Sequences

Can be described by:

- words
- several terms $a_1, a_2, a_3, a_n, \ldots$
- recursive relation
- general (explicit) formula

Examples

1) Given the first four terms

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$$

write down the next two terms of the sequence, find the general formula.

Solution Since $2 = \frac{2}{1}$ we see that $n$th number in the sequence is $\frac{n+1}{n}$.

The general formula: $a_n = \frac{n+1}{n}$.

After $a_4 = \frac{5}{4}$ we have $a_5 = \frac{6}{5}$, $a_6 = \frac{7}{6}$.

Answer $\frac{6}{5}, \frac{7}{6}$, $a_n = \frac{n+1}{n}$.

2) A sequence starts with $a_1 = 2$ and $a_2 = 0.5$.

Each next term is a half difference of the previous and second previous. Write the recursive formula and find $a_3, a_4, a_5$.

Solution $a_{n+1} = \frac{1}{2} (a_n - a_{n-1})$. 
2. 

\[ a_3 = \frac{1}{2} (a_2 - a_1) = \frac{1}{2} (0.5 - 2) = -0.75 \]

\[ a_4 = \frac{1}{2} (a_3 - a_2) = \frac{1}{2} (-0.75 - 0.5) = -0.625 \]

\[ a_5 = \frac{1}{2} (a_4 - a_3) = \frac{1}{2} (-0.625 - (-0.75)) = 0.0625 \]

**Answer:** \[ a_{n+1} = \frac{1}{2} (a_n - a_{n-1}) \]

\[ a_3 = -0.75, \quad a_4 = -0.625, \quad a_5 = 0.0625 \]

Linear growth model means that the population changes by the same amount in equal time periods.

Population sequence is of the form:

- \( P_0, P_0 + d, P_0 + 2d, P_0 + 3d, \ldots \)
- Recursive formula: \( P_{n+1} = P_n + d \)
- General formula: \( P_n = P_0 + nd \)

\( d \) is the common difference.

The sequence \( P_n \) is an arithmetic sequence.

**Example** The population of deers in a forest was 100 in 2010 and 130 in 2016. Assuming linear growth in which year it will reach 200?

**Solution** \( P_n = P_0 + nd \). Since the question is "which year" the equal time periods are one year. \( P_0 = 100 \) in 2010. Let \( P_n \) be the population \( n \) years after 2010.
Then \( P_6 = 130 \) in 2016.

\[
\begin{align*}
    P_n &= P_0 + n \cdot d \\
    P_6 &= P_0 + 6d \\
    130 &= 100 + 6d \\
    6d &= 30 \\
    d &= 5.
\end{align*}
\]

\[
P_n = 100 + 5n
\]

When \( P_n = 200 \):

\[
200 = 100 + 5n
\]

\[5n = 100, \quad n = 20.
\]

Answer in 2030.

**Arithmetic sum formula:**

\[
P_0 + P_1 + \ldots + P_{n-1} = \frac{P_0 + P_{n-1}}{2} \cdot n
\]

**Example** Find \(3 + 5 + 7 + \ldots + 37\).

**Solution** \(P_0 = 3, \quad d = 2\)

\[
P_n = P_0 + n \cdot d = 3 + 2n
\]

We want: \(P_{n-1} = 37\)

\[
3 + 2(n-1) = 37, \quad 2n = 36, \quad n = 18.
\]

Thus, \(3 + 5 + 7 + \ldots + 37 = P_0 + P_1 + \ldots + P_{17} = \frac{P_0 + P_{17}}{2} \cdot 18 = \frac{3 + 37}{2} \cdot 18 = 360\).
4. The exponential growth model means that in equal time periods the population grows by the same constant factor $R$. The population sequence has the form:

- $P_0, R P_0, R^2 P_0, R^3 P_0, \ldots$
- Recursive formula: $P_{n+1} = R P_n$
- General formula: $P_n = R^n P_0$

$R$ is called the common ratio.

The sequence $P_n$ is a geometric sequence.

**Example** In 2010, the population of Manhattan was $\approx 1,585,873$, in 2013 $\approx 1,626,159$. Assuming exponential growth, what will be the population in 2022?

**Solution** 2013 is 3 years after 2010.

2022 is 12 years after 2010.

3 divides 12, so we can choose time period 3 years. $P_n$ = the population 3n years after 2010.

In 2010: $P_0 = 1,585,873$

In 2013: $P_1 = 1,626,159$

In 2022: $P_4 = ?$

We have: $P_1 = R P_0$, so $R = \frac{P_1}{P_0} = \frac{1,626,159}{1,585,873} \approx 1.025403$

$P_4 = R^4 P_0 = 1.025403^4 \cdot 1,585,873 \approx 1,753,262$

Answer: about 1,753,262
5. Geometric sum formula:
\[ P_0 + R P_0 + R^2 P_0 + \ldots + R^{n-1} P_0 = \frac{R^n - 1}{R - 1} P_0. \]

**Example** Find
\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{10}} \]

**Solution** \( P_0 = 1 \), \( R = \frac{1}{2} \), \( P_n = 1 \left( \frac{1}{2} \right)^n = \frac{1}{2^n} \)

We want: \( R^{n-1} P_0 = \frac{1}{2^{10}} \)
\( \left( \frac{1}{2} \right)^{n-1} = \frac{1}{2^{10}}, \) so \( n-1 = 10 \), \( n = 11 \).

By the formula,
\[ 1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^{10}} = \frac{R^n - 1}{R - 1} P_0 = \frac{\left( \frac{1}{2} \right)^{11} - 1}{\frac{1}{2} - 1} = 2 - \frac{1}{2^{10}} \]

**Answer:** \( 2 - \frac{1}{2^{10}} \)

**Logistic growth model**

Elements: maximal carrying capacity \( C \), population value \( P_n \), natural growth parameter \( R \).

Population satisfies the recurrent formula
\[ p_{n+1} = R \left( 1 - \frac{p_n}{C} \right) p_n \]

**Example** \( R = 0.8 \), \( P_0 = 9000 \), \( C = 10000 \).
Find \( P_1, P_2, P_3, P_4, P_5 \). Predict the long term behavior.
6. Solution  \[ p-value \quad p_0 = \frac{P_0}{C} = \frac{9000}{10000} = 0.9. \]
\[ p_1 = R (1-p_0)p_0 = 0.8 \cdot 0.1 \cdot 0.9 = 0.072, \quad P_1 = Cp_1 = 720 \]
\[ p_2 = R (1-p_1)p_1 = 0.8 \cdot (1-0.072) \cdot 0.072 \approx 0.0535, P_2 = Cp_2 = 535 \]
\[ p_3 = R (1-p_2)p_2 \approx 0.0405, \quad P_3 = Cp_3 = 405 \]
\[ p_4 \approx 0.0311, \quad P_4 = 311 \]
\[ p_5 \approx 0.0241, \quad P_5 = 241. \]

We see that the population constantly decreases. Prediction: the population will decrease to extinction.

percentages

- P% as decimal is \( p = \frac{P}{100} \)
- P% of B is \( p \cdot B = \frac{P \cdot B}{100} \)
- Starting with baseline value B adding P% you get \( F = (1+p)B = (1+\frac{P}{100})B \)

Example Kevin's salary was $60,000. First in increased by 5%, then by 8%. But later it decreased by 3%. What's the final salary?

Solution \( p_1 = 0.05, \ p_2 = 0.08, \ p_3 = -0.03 \)

\[
\begin{align*}
B &= 60,000. \\
F_1 &= B(1+p_1) = 60000 \cdot 1.05 = 63000 \\
F_2 &= F_1(1+p_2) = 63000 \cdot 1.08 = 68040 \\
F_3 &= F_2(1+p_3) = 68040 \cdot 0.97 = 65999 = 66000
\end{align*}
\]

Answer: about $66,000

Interest

Basic elements

- Principal P, final value F, total interest I = F - P
- Interest rate r
- Term t
Simple interest means that the interest rate is applied to the principal value only and does not accumulate.

**Simple interest formula:**
\[ F = P (1 + r) \]

- **F** = final principal
- **P** = principal
- **r** = periodic interest rate term (number of times interest rate applied)

**Example** A government bond has price $5000 and the future value after 4 years is $6000. What is the annual percentage rate?

**Solution** \( P = 5000, \ F = 6000, \ t = 4 \).

\[ 6000 = 5000 \cdot (1 + r \cdot 4) \]
\[ 1000 = 5000 \cdot r \cdot 4 \]
\[ r = \frac{1000}{5000 \cdot 4} = 0.05 \]

The percentage rate is \( 100 \cdot 0.05 = 5\% \).

**Answer** APR is 5\%. 
Compound interest means that interest is applied to both the principal value and the previously accumulated interest.

\[ F = P(1+r)^t \]

Example: Borrowing $5000 for two years with monthly compounding, APR = 3.6%.

How much to return?

Solution: \( P = 5000 \), decimal value of APR is \( \frac{3.6}{100} = 0.036 \). Monthly interest rate is \( r = \frac{0.036}{12} = 0.003 \). The term is \( t = 2 \times 12 = 24 \) months. Thus

\[ F = P(1+r)^t = 5000 \cdot (1.003)^{24} \approx 5373 \]

Answer: 5373.

Annual percentage yield is the actual annual interest rate. If APR decimal value is \( r \) then APY decimal value is \( \left(1 + \frac{r}{n}\right)^n - 1 \) if compounded \( n \) times per year.
10. Example Which rate gives more interest:
   a) 10% compounded annually or
   b) 9.8% compounded monthly

Solution Let's find the effective interest rate APY in b) to compare it with a).

APY decimal value is

\[
(1 + \frac{r}{n})^n - 1 \quad \text{with} \quad r = \frac{9.8}{100} = 0.098, \; n=12.
\]

\[
(1 + \frac{0.098}{12})^{12} - 1 \approx 0.1025
\]

Thus, in b) APY is \(\approx 10.25\% > 10\%\).

So, even though in a) APR is higher, the accumulated interest in a) is lower than in b).

Answer 9.8% compounded monthly is higher
11. **Installment loans**

Amortization formula for monthly payments:

\[ M = P \frac{r(1+r)^T}{(1+r)^T-1} \]

where \( r = \frac{r}{12} \) is the monthly interest rate.

**Example**

Buying a house for $500,000 financing for 27 years at 4% APR. How much will be paid in total?

**Solution**

\[ \begin{align*}
M &= \frac{P \cdot r}{1 - \left(1 + \frac{r}{12}\right)^{-1}} \\
&= \frac{500000 \cdot \frac{0.04}{12}}{1 - \left(1 + \frac{0.04}{12}\right)^{-324}} \\
&= \frac{500000 \cdot 0.003333}{1.003333^{324} - 1} \\
&\approx 2562.38 \\
\end{align*} \]

**Total payment**

\[ T \cdot M = 324 \cdot 2562.38 \approx 830211 \]
Basic facts:
- $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$
- $F_1 = F_2 = 1$, recursive formula $F_{n+1} = F_n + F_{n-1}$
- General formula (Binet) $F_n = \left[\frac{(\sqrt{5}+1)^n}{\sqrt{5}}\right]$
- $\frac{F_{n+1}}{F_n}$ approach the golden ratio $\phi = \frac{\sqrt{5}+1}{2} \approx 1.618$

Examples:

1. Simplify $2F_{n+1} - F_n + F_{n-2}$

Solution:

$2F_{n+1} - F_n + F_{n-2} = F_{n+1} + F_{n+1} - F_n + F_{n+2}$

$F_{n+1} + \frac{F_{n+1} + F_{n-2}}{F_n} = F_{n+1} + F_n = F_{n+2}$

Answer: $F_{n+2}$.

2. Find approximate value of $\frac{F_{31}}{F_{30}}$ up to three decimal digits.

Solution:

$\frac{F_{31}}{F_{30}}$ approaches $\phi = \frac{\sqrt{5}+1}{2} \approx 1.618$

Answer: $1.618$
13. 3. Find \( F_{18} \)

**Solution** \( F_{18} = \lfloor (\frac{\sqrt{5}+1}{2})^{18} / \sqrt{5} \rfloor = \lfloor 2584.00008... \rfloor = 2584 \).

**Answer** 2584.

**Golden ratio** \( \varphi = \frac{\sqrt{5}+1}{2} \approx 1.618 \) is a positive solution of \( \varphi^2 = \varphi + 1 \).

\[ \varphi^n = F_n \cdot \varphi + F_{n-1} \]

**Example** Find \( \varphi^7 \) without calculating powers of numbers. Write three decimal digits.

**Solution** \( \varphi^7 = F_7 \cdot \varphi + F_6 = 13 \cdot \frac{\sqrt{5}+1}{2} + 8 \approx 29.034 \)

**Answer** 29.034

**Sum of the first \( n \) Fibonacci numbers**

\[ F_1 + F_2 + \ldots + F_n = F_{n+2} - 1 \]

**Sum of the squares**

\[ F_1^2 + F_2^2 + \ldots + F_n^2 = F_n \cdot F_{n+1} \]
14. **Pairwise-comparison method**

**Example** Find the complete ranking in the following elections using pairwise comparison method

<table>
<thead>
<tr>
<th>Number of voters</th>
<th>2</th>
<th>6</th>
<th>3</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st</td>
<td>A</td>
<td>B</td>
<td>B</td>
<td>D</td>
</tr>
<tr>
<td>2nd</td>
<td>B</td>
<td>D</td>
<td>D</td>
<td>C</td>
</tr>
<tr>
<td>3rd</td>
<td>D</td>
<td>A</td>
<td>C</td>
<td>A</td>
</tr>
<tr>
<td>4th</td>
<td>C</td>
<td>C</td>
<td>A</td>
<td>B</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th></th>
<th>Count</th>
<th>Winner</th>
</tr>
</thead>
<tbody>
<tr>
<td>A v B</td>
<td>10 : 9</td>
<td>A</td>
</tr>
<tr>
<td>A v C</td>
<td>8 : 11</td>
<td>C</td>
</tr>
<tr>
<td>A v D</td>
<td>2 : 17</td>
<td>D</td>
</tr>
<tr>
<td>B v C</td>
<td>11 : 8</td>
<td>B</td>
</tr>
<tr>
<td>B v D</td>
<td>11 : 8</td>
<td>B</td>
</tr>
<tr>
<td>C v D</td>
<td>0 : 19</td>
<td>D</td>
</tr>
</tbody>
</table>

A : 1, B: 2, C: 1, D: 2

**Answer** B, D share first-second, A, C share third-fourth
Banzhaf power

Basic elements: winning coalition, critical player, critical counts \( B_1, B_2, \ldots B_N \), total critical count \( T = B_1 + B_2 + \ldots + B_N \), Banzhaf power indexes
\[
\beta_1 = \frac{B_1}{T}, \quad \beta_2 = \frac{B_2}{T}, \quad \ldots \quad \beta_N = \frac{B_N}{T}.
\]

Example: Find Banzhaf power in the weighted voting system
\[[8: 5, 5, 3, 1]\]

Solution

<table>
<thead>
<tr>
<th>Winning coalitions</th>
<th>Weights</th>
</tr>
</thead>
<tbody>
<tr>
<td>( {P_1, P_2} )</td>
<td>10</td>
</tr>
<tr>
<td>( {P_1, P_3} )</td>
<td>8</td>
</tr>
<tr>
<td>( {P_2, P_3} )</td>
<td>8</td>
</tr>
<tr>
<td>( {P_1, P_2, P_3} )</td>
<td>13</td>
</tr>
<tr>
<td>( {P_1, P_2, P_4} )</td>
<td>11</td>
</tr>
<tr>
<td>( {P_1, P_3, P_4} )</td>
<td>9</td>
</tr>
<tr>
<td>( {P_2, P_3, P_4} )</td>
<td>9</td>
</tr>
<tr>
<td>( {P_1, P_2, P_3, P_4} )</td>
<td>14</td>
</tr>
</tbody>
</table>

\( B_1 = 4, \quad B_2 = 4, \quad B_3 = 4, \quad B_4 = 0, \quad T = 12 \)

Answer: \( \beta_1 = \beta_2 = \beta_3 = \frac{4}{12} = \frac{1}{3}, \beta_4 = 0 \).
16. Method of markers

Fair division method for fine-grained or continuous asset.

Example: Describe the division and the leftover.

```
A      B      C
 1 2 3 4 5 6 7 8 9 10 11 12 13 14
A1 B1 D1 A2 B2 C2 B3 A3 D3 C3 D2
```

Answer: A gets 1-2, B gets 4-7, C gets 13-14, D gets 10-11. Leftover: 3, 8, 9, 12.

Eulerization is doubling some edges of the graph to make all vertices even degree. After Eulerization in a connected graph it is possible to find a route covering each edge once and returning to the initial vertex (Euler circuit).

Example: Find an effective route for a security starting and ending at A.

Solution: There are 4 odd vertices: A, B, D, F. Need to double some edges to make them even.
On the new graph, we can find an Euler's path using Fleury's algorithm.

Double edges mean security will need to cover corresponding segments twice.

Nearest Neighbor algorithm
An approximate algorithm for solving Traveling Salesman Problem. Each step go to the "closest" among remaining vertexes.

Example
Find an effective route visiting each city starting and ending at C. Find the cost.

Solution
\[ C \rightarrow D \rightarrow A \rightarrow B \rightarrow E \rightarrow C \]
Cost: \(4 + 5 + 6 + 7 + 9 = 31\)
7)

Peter,

Chocolate: $24 - $12 = $12
Cheese: $18 - $18 = $0

1) Peter

Paul

2) Peter

3) Mary

Peter

Paul

Mary

Paul

Mary

Paul + Mary = $30
Arthur, Brian, Carl

Arthur: likes C = 0, hates S, V

Brian: likes C = S, hates 0, V

Carl: likes C = V, hates 0, S

Carl, Arthur are dividers, Carl divides:

![Diagram of preferences]

Division
10) Subdivisions:

Arthur

Carl

Values:

Arthur

Carl

$\frac{1}{2}$ of $C$ +
$\frac{1}{3}$ of $V$ = $33.3\%$ of the total value
Whole $S + 60^\circ$ of $C = 150^\circ$ of the total $180^\circ$.

\[ \frac{150^\circ}{180^\circ} \times 100\% \approx 83.3\% . \]