2 + 1) - 2 \), and dividing both terms by \( x^2 + 1 \)). Thus

\[
\int \frac{2x^2}{x^2 + 1} \, dx = \int \left( 2 - \frac{2}{x^2 + 1} \right) \, dx = 2x - 2 \tan^{-1} x + C.
\]

(b) \[
\int \frac{2x}{x^2 + 1} \, dx
\]

**Solution.** To evaluate this integral, we use the substitution \( u = 1 + x^2 \), so that \( du = 2x \, dx \) (compare with the last integral in problem 2 (d)). We have

\[
\int \frac{2x}{x^2 + 1} \, dx = \int \frac{du}{u} = \ln |u| + C = \ln(1 + x^2) + C.
\]
5. (a) Write a formula for $\tan x$ in terms of $\sin x$ and $\cos x$.

Solution.

$$\tan x = \frac{\sin x}{\cos x}.$$ 

(b) Evaluate

$$\int \tan x \, dx$$

Solution. Using part (a) we have

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx,$$

which suggests the substitution $u = \cos x$. We have $du = -\sin x \, dx$, so that

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{du}{u} = -\ln |u| + C = -\ln |\cos x| + C.$$

6. Evaluate

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \frac{x}{\sin^2 x} \, dx$$

Solution. We use integration by parts with $u = x$ and

$$dv = \frac{1}{\sin^2 x} \, dx,$$

so that $du = dx$ and $v = -\cot x$. We have, using that antiderivative of $\cot x$ is $\ln |\sin x|$ (compare with part (a)),

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \frac{x}{\sin^2 x} \, dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} u dv = uv\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} v du$$

$$= -x \cot x\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x \, dx$$

$$= \frac{\pi}{4} + \ln |\sin x|\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{2} \ln 2,$$

where we have used that $\cot \frac{\pi}{4} = 1$, $\cot \frac{\pi}{2} = 0$ and $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$, $\sin \frac{\pi}{2} = 1$.

7. (a) Set

$$f(x) = \int_1^{x^2} \sin t^3 \, dt + x^3$$

Find $f(1)$ and $f'(x)$.

Solution. First, $f(1) = 1^3 = 1$, since in this case the interval of integration shrinks to a point and the integral
is zero. Second, we get by the FTC and the chain rule, setting $u = x^2$,
\[ f'(x) = \frac{du}{dx} \frac{d}{du} \left( \int_1^u \sin t^3 dt \right) \bigg|_{u=x^2} + 3x^2 \\
= 2x (\sin u^3) \bigg|_{u=x^2} + 3x^2 = 2x \sin x^6 + 3x^2. \]

(b) Set
\[ f(x) = \int_{\sqrt{x}}^{x^2} \tan^2 t \, dt \]

Find $f(4)$ and $f'(x)$

Solution. We have
\[ f(4) = \int_{\sqrt{4}}^{4-2} \tan^2 t \, dt = \int_{2}^{2} \tan^2 t \, dt = 0. \]

To find $f'(x)$, we write
\[ f(x) = \int_{\sqrt{x}}^{x^2} \tan^2 t \, dt = \int_{\sqrt{x}}^{0} \tan^2 t \, dt + \int_{0}^{x^2} \tan^2 t \, dt \]
\[ = - \int_{0}^{\sqrt{x}} \tan^2 t \, dt + \int_{0}^{x^2} \tan^2 t \, dt, \]
and apply the FTC and the chain rule (for the first integral we use $u = \sqrt{x}$, and for the second integral we use $u = x - 2$).

We get
\[ f'(x) = -\frac{du}{dx} \frac{d}{du} \left( \int_{0}^{u} \tan^2 t \, dt \right) \bigg|_{u=\sqrt{x}} + \frac{du}{dx} \frac{d}{du} \left( \int_{0}^{u} \tan^2 t \, dt \right) \bigg|_{u=x-2} \]
\[ = -\frac{1}{2\sqrt{x}} \tan^2(\sqrt{x}) + \tan^2(x - 2). \]