1. Determine these EASY antiderivatives. You should be able to do these very well. In these problems, no justification is needed. Remember the '+C'.

(a) \[ \int \frac{2}{x} \, dx \]
Solution:
\[ \int \frac{2}{x} \, dx = 2 \ln |x| + C \]

(b) \[ \int 2 \sin(x) \, dx \]
Solution:
\[ \int 2 \sin(x) \, dx = -2 \cos(x) + C \]

(c) \[ \int e^{4x} \, dx \]
Solution:
\[ \int e^{4x} \, dx = \frac{1}{4} e^{4x} + C \]

(d) \[ \int \frac{2}{t^2 + 1} \, dt \]
Solution:
\[ \int \frac{2}{t^2 + 1} \, dt = 2 \arctan(t) + C \]

(e) \[ \int \frac{1}{\sqrt{1-u^2}} \, du \]
Solution:
\[ \int \frac{1}{\sqrt{1-u^2}} \, du = \arcsin(u) + C \]
2. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

15 pts (a) Suggested method: substitution \( \int \frac{y}{1 + y^2} \, dy \)

**Solution:** Make the substitution \( u = 1 + y^2 \), so that \( du = 2\,dy \), or \( \frac{1}{2} du = dy \). Then

\[
\int \frac{y}{1 + y^2} \, dy = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln|u| + C = \frac{1}{2} \ln|1 + y^2| + C
\]

Note that since \( 1 + y^2 > 0 \) for all \( y \), the absolute value is not necessary; the answer \( \frac{\ln(1 + y^2)}{2} + C \) is fine, too.

15 pts (b) Suggested method: substitution \( \int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} \, dx \)

**Solution:** Make the substitution \( u = \sqrt{x+1} \). Then

\[
du = \frac{1}{2\sqrt{x+1}} \, dx \quad \text{or} \quad 2\,du = \frac{dx}{\sqrt{x+1}}.
\]

Thus,

\[
\int \frac{e^{\sqrt{x+1}}}{\sqrt{x+1}} \, dx = 2 \int e^u \, du = 2e^u + C = 2e^{\sqrt{x+1}} + C
\]

15 pts (c) Suggested method: substitution \( \int \frac{\ln(z)}{z} \, dz \)

**Solution:** Here, we let \( u = \ln(z) \) and so \( du = \frac{dz}{z} \). This means we have

\[
\int \frac{\ln(z)}{z} \, dz = \int u \, du = \frac{u^2}{2} + C = \frac{(\ln(z))^2}{2} + C
\]
3. In this question we tell you which method we suggest you use. Use the back of the previous page if you need more space.

(a) Suggested method: integration by parts \( \int x^6 \ln(x) \, dx \)

**Solution:** Take \( u = \ln(x) \) and \( dv = x^6 \, dx \). Then \( du = \frac{1}{x} \, dx \) and \( v = \frac{x^7}{7} \). So:

\[
\int x^6 \ln(x) \, dx = \frac{x^7 \ln(x)}{7} - \frac{1}{7} \int x^7 \cdot \frac{1}{x} \, dx = \frac{x^7 \ln(x)}{7} - \frac{1}{7} \int x^6 \, dx = \frac{x^7 \ln(x)}{7} - \frac{x^7}{49} + C
\]

(b) Suggested method: integration by parts \( \int xe^{2x} \, dx \)

**Solution:** Take \( u = x \) and \( dv = e^{2x} \, dx \). Then \( du = dx \) and \( v = \frac{1}{2} e^{2x} \), and so we have

\[
\int xe^{2x} \, dx = \frac{x e^{2x}}{2} - \frac{1}{2} \int e^{2x} \, dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} + C
\]

(c) Suggested method: integration by parts \( \int \sin(x)e^{3x} \, dx \)

**Solution:** Take \( u = e^{3x} \) and \( dv = \sin(x) \, dx \). Then \( du = 3e^{3x} \, dx \) and \( v = -\cos(x) \). So we have

\[
\int \sin(x)e^{3x} \, dx = -\cos(x)e^{3x} + 3 \int \cos(x)e^{3x} \, dx
\]

(we have a + before the integral because we were subtracting a negative). To do the second integral, we take \( u = e^{3x} \) and \( dv = \cos(x) \, dx \). Then \( du = 3e^{3x} \, dx \) and \( v = \sin x \). This gives us

\[
\int \sin(x)e^{3x} \, dx = -\cos(x)e^{3x} + 3 \left( \sin(x)e^{3x} - 3 \int \sin(x)e^{3x} \, dx \right)
\]

Multiplying out gives

\[
\int \sin(x)e^{3x} \, dx = -\cos(x)e^{3x} + 3\sin(x)e^{3x} - 9 \int \sin(x)e^{3x} \, dx
\]

or, equivalently,

\[
10 \int \sin(x)e^{3x} \, dx = -\cos(x)e^{3x} + 3\sin(x)e^{3x} + C
\]

Thus, we have

\[
\int \sin(x)e^{3x} \, dx = \frac{-\cos(x)e^{3x} + 3\sin(x)e^{3x}}{10} + C
\]
4. Determine the following antiderivatives. Use the back of the previous page if you need more space.

(a) \[ \int \sin^3(x) \, dx \]

**Solution:** We use the identity \( \sin^2(x) = 1 - \cos^2(x) \) to get

\[ \int \sin^3(x) \, dx = \int (1 - \cos^2(x)) \sin(x) \, dx. \]

Now take \( u = \cos(x) \) and \( du = -\sin(x) \, dx \), giving

\[ \int \sin^3(x) \, dx = -\int (1 - u^2) \, du = -u + \frac{u^3}{3} + C = \frac{\cos^3(x)}{3} - \cos(x) + C \]

(b) \[ \int \frac{1}{\sec(2x)} \, dx \]

**Solution:**

\[ \int \frac{1}{\sec(2x)} \, dx = \int \cos(2x) \, dx = \frac{1}{2} \sin(2x) + C \]

(c) \[ \int \frac{1}{x^2 \sqrt{x^2 - 1}} \, dx \]

**Solution:** Take \( x = \sec \theta \) so \( dx = \sec \theta \tan \theta \, d\theta \). Then we have

\[ \int \frac{1}{x^2 \sqrt{x^2 - 1}} \, dx = \int \frac{\sec \theta \tan \theta \, d\theta}{\sec^2 \theta \sqrt{\sec^2 \theta - 1}} = \int \frac{\tan \theta \, d\theta}{\sec \theta \tan \theta} = \int \frac{d\theta}{\sec \theta} = \int \cos \theta \, d\theta \]

This means we have \( \sin \theta + C \) as our answer, but of course we need the answer in terms of \( x \). Recall that we took \( x = \sec \theta \), and so \( \sin \theta = \frac{\sqrt{x^2 - 1}}{x} \) (see figure). Thus, we have shown

\[ \int \frac{1}{x^2 \sqrt{x^2 - 1}} \, dx = \frac{\sqrt{x^2 - 1}}{x} + C \]
5. Evaluate these definite integrals. Use the back of the previous page if you need more space.

15 pts  
(a) \[ \int_{-1}^{1} \frac{1}{1-x^2} \, dx \]

**Solution:** We use partial fractions:

\[ \frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x} \]

so \[ 1 = A(1-x) + B(1+x). \] Thus

\[ A + B = 1 \]
\[ -A + B = 0 \]

hence \[ A = \frac{1}{2}, \ B = \frac{1}{2} \]

\[ \int_{-1}^{1} \frac{1}{1-x^2} \, dx = \int_{-1}^{1} \frac{1/2}{1+x} + \frac{1/2}{1-x} \, dx = \frac{1}{2} \ln |1+x| - \frac{1}{2} \ln |1-x| \bigg|_{-1/2}^{1/2} \]

\[ = \frac{1}{2} \left[ \ln \left( \frac{3}{2} \right) - \ln \left( \frac{1}{2} \right) - \ln \left( \frac{1}{2} \right) + \ln \left( \frac{3}{2} \right) \right] \]

\[ = \ln \left( \frac{3}{2} \right) - \ln \left( \frac{1}{2} \right) = \ln(3) \]

15 pts  
(b) \[ \int_{-100}^{100} \frac{\sin^{21}(x)}{1 + e^{x^2}} \, dx \]

**Solution:** Since \( \frac{\sin^{21}(x)}{1 + e^{x^2}} \) is an odd function and the bounds are symmetric with respect to 0, the value of the integral is 0.

15 pts  
(c) \[ \int_{0}^{1} \frac{x}{\sqrt{4-x^2}} \, dx \]

**Solution:** Let \( u = 4 - x^2 \) so that \( du = -2x \, dx \). When \( x = 0, u = 4 \) and when \( x = 1, u = 3 \). Thus we have

\[ \int_{0}^{1} \frac{x}{\sqrt{4-x^2}} \, dx = -\int_{4}^{3} \frac{du}{2\sqrt{u}} = -\sqrt{u} \bigg|_{4}^{3} = -\sqrt{3} + \sqrt{4} = 2 - \sqrt{3}. \]
6. Since \( \int_0^1 \frac{1}{1 + x^2} \, dx = \arctan(1) = \frac{\pi}{4} \), evaluating the integral \( \int_0^1 \frac{4}{1 + x^2} \, dx \) gives \( \pi \).

(a) Use Simpson’s rule with 2 intervals to estimate \( \int_0^1 \frac{4}{1 + x^2} \, dx \).

**Solution:** Since there are two intervals, the width of each is \( \frac{1}{2} \). Thus, Simpson’s rule gives:

\[
\frac{1}{3} \cdot \frac{1}{2} \left( f(0) + 4f(1/2) + f(1) \right) = \frac{1}{6} \left( 4 + 4 \left( \frac{4}{1 + 1/4} \right) + 2 \right) = \frac{94}{30} \approx 3.1333
\]

(b) How many intervals are needed to estimate \( \int_0^1 \frac{4}{1 + x^2} \, dx = \pi \) within .0001 using the trapezoid rule?

**Solution:** We use the information in the footnote. We need to determine \( n \) so that

\[
\frac{1}{12n^2} K \leq .0001
\]

where \( K \) is the maximum of the absolute value of the second derivative of \( 4/(1+x^2) \) for \( x \) between 0 and 1. Since \( \left| \frac{4(6x^2 - 2)}{(1 + x^2)^3} \right| \) is a decreasing function on this interval, the maximum occurs at \( x = 0 \), so we take \( K = |-8/1| = 8 \).

To solve \( \frac{8}{12n^2} \leq .0001 \), we multiply both sides by \( 10000n^2 \) to get

\[
\frac{80000}{12} \leq n^2,
\]

so \( n \) is the smallest integer bigger than \( \sqrt{20000/3} \approx 81.6 \).
Thus, \( n = 82 \).

---

1Use the following estimate for \( E_T \) using \( n \) intervals: If \( |f''(x)| \leq K \) then \( E_T \leq K \frac{(b-a)^3}{12n^2} \).

If \( f(x) = \frac{1}{1 + x^2} \), then \( f''(x) = \frac{6x^2 - 2}{(1 + x^2)^3} \).