ALEXANDROV IMMERSIONS, HOLONOMY AND MINIMAL SURFACES IN S³

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ABSTRACT. We prove that compact 3-manifolds M of constant curvature $+1$ with boundary a minimal surface are locally naturally parametrized by the conformal class of the boundary metric γ in the Teichmüller space of ∂M when $genus(\partial M) \geq 2$. Stronger results are obtained in the case of genus 1, giving in particular a new proof of Brendle's solution of the Lawson conjecture. The results generalize to constant mean curvature surfaces, and surfaces in flat and hyperbolic 3-manifolds.

1. INTRODUCTION

In this paper, we consider compact embedded minimal surfaces in $S^3 = S^3(1)$ and related spaces of constant positive curvature. After earlier work on uniqueness of minimal spheres S^2 immersed in S^3 (the Almgren-Hopf uniqueness theorem [2]), this subject essentially began in earnest with the groundbreaking work of Lawson [22], who constructed the first examples of embeddded minimal surfaces of arbitrary genus in S^3. Since then, there have been many further examples and constructions of such surfaces, such as those in [12], [18], [19]; we refer to [11] for an excellent recent survey of the area. However, a full classification of embedded minimal surfaces is still far from being understood.

An important recent breakthrough on this issue is the solution of the Lawson conjecture by Brendle [9] that the Clifford torus is the unique minimal embedded torus in S^3, (up to rigid motions). This settles the classification issue for embedded minimal tori $T^2 \subset S^3$.

We note that the consideration of compact embedded minimal surfaces as opposed to general immersed surfaces is a strong restriction. There are many more immersed minimal surfaces than embedded ones. An important intermediate class of surfaces between embeddings and immersions are the *Alexandrov immersed* minimal surfaces. Recall that an immersed compact surface $f : \Sigma \rightarrow S^3$ is Alexandrov immersed if there is a compact 3-manifold M with $\partial M = \Sigma$ and an immersion $F : M \rightarrow S^3$ such that $F|_{\partial M} = f$. This is a natural generalization of embeddings, since an embedded minimal surface $\Sigma \subset S^3$ divides S^3 into two components $M_1 \cup M_2$ of the complement $S^3 \setminus \Sigma$. Hence the embedding $f : \Sigma \subset S^3$ extends to a pair of embeddings $F_i : M_i \subset S^3$, $i = 1, 2$.

In this work, we study the space of minimal embeddings, or more generally Alexandrov immersions, of a surface in S^3, and in more general spaces of constant positive curvature. The main point of view is to focus on the geometry of the constant curvature 3-manifolds M with boundary $\partial M = \Sigma$ rather than on the embeddings $\Sigma \subset S^3$ themselves. Thus, the analysis is naturally adapted to the class of Alexandrov immersed minimal surfaces in S^3.

To describe the point of view, let for the moment M be an arbitrary compact $(n+1)$-dimensional manifold with non-empty boundary ∂M, and let $\mathcal{E} = \mathcal{E}^{m,\alpha}$ be the moduli space of Einstein metrics

$$(1.1) \qquad\qquad Ric_g = \lambda g$$

on M which are $C^{m,\alpha}$ up to the boundary ∂M; here $\lambda \in \mathbb{R}$ is arbitrary but fixed and $m \geq 3$, $\alpha \in (0, 1)$. The space $\mathcal{E}^{m,\alpha}$ is defined to be the space of all λ-Einstein metrics $\mathbb{E}^{m,\alpha}$, modulo the

action of diffeomorphisms $\text{Diff}_1^{m+1,\alpha}(M)$ of M equal to the identity on ∂M. Let $\mathcal{C} = \mathcal{C}^{m,\alpha}$ be the space of (pointwise) conformal classes $[\gamma]$ of $C^{m,\alpha}$ metrics γ on ∂M and let $C(\partial M) = C^{m-1,\alpha}(\partial M)$ be the space of $C^{m-1,\alpha}$ functions on ∂M. A metric $g \in \mathcal{E}$ induces by restriction a metric γ on ∂M. Let H denote the mean curvature of ∂M in (M, g) with respect to the outward normal N, (so that $H > 0$ means the area is increasing in the outward direction).

It is proved in [3] that $\mathcal{E}$ is a smooth Banach manifold (at least under the topological assumption $\pi_1(M, \partial M) = 0$) and, if $\mathcal{E}$ is non-empty, the natural boundary map

(1.2)
$$\Pi : \mathcal{E} \to \mathcal{C} \times C(\partial M),$$
$$\Pi(g) = ([\gamma], H),$$

is Fredholm, of Fredholm index 0. The map Π associates to the ambient Einstein manifold (M, g) the pointwise conformal class of the induced metric γ and mean curvature H of the boundary ∂M in M.

Consider now the case $n = 2$, so M is a 3-manifold, with boundary a surface $\Sigma = \partial M$ and suppose $\lambda = 2$ so that (M, g) is of constant curvature $+1$. A $C^{m+1,\alpha}$ Alexandrov immersion $f : \Sigma \to S^3$ gives an element $(M, g) \in \mathcal{E}^{m,\alpha}$ with $g = F^*(g_{S^3})$, where $F : M \to S^3$ is a $C^{m+1,\alpha}$ extension of f. Observe that the data (M, g) is in fact somewhat more general than an Alexandrov immersion. Namely the developing map of (M, g) gives an isometric immersion of the universal cover $(\widetilde{M}, g) \to S^3$; one obtains an isometric immersion of (M, g) itself into S^3 only when the holonomy of (M, g) is trivial.

Suppose first $genus(\partial M) \geq 2$. Let $\text{Diff}_0(M) = \text{Diff}_0^{m+1,\alpha}(M)$ be the group of $C^{m+1,\alpha}$ diffeomorphisms of M isotopic to the identity and mapping the boundary ∂M into itself. Thus $\text{Diff}_1(M) \subset \text{Diff}_0(M)$ and the quotient $\text{Diff}_0(M)/\text{Diff}_1(M) \simeq \text{Diff}_0(\partial M)$ is the group of diffeomorphisms of the boundary isotopic to the identity. The group $\text{Diff}_0(\partial M)$ also acts on $\mathcal{C}$, and it is well known that the action is free with quotient the Teichmüller space $\mathcal{T}(\partial M)$ of ∂M.

When $\partial M = T^2$, let $\text{Diff}_0(M)$ denote the diffeomorphisms of M isotopic to the identity, mapping ∂M to ∂M, and which fix a given point $p_0 \in \partial M$. Then again $\mathcal{C}/\text{Diff}_0(\partial M) = \mathcal{T}(\partial M) = \mathcal{T}(T^2)$, cf. [13] for instance.

Let

(1.3)
$$\widetilde{\mathcal{E}} = \mathbb{E}/\text{Diff}_0(M),$$

be the quotient space of $\mathbb{E}$ by the action of $\text{Diff}_0(M)$. In general, the boundary map Π does not descend to a map on $\widetilde{\mathcal{E}}$, since the mean curvature H is not invariant under the action of $\text{Diff}_0(\partial M)$. However, minimal surface boundaries $H = 0$, or more generally constant mean curvature boundaries, are invariant under reparametrizations or diffeomorphisms of ∂M.

Thus, let

(1.4)
$$\mathcal{M} = \Pi^{-1}(\mathcal{C}, 0)/\text{Diff}_0(M),$$

be the moduli space of constant curvature $+1$ spaces $(M, g) \in \widetilde{\mathcal{E}}$ with minimal surface boundary. By [22], $\mathcal{M}$ is non-empty. The boundary map Π descends to a map

$$\Pi : \mathcal{M} \to \mathcal{T}(\partial M),$$

associating to $(M, g) \in \mathcal{M}$ the conformal class of $(\partial M, \gamma)$ in Teichmüller space.

The main result of this paper is the following "regular value theorem":

Theorem 1.1. *Suppose $genus(\partial M) \geq 2$. Then for any conformal class $[\gamma] \in \mathcal{C}$, $([\gamma], 0)$ is a regular value of the boundary map Π in (1.2), so that Π is a local diffeomorphism whenever ∂M is minimal.*

Consequently, the space $\mathcal{M}$ is a smooth manifold and the smooth map

(1.5)
$$\Pi : \mathcal{M} \to \mathcal{T}(\partial M),$$

is everywhere a local diffeomorphism. In particular the linearization

$$D\Pi : T(\mathcal{M}) \to T(\mathcal{T}(\partial M))$$

is an isomorphism, at any $(M, g) \in \mathcal{M}$ and

$$dim\,\mathcal{M} = dim(\mathcal{T}(\partial M)).$$

This result shows that the space of compact, oriented 3-manifolds of constant curvature $+1$ with minimal surface boundary is naturally locally parametrized by the conformal class of the boundary metric γ in the Teichmüller space $\mathcal{T}(\partial M)$ of the boundary ∂M. The mapping class group $\Gamma(\partial M)$ of ∂M acts on both factors and Π is equivariant with respect to these actions, so that Π in (1.5) descends to map of the corresponding moduli spaces:

$$(1.6) \qquad\qquad \Pi : \mathcal{M}/\Gamma(\partial M) \to \mathcal{T}(\partial M)/\Gamma(\partial M).$$

The global behavior of the map Π in (1.5) will not be studied here (except see Theorems 1.2-1.3 below for the $genus(\partial M) = 1$ case); we hope to discuss this in detail elsewhere. Note that $\mathcal{M}$ may have many and possibly infinitely many components; for instance one expects that the components of $S^3 \setminus \Sigma$ where S is an embedded minimal surface in S^3 lie in different components of $\mathcal{M}$.

Theorem 1.1 is not true as it stands for $genus(\partial M) = 1$; this case is discussed further in Theorems 1.2 and 1.3 below.

Theorem 1.1 implies in particular that any Alexandrov immersed minimal surface $\Sigma \subset S^3$ may be perturbed to a "minimal boundary" $\Sigma = \partial M$ in a space of constant curvature (M, g). For instance, all Jacobi fields on Σ are "integrable", i.e. tangent to a curve of Alexandrov immersed minimal surfaces, if one allows the ambient geometry to vary within the class of constant curvature $+1$ metrics. However, generically (M, g) will not isometrically immerse in S^3. Namely, any $(M, g) \in \widetilde{\mathcal{E}}$ has a holonomy map

$$(1.7) \qquad\qquad \rho : \pi_1(M) \to SO(4) = Isom^+(S^3).$$

The map ρ is a homomorphism and the space $Hom(\pi_1(M), SO(4))/Ad$ of all such homomorphisms modulo conjugacy is the representation variety $\mathcal{R}(M)$ of M, (also sometimes called the character variety of M);

$$(1.8) \qquad\qquad \mathcal{R}(M) = Hom(\pi_1(M), SO(4))/Ad,$$

cf. [15], [21], [27]. The space (M, g) isometrically immerses in S^3 only when $\rho = e$ is the trivial map. As shown in Section 2.2, the space $\mathcal{R}(M)$ has the same topological dimension as the Teichmüller space $\mathcal{T}(\partial M)$ of the boundary,

$$dim(\mathcal{R}(M)) = dim(\mathcal{T}(\partial M)).$$

However, $\mathcal{R}(M)$ is compact, while $\mathcal{T}(\partial M)$ is always non-compact. Moreover, $\mathcal{T}(\partial M)$ is a smooth manifold, while $\mathcal{R}(M)$ is not; topologically it is a stratified manifold with non-trivial strata.

One has a canonical projection map

$$\pi : \widetilde{\mathcal{E}} \to \mathcal{R}(M),$$

associating to (the isometry class of) each constant curvature $+1$ metric g its holonomy $\rho = \rho(g)$. Letting $\iota : \mathcal{M} \to \widetilde{\mathcal{E}}$ denote the inclusion, one thus has a natural map

$$(1.9) \qquad\qquad \chi : \mathcal{M} \to \mathcal{R}(M), \quad \chi = \pi \circ \iota.$$

Of course χ will not be a local diffeomorphism everywhere, since $\mathcal{R}(M)$ is not smooth. In particular, $\mathcal{R}(M)$ is not smooth at the trivial holonomy map $\rho = e$ corresponding to $(M, g) \subset S^3$. However the singularity at e is quite simple (mainly since $\pi_1(M)$ has a simple structure) and is described by the Zariski tangent space to $\mathcal{R}(M)$, cf. Section 2.2 for further discussion.

The proof of Theorem 1.1 is partly based on a detailed study of the natural relations between the linearizations of Π and χ at any $(M, g) \in \mathcal{M}$. Another key ingredient in the proof is a detailed analysis of the second variation of the Einstein-Hilbert action (total scalar curvature functional), giving rise to a duality between Dirichlet and Neumann data associated to the boundary map (1.2). This is of course related to a study of the corresponding Dirichlet-to-Neumann map. Further motivation and explanation of the ideas of the proof are given at suitable places in the course of the proof in Section 3.

Next we turn to the case of toral boundary, $genus(\partial M) = 1$. Theorem 1.1 is not true in general in this case, due to the presence of tangential conformal Killing fields on T^2; these generate elements in the kernel of $D\Pi$. Nevertheless, almost all of the proof of Theorem 1.1 applies without change to the case of $genus(\partial M) = 1$ and, suitably modified, gives in fact stronger results.

In particular, as a by-product of the proof of Theorem 1.1, we obtain a new proof of Brendle's solution of the Lawson conjecture on the uniqueness of the Clifford torus in S^3.

Theorem 1.2. *[9], [10]. An embedded minimal torus $T^2 \subset S^3$ is congruent to the Clifford torus. Moreover, an Alexandrov immersed minimal torus in S^3 is necessarily a "surface of revolution", i.e. is invariant under an isometric S^1 action for some $S^1 \subset Isom(S^3)$.*

We note that the S^1-invariant tori in S^3 are fully classified by work of Hsiang-Lawson [17]. The analog of Theorem 1.1 in this setting is:

Theorem 1.3. *If $genus(\partial M) = 1$, then $\mathcal{M}$ is a smooth 2-dimensional manifold and the map Π in (1.5) is smooth. At any $(M, g) \in \mathcal{M}$, the derivative $D\Pi : T(\mathcal{M}) \to T(\mathcal{T}(\partial M))$ is either an isomorphism or has rank 1, so $dimKer(D\Pi) = dimCoker(D\Pi) = 1$.*
The same statement also holds for the full boundary map Π in (1.2) at minimal boundaries.

We refer to Theorems 3.15 and 3.16 for a more detailed description. In the first case where $dim\, KerD\Pi = 0$, all elements of the component of $\mathcal{M}$ are all "Clifford tori", while in the case $dim\, KerD\Pi = 1$, all elements of the component of $\mathcal{M}$ are surfaces of revolution. Of course generically, elements of $\mathcal{M}$ do not isometrically immerse in S^3.

These results also hold for constant mean curvature boundaries, with the same proofs. Thus in analogy to (1.4), for $c \in \mathbb{R}$, let $\mathcal{M}_c = \Pi^{-1}(\mathcal{C}, c)/\mathrm{Diff}_0(M)$.

Theorem 1.4. *Theorems 1.1 - 1.3 hold for boundaries ∂M of constant mean curvature $H = const. = c$, provided $H \geq 0$ (with respect to the outward normal), with $\mathcal{M}$ replaced by $\mathcal{M}_c$ for any fixed $c \geq 0$.*

Theorem 1.2 has been recently been proved, by different methods, for Alexandrov immersions of constant mean curvature tori in S^3 by Andrews-Li, [6], and by Brendle, [11]. Again there is a full classification of Alexandrov immersed constant mean curvature tori in S^3, cf. [24], [6].

Finally, very similar results hold for minimal or constant mean curvature boundaries in flat and hyperbolic space forms $\mathbb{R}^3$, $\mathbb{H}^3$. These are discussed in Section 4; cf. Theorem 4.1 for the hyperbolic case. Again, we point out that all of these further results (Theorems 1.2-1.4) are reasonably simple consequences of the methods used to prove Theorem 1.1. Thus the proof of Theorem 1.1 is the central focus of the paper.

A brief summary of the contents of the paper is as follows. In Section 2, we discuss preliminary results and background information in order to prove the main results. This includes discussion of Einstein metrics, the Einstein-Hilbert action and its second variation, and aspects of holonomy and the representation variety. Section 3 is devoted to the proof of Theorems 1.1-1.3, while Section 4 discusses the extension of the results to constant mean curvature boundaries and spaces of constant curvature 0 or -1.

2. Preliminaries

In this section, we discuss preliminary material and results needed for the work to follow. Throughout the paper M denotes a compact, connected and oriented 3-dimensional manifold with non-empty boundary ∂M. Some of the results are valid in all dimensions, and we will occasionally point this out.

2.1. Let $Met^{m,\alpha}(M)$ be the space of metrics on M, $C^{m,\alpha}$ smooth up to ∂M, and let $S^{m,\alpha}(M)$ be the corresponding space of symmetric bilinear forms. Let

$$(2.1) \qquad E(g) = Ric_g - \frac{R}{2}g + \Lambda g : Met^{m,\alpha}(M) \to S^{m-2,\alpha}(M),$$

be the Einstein tensor. If $E(g) = 0$, so that g is an Einstein metric, then the scalar curvature R is given by $R = 6\Lambda$, so

$$(2.2) \qquad Ric_g = \lambda g,$$

with $\lambda = 2\Lambda$. The main focus is on $\Lambda = 1$, but the cases $\Lambda = 0, -1$ (flat and hyperbolic metrics) will also be considered briefly in Section 4. Of course the Einstein metrics satisfying (2.2) are of constant curvature Λ.

For (M, g) Einstein, one has the divergence and scalar constraint equations on ∂M, (equivalent to the Gauss-Codazzi and Gauss equations):

$$(2.3) \qquad \delta_\gamma(A - H\gamma) = -Ric(N, \cdot) = 0,$$

$$(2.4) \qquad |A|^2 - H^2 + R_\gamma = R_g - 2Ric(N, N) = (n-1)\lambda = 2\Lambda.$$

Here A is the 2$^{\text{nd}}$ fundamental form of ∂M in M, R_γ and R_g are the scalar curvatures of γ and g respectively, $\delta_\gamma = -tr\nabla$ is the divergence with respect to γ and N is the unit outward normal.

Let $\mathbb{E}^{m,\alpha}$ denote the space of all Einstein metrics, $C^{m,\alpha}$ up to ∂M, and let $\mathcal{E}^{m,\alpha} = \mathbb{E}^{m,\alpha}/\mathrm{Diff}_1^{m+1,\alpha}(M)$ be the quotient of $\mathbb{E}^{m,\alpha}$ by the group of $C^{m+1,\alpha}$ diffeomorphisms of M equal to the identity on ∂M. The action of $\mathrm{Diff}_1^{m+1,\alpha}(M)$ on $\mathbb{E}^{m,\alpha}$ is smooth, since Einstein metrics are real-analytic in the interior and the diffeomorphisms fix the boundary ∂M, cf. [3] for further details.

As noted in the Introduction, the space $\mathcal{E}^{m,\alpha}$ is a smooth Banach manifold (at least when $\pi_1(M, \partial M) = 0$) and the boundary map

$$(2.5) \qquad \Pi : \mathcal{E}^{m,\alpha} \to \mathcal{C}^{m,\alpha} \times C^{m-1,\alpha},$$

$$\Pi(g) = ([\gamma], H),$$

is a smooth Fredholm map, of Fredholm index 0, when $\mathcal{E}^{m,\alpha} \neq \emptyset$.

We note that the action of the larger group $\mathrm{Diff}_0^{m+1,\alpha}(M)$ of diffeomorphisms of M mapping $\partial M \to \partial M$ on $\mathbb{E}^{m,\alpha}$ is no longer smooth in general. Thus, if $\psi \in \mathrm{Diff}_0^{m+1,\alpha}(M)$ and $g \in \mathbb{E}^{m,\alpha}$ then $\psi^* g \in \mathbb{E}^{m,\alpha}$, so the action is well-defined and continuous. However, if $Y^T \in \chi^{m+1,\alpha}(M)$ is a $C^{m+1,\alpha}$ smooth vector field on M, tangent to ∂M, then

$$(2.6) \qquad (\mathcal{L}_{Y^T}g)(A, B) = Y^T(g(A, B)) - g([Y^T, A], B) - g(A, [Y^T, B]).$$

While the last two terms are $C^{m,\alpha}$ smooth, the first term is only $C^{m-1,\alpha}$ smooth at ∂M. Thus, one loses one derivative. (This is closely related to the well-known loss of derivative in isometric embedding problems).

However, suppose ∂M is minimal, so that $H = 0$. As noted in the Introduction, this condition is invariant under the action of the diffeomorphism group $\mathrm{Diff}_0^{m+1,\alpha}(M)$. It is well-known that minimal surfaces (or minimal boundaries in this context) are real-analytic. Namely, any such boundary can be locally graphed over its tangent totally geodesic sphere $S^2 \subset S^3$, by a normal graphing function f. The sphere S^2 is analytic, and the equation $H(f) = 0$ is a non-linear elliptic

equation with analytic coefficients. Hence, by elliptic regularity, the solution f is analytic. Thus, when $H = 0$, the induced metric γ on ∂M is real analytic. Hence, so are also the normal vector field N and the second fundamental form A. In this case, examination of (2.6) shows that

$$2\delta^*(Y^T) = \mathcal{L}_{Y^T} g \in S_2^{m,\alpha}(M).$$

This also holds for arbitrary $C^{m+1,\alpha}$ vector fields $Y \in \chi^{m+1,\alpha}(M)$ on M, not necessarily tangent to ∂M. Hence, the action of $\text{Diff}_0^{m+1,\alpha}(M)$ and of the full diffeomorphism group $\text{Diff}^{m+1,\alpha}(M)$ is smooth on $\mathcal{E}^{m,\alpha}$ at configurations where $H = 0$ (or $H = const$).

The boundary data $([\gamma], H)$ in (2.5) arise from a natural Lagrangian. Thus, consider the functional

(2.7) $$\widetilde{I} : Met^{m,\alpha}(M) \to \mathbb{R},$$

$$\widetilde{I}(g) = \int_M (R - 2\Lambda) + \tfrac{2}{n} \int_{\partial M} H,$$

where $n = dim\, \partial M$. This is a modification of the Einstein-Hilbert action with Gibbons-Hawking-York boundary term [14], [30]:

$$I(g) = \int_M (R - 2\Lambda) + 2 \int_{\partial M} H.$$

The first variation of $\widetilde{I}$ is given by

(2.8) $$d\widetilde{I}_g(h) = -\int_M \langle E_g, h \rangle - \int_{\partial M} \langle A, h_0^T \rangle + (2 - \tfrac{2}{n}) H_h',$$

cf. [4]. Here h^T is the restriction of h to ∂M, (the tangential part of h), h_0^T is its trace-free part with respect to γ, $E_g = E(g)$ and H_h' is the variation of the mean curvature H in the direction h. Thus critical points of $\widetilde{I}$ among metrics with fixed boundary data $([\gamma], H)$ are exactly the Einstein metrics. The first variation of I is

(2.9) $$dI_g(h) = -\int_M \langle E_g, h \rangle - \int_{\partial M} \langle \tau, h^T \rangle,$$

where $\tau = A - H\gamma$ is the conjugate momentum to γ. These formulas are derived from standard formulas for the variation of R, integration by parts, and the identity $2H_k' = N(tr_g k) + (\delta k)(N) + \delta(k(N)^T) - \langle A, k \rangle$, cf. (2.17) below.

Let $g_{s,t} = g + sh + tk$ be a 2-parameter variation of $g \in \mathcal{E}$, so that one has

$$\frac{\partial^2 I(g_{s,t})}{\partial_s \partial_t} = \frac{\partial^2 I(g_{s,t})}{\partial_t \partial_s}.$$

This gives the identity

(2.10) $$\int_M \langle E_k', h \rangle + \int_{\partial M} \langle \tau_k', h^T \rangle + \langle a(k), h^T \rangle = \int_M \langle E_h', k \rangle + \int_{\partial M} \langle \tau_h', k^T \rangle + \langle a(h), k^T \rangle;$$

cf. [4] for details. Here $a(k) = -2\tau \circ k^T + \tfrac{1}{2}(tr k^T)\tau$, $(\tau \circ \ell)(A, B) = \tfrac{1}{2}(\langle \tau(A), \ell(B) \rangle + \langle \tau(B), \ell(A) \rangle)$ and E_k' is the linearization of E at g in the direction k. The formula (2.10) will play a central role in Section 3. Of course the bulk terms in (2.10) vanish when k and h are infinitesimal Einstein deformations (tangent to $\mathcal{E}$).

The operator E in (2.1) is not elliptic, due to the diffeomorphism invariance of the Einstein operator and to obtain an elliptic operator, one needs to introduce a gauge. The most natural gauge for our purposes is the divergence-free gauge. Thus, given any background Einstein metric $\widetilde{g}$, consider the operator

$$\Phi_{\widetilde{g}} : Met^{m,\alpha}(M) \to S^{m-2,\alpha}(M),$$

(2.11)
$$\Phi_{\widetilde{g}}(g) = Ric_g - \frac{R}{2}g + \Lambda g + \delta_g^* \delta_{\widetilde{g}}(g).$$

The linearization of Φ at $g = \widetilde{g}$ is

(2.12)
$$L(h) = \tfrac{1}{2}[D^*Dh - 2R(h) - D^2trh - (\delta\delta h)g + \Delta trh\, g + \lambda trh\, g],$$

where D^2 is the Hessian and $\Delta = trD^2$ the Laplacian (with respect to g); the notation is otherwise standard, as in [7]. It is straightforward to see that L is an elliptic operator (in the interior of M). Moreover, since $\Phi_{\widetilde{g}} = E + \delta^*\delta$, one has

(2.13)
$$E' = L - \delta^*\delta.$$

Of course solutions of $L(h) = 0$ with $\delta h = 0$ on M are infinitesimal Einstein deformations.

The following simple Lemma gives the converse.

Lemma 2.1. *If $L(h) = 0$ on M and $\delta h = \delta_M h = 0$ on ∂M, then*
$$\delta h = 0 \quad \text{on} \quad M,$$

so that h is an infinitesimal Einstein deformation.

For any $h \in S^{m,\alpha}(M)$ there is a $C^{m+1,\alpha}$ vector field V on M with $V = 0$ on ∂M such that $\widetilde{h} = h + \delta^*V$ satisfies
$$\delta\widetilde{h} = 0.$$

Proof: By (2.13), $\delta(L(h)) = \delta\delta^*(\delta h)$, since $\delta E' = 0$, by the linearized Bianchi identity. Thus
$$\delta\delta^*(\delta h) = 0.$$

Pairing this with δh and applying the divergence theorem gives the first result.

For the second result, consider the equation $\delta\delta^*V = -\delta h$ with Dirichlet boundary condition $V = 0$. This is an elliptic boundary value problem, with trivial kernel, and so has a unique solution. This gives the second result. ∎

The tangent space $T_g\mathbb{E}$ is given by $KerE'$, i.e. the space of infinitesimal Einstein deformations. The kernel $K = KerD\Pi$ of $D\Pi$ in (2.5) is given by infinitesimal Einstein deformations κ satisfying

(2.14)
$$\kappa^T = \varphi\gamma, \quad H'_\kappa = 0,$$

at ∂M, for some $\varphi \in C^{m,\alpha}(\partial M)$. The study of K will be the central issue throughout the paper. Theorem 1.1 is essentially the statement that $K = 0$ when $genus(\partial M) \geq 2$.

Define the *superkernel* $\mathbf{K} \subset K$ to be space of infinitesimal Einstein deformations κ as above satisfying the stronger condition

(2.15)
$$\kappa^T = \varphi\gamma, \quad \text{and} \quad (A'_\kappa)^T = \frac{\varphi H}{2}\gamma.$$

Since $tr(A'_\kappa)^T = H'_\kappa + \varphi H$, this implies $H'_\kappa = 0$ so that $\mathbf{K} \subset K$. The second equation in (2.15) is equivalent to the statement that $(A'_\kappa)^T$ is trace-free. Of course $H = 0$ implies $(A'_\kappa)^T = 0$ when $\kappa \in \mathbf{K}$.

To compute the variation A'_k of A, let $g_s = g + sk$ be a variation of g. Since $A = \tfrac{1}{2}\mathcal{L}_N g$, one has $2A'_k \equiv 2\frac{d}{ds}A_{g_s}|_{s=0} = (\mathcal{L}_{N_s}g_s)'|_{s=0} = \mathcal{L}_N k + \mathcal{L}_{N'}g$. A simple computation gives $N' = -k(N)^T - \tfrac{1}{2}k_{00}N$, where $k(N)^T$ is the component of $k(N)$ tangent to ∂M and $k_{00} = k(N,N)$. Using the standard identity $\mathcal{L}_N k = \nabla_N k + 2A \circ k$, it follows that

(2.16)
$$2A'_k = \nabla_N k + 2A \circ k - 2\delta^*(k(N)^T) - \delta^*(k_{00}N).$$

Since $H = trA$, $H'_k = trA'_k - \langle A, k \rangle$, so that
$$2H'_k = N(tr_g k) + 2\delta_\gamma(k(N)^T) - k_{00}H - N(k_{00}).$$

A straightforward calculation gives $(\delta k)(N) = -N(k_{00}) + \delta(k(N)^T) + \langle A, k \rangle - k_{00}H$, so that this is equivalent to

$$(2.17) \qquad 2H'_k = N(tr_g k) + (\delta k)(N) + \delta_\gamma(k(N)^T) - \langle A, k \rangle.$$

The formula (2.17) will be used later in deriving (3.32).

Lemma 2.2. *Let $g \in \mathbb{E}^{m,\alpha}$ and suppose k is an infinitesimal Einstein deformation satisfying*

$$(2.18) \qquad k^T = (A'_k)^T = 0$$

at ∂M. Then k is pure gauge near ∂M, i.e.

$$(2.19) \qquad k = \delta^* V \quad near \quad \partial M,$$

with $V = 0$ on ∂M.

Proof: This is a unique continuation result for the linearized Einstein equations. It is proved in [5] for $m \geq 5$ (in all dimensions), cf. also [8] for the C^∞ case. In the main case of interest here where ∂M is minimal (or $H = const.$), the data and boundary are all analytic. The result is then a simple consequence of the Cauchy-Kovalevsky theorem. Alternately, the result also follows from the linearized version of the fundamental theorem for surfaces in space-forms - that a surface is uniquely determined up to rigid motion by its first and second fundamental forms (γ, A). ∎

This leads easily to the following global result.

Corollary 2.3. *Let $g \in \mathbb{E}^{m,\alpha}$ and suppose κ is an infinitesimal Einstein deformation of (M, g). If $\pi_1(M, \partial M) = 0$ and (2.18) holds, then κ is pure gauge on M, i.e. there exists a vector field V on M with $V = 0$ on ∂M such that*

$$(2.20) \qquad \kappa = \delta^* V \quad on \quad M.$$

If κ is in divergence-free gauge, so that $L(\kappa) = 0$, then

$$\kappa = 0 \quad on \quad M.$$

Proof: The hypotheses and Lemma 2.2 imply that the form κ on M is pure gauge near ∂M, so that (2.20) holds in a neighborhood Ω of ∂M.

It then follows from a well-known analytic continuation argument in the interior of M that the vector field V may be extended so that (2.20) holds on all of M, cf. [20, VI.6.3] for instance. A detailed proof of this is also given in [4, Lemma 2.6]. This analytic continuation argument requires the topological hypothesis $\pi_1(M, \partial M) = 0$ to obtain a well-defined (single-valued) vector field V on M. Moreover, since ∂M is connected, the condition $V = 0$ on ∂M remains valid in the analytic continuation.

For the second statement, if in addition $\delta \kappa = 0$, then $\delta \delta^* V = 0$ on M with $V = 0$ on ∂M. It then follows as in the proof of Lemma 2.1 that $V = 0$ on M and hence $\kappa = 0$ on M, as claimed. ∎

In view of (2.5) and (2.14) it is natural to consider a conformal generalization of Corollary 2.3, i.e. ask whether the conditions

$$(2.21) \qquad k^T = \varphi\gamma, \quad (A'_k)^T = \frac{\varphi H}{2}\gamma,$$

on ∂M imply that $k = 0$ on M, in divergence-free gauge, i.e. $\mathbf{K} = 0$ in this gauge. However, this is not true in general. Namely, observe that any $(M, g) \in \mathcal{E}$ with $\partial M = T^2$ and $H = 0$ (or $H = const$) has a 2-dimensional space of conformal Killing fields T, generated by translation along the two lattice directions defining the conformal structure of ∂M. These may be extended to vector

fields on M and so give a space $\mathcal{T}$ of infinitesimal Einstein deformations $k = \delta^* T$, satisfying the boundary conditions (2.14). Thus

$$\mathcal{T} \subset Ker D\Pi,$$

so that

$$(2.22) \qquad\qquad dim(Ker D\Pi) \geq 2 - i,$$

whenever $\partial M = T^2$ with $H = const$; here i is the number of linearly independent Killing fields of $(\partial M, \gamma)$. Moreover, the trace-free part A_0 of the second fundamental form A of ∂M is a holomorphic quadratic differential on T^2 and hence has constant coefficients with respect to the basis lattice vectors T_1, T_2. Thus

$$(2.23) \qquad\qquad \mathcal{L}_{T_i} A_0 = 0.$$

It is easy to see that $\mathcal{L}_V A = 2(A'_{\delta^* V})^T$ whenever V is tangent to ∂M and so (2.21) holds on $\partial M = T^2$. Hence

$$(2.24) \qquad\qquad dim\, \mathbf{K} \geq 2 - i,$$

whenever $H = const$.

This situation shows a marked difference in the structure of $D\Pi$ when $genus(\partial M) \geq 2$ and $genus(\partial M) = 1$.

2.2. In this section we discuss holonomy of constant curvature metrics and so set $n = 2$ where Einstein metrics are of constant curvature. We consider here only Einstein metrics with positive scalar curvature, and choose the normalization $\Lambda = 1$, so

$$Ric_g = 2g.$$

Although $(M, g) \in \mathcal{E}$ thus has constant curvature $+1$, (M, g) does not necessarily embed or immerse in $S^3 = S^3(1)$; even if it does, arbitrarily close metrics in $\mathcal{E}$ will not. The developing map D is defined only on the universal cover $\widetilde{M}$ of M, and gives an isometric immersion

$$(2.25) \qquad\qquad D : \widetilde{M} \to S^3,$$

with $\widetilde{g} = D^*(g_{S^3})$, where $\widetilde{g}$ is the lift of g to $\widetilde{M}$. Thus, while (M, g) is locally isometric to S^3, i.e. small balls in (M, g) are isometric to balls in S^3, such local isometries may not patch together consistently to give a global isometric immersion. This lack of consistency is measured by the holonomy representation or homomorphism

$$(2.26) \qquad\qquad \rho : \pi_1(M) \to SO(4),$$

cf. [27], [21], [15]. The configuration (M, g) isometrically immerses in S^3 if and only if $\rho = \{e\}$ is the trivial homomorphism. Also metrics g, g' with conjugate holonomy $\rho, \rho' = g\rho g^{-1}$, $g \in SO(4)$, are isometric, modulo deformations of the boundary, cf. [27].

Now recall a result of Frankel-Lawson [23] that any compact $(n+1)$-manifold of positive Ricci curvature with boundary ∂M satisfying $H_{\partial M} \geq 0$ satisfies

$$(2.27) \qquad\qquad \pi_1(M, \partial M) = 0.$$

Thus, $\pi_1(\partial M)$ surjects onto $\pi_1(M)$ and ∂M is connected. This applies in particular to any $(M, g) \in \mathcal{M}$. In dimension 3, this implies that M is a handlebody, cf. again [23]; if $\partial M = \Sigma_g$ is a surface of genus g, then

$$(2.28) \qquad\qquad \pi_1(M) = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{g},$$

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the free group on g generators. Choosing a set of generators σ_i, $i = 1, \cdots, g$ the holonomy map ρ is determined by its images $\rho(\sigma_i) \in SO(4)$. The space

$$(2.29) \qquad \mathcal{R}(M) = Hom(\pi_1(M), SO(4))/Ad$$

of all holonomy representations modulo conjugacy is thus naturally identified with the quotient space

$$(2.30) \qquad \mathcal{R}(M) = \underbrace{[SO(4) \times SO(4) \times \cdots \times SO(4)]}_{g}/SO(4),$$

where $SO(4)$ acts on the product diagonally by conjugation.

Consider first the case $\partial M = T^2$ in detail. One has $\pi_1(M) = \mathbb{Z}$ and

$$\mathcal{R}(M) = SO(4)/Ad,$$

the quotient of $SO(4)$ by its adjoint action. Any element in $SO(4)$ is conjugate to an element in its maximal torus T^2 and

$$(2.31) \qquad \mathcal{R}(M) = T^2/W,$$

where W is the Weyl group of $SO(4)$. This is isomorphic to $\mathbb{Z}_2 \oplus Z_2$, acting on $\mathbb{R}^2$, the Lie algebra of the maximal torus T^2, by $x_i' = \pm x_{\sigma_i}$, where σ_i is a permutation of $(1,2)$ and the product of the signs is 1, cf. [1] for example. The Weyl chamber in $\mathbb{R}^2$ has two walls $x_1 = \pm x_2$ and integer translates thereof. The exponential map takes the square $[0,1] \times [0,1] \subset \mathbb{R}^2$ onto the maximal torus T^2 and a fundamental domain for the action of W on $\mathbb{R}^2$ is the 2-simplex S bounded by $x_2 = 0$, and the two walls $x_2 = x_1$, $x_2 = -x_1 + 1$, with $0 \leq x_2 \leq 1$, with vertices at $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{1}{2})$. The exponential map maps S to a 2-simplex with the two vertices $(0,0)$, $(1,0)$ identified, representing $Id \in SO(4)$, and with $(\frac{1}{2}, \frac{1}{2})$ mapping to $-Id \in SO(4)$.

The interior of the Weyl chamber consists of regular elements $g \in SO(4)$, where the dimension of the normalizer $N(g)$ of g satisfies $dim N(g) = 2$. Generic singular elements g are points on the two walls of the Weyl chamber, where $dim N(g) = 4$, while the two singular points $\pm Id$ satisfy $dim N(\pm Id) = 6$. Topologically $\mathcal{R}(M)$ is thus a 2-manifold with corners; the boundary is formed by the wedge two circles, with corners at the two singular points $\pm Id$.

We note that

$$(2.32) \qquad dim\, N(g) = dim\, \mathcal{Z},$$

where $\mathcal{Z}$ is the space of Killing fields on (M, g).

When the genus $g \geq 2$, the diagonal action of $SO(4)$ by conjugacy on $SO(4) \times SO(4) \times \cdots \times SO(4)$ is effective, and hence

$$dim\, \mathcal{R}(M) = 6g - 6.$$

Here the dimension is the topological dimension. Thus, in all cases, one has

$$(2.33) \qquad dim\, \mathcal{R}(M) = dim\, \mathcal{T}(\partial M),$$

where $\mathcal{T}(\partial M)$ is the Teichmüller space of the boundary $\mathcal{T}(\partial M)$.

However, there are strong differences between the spaces $\mathcal{R}(M)$ and $\mathcal{T}(\partial M)$. First, $\mathcal{R}(M)$ is clearly compact, while $\mathcal{T}(\partial M)$ is never compact. Also, while $\mathcal{T}(\partial M)$ is a smooth manifold, (diffeomorphic to a ball of dimension $6g - 6$ when $g \geq 2$), the variety $\mathcal{R}(M)$ is only smooth at the regular points $h \in G = SO(4) \times \cdots \times SO(4)$. A point is regular if its orbit is a maximal orbit for the diagonal action of $SO(4)$ on G; these form an open dense set in the orbit space $\mathcal{R}(M)$. A point is singular otherwise, and so is a point with non-trivial stabilizer; in the case $\partial M = T^2$ the stabilizer of the action at g is the normalizer $N(g)$ of $g \in SO(4)$. Thus in general $\mathcal{R}(M)$ is a stratified manifold.

The space $\mathcal{R}(M)$ also has the structure of a real analytic, in fact real algebraic, variety. The ring of functions on $\mathcal{R}(M)$ is defined to be algebraic functions on $G = SO(4) \times \cdots \times SO(4)$ invariant under the action $h \to ghg^{-1}$ of $SO(4)$ on G. However, as is common in this setting (geometric invariant theory) the algebraic quotient is not modeled well by the topological quotient; for instance the topological quotient in (2.31) does not effectively describe $\mathcal{R}(M)$ as an algebraic variety at the singular points.

At the singular points of $\mathcal{R}(M)$, the tangent space $T\mathcal{R}(M)$ is defined to be the Zariski tangent space of the algebraic variety $\mathcal{R}(M)$. By a theorem of Weil [29],

$$(2.34) \qquad\qquad T_\rho \mathcal{R}(M) = H^1(\pi_1(M), Ad\rho),$$

where $\rho : \Gamma \to SO(4)$ is the holonomy representation. In more detail, let $\Gamma = \pi_1(M)$ and let $Ad\rho$ be the Γ-module $\mathcal{L}(SO(4))$ with Γ action given by $Ad \circ \rho$. The 1-cocycles $Z^1(\Gamma, Ad\rho)$ consist of maps $f : \Gamma \to \mathcal{L}(SO(4))$ satisfying the cocycle condition

$$(2.35) \qquad\qquad f(ab) = f(a) + Ad(\rho(a))f(b),$$

for $a, b \in \Gamma$. The 1-coboundaries $B^1(\Gamma, Ad\rho)$ consist of maps $f : \Gamma \to \mathcal{L}(SO(4))$ satisfying the coboundary condtion

$$(2.36) \qquad\qquad f(a) = v - Ad\rho(a)v,$$

for some $v \in \mathcal{L}(SO(4))$. Then $H^1(\Gamma, Ad\rho) = Z^1(\Gamma, Ad\rho)/B^1(\Gamma, Ad\rho)$.

A smooth curve $\rho_t : \pi_1(M) \to SO(4)$ of holonomy maps is determined uniquely by $g = genus(\partial M)$ smooth curves in $SO(4)$ - the values of ρ_t on the generators. The derivative $\rho' : \pi_1(M) \to \mathcal{L}(SO(4))$ satisfies the cocycle condition (2.35). The derivative ρ' is a coboundary if and only if ρ_t is conjugate to ρ, to first order in t.

An important case is $\rho = e$ the trivial representation, corresponding to $(M, g) \subset S^3$. A simple calculation from (2.35) and (2.36) gives

$$(2.37) \qquad\qquad dim(T_e \mathcal{R}(M)) = dim(H^1(\Gamma, Ad\, e)) = 6g.$$

Namely one may define f arbitrarily on the generators $f(\sigma_i) = a_i \in \mathcal{L}(SO(4))$ and then extend such f to satisfy the cocyle condition (2.35); the only coboundary in this case is the zero map. Analogous to (2.32), for $genus(\partial M) \geq 2$, one has

$$(2.38) \qquad\qquad dim\, \mathcal{Z} = dim\, N(\rho) = dim\, T_\rho \mathcal{R}(M) - (6g - 6),$$

where $\mathcal{Z}$ is the space of Killing fields on (M, g), ρ is the holonomy of (M, g) and $N(\rho) = \{r \in SO(4) : r^{-1}\rho r = \rho\}$. We refer to [21] for further background on $\mathcal{R}(M)$.

Next we consider the description of the variation of holonomy in terms of variation of the metric. Let g_t be a curve in $\mathbb{E}$. Passing to the universal cover $\widetilde{M}$ of M gives then a curve of developing maps

$$D_t : \widetilde{M} \to S^3.$$

Let $\widetilde{g}_t$ be the curve of lifted metrics on $\widetilde{M}$; the fundamental group $\pi_1(M)$ acts by isometries of $\widetilde{g}_t$. The derivative D' of D_t is a vector field W on S^3 along $D = D_0$, so $W : \widetilde{M} \to T(S^3)$. Since D is an isometric immersion, the pullback D^*W is a well-defined vector field, also called W, on $\widetilde{M}$. In general W will not be invariant under the action of $\pi_1(M)$ and so will not descend to a vector field on M. The derivative $\widetilde{k} = d\widetilde{g}_t/dt$ is given by $\widetilde{k} = \delta^*W$ on $\widetilde{M}$. The form $\widetilde{k}$ is invariant under the action of $\pi_1(M)$ and so descends to a symmetric form k on M, with $k = dg_t/dt$. Thus any infinitesimal Einstein (i.e. constant curvature $+1$) deformation of g on M is locally of the form δ^*W, so locally pure gauge. These correspond to the cocycles in $H^1(\pi_1(M), Ad\rho)$ while the coboundaries correspond to forms δ^*Y with Y a globally defined vector field on M. We refer

also to [16] where a very similar discussion is given in the case of $SL(2, \mathbb{C})$ in place of $SO(4)$, corresponding to deformations of hyperbolic metrics.

We summarize the previous discussion as follows. The tangent space $T_g\mathbb{E}$ to $\mathbb{E}$ at g consists of symmetric forms $k = \delta^*W$, where W is a locally defined vector field on M; W is globally defined on the universal cover $\widetilde{M}$, but is multi-valued on M. The tangent space $T_g\mathcal{E}$ to $\mathcal{E}$ at g consists of equivalence classes symmetric forms $[k]$ as above, where $k \sim k + \delta^*V$ and V is a global vector field on M, vanishing on ∂M. Similarly, the tangent space $T_g\widetilde{\mathcal{E}}$ to $\widetilde{\mathcal{E}}$ as in (1.3) at g consists of equivalence classes symmetric forms $[k]$, where $k \sim k + \delta^*Y^T$ and Y^T is a global vector field on M, tangent to ∂M at ∂M. Finally, the tangent space to the quotient $\mathcal{R}(M)$ consists of equivalence classes symmetric forms $[k]$, where $k \sim k + \delta^*Y$, with Y a globally defined vector field on M, not necessarily tangent to ∂M at ∂M.

3. Minimal Surface Boundaries

In this section, we prove the main results in the Introduction. Most of the work concerns the proof of Theorem 1.1. The other results are relatively straightforward consequences of the methods used to prove Theorem 1.1. Motivation and ideas of the proof are presented in several places during the course of the proof.

Given $(M, g) \in \mathcal{E}$, suppose k is an infinitesimal Einstein deformation, so $k \in T\mathcal{E}$. Then

$$(3.1) \qquad \qquad E'_k = 0,$$

i.e. the variation of the Einstein tensor (2.1) in the direction k vanishes. One also has the induced variations of the boundary data $([\gamma], H)$, given by (k_0^T, H'_k); here k_0^T is the trace-free part of k^T and H'_k is given as in (2.17). As discussed in Section 2, in dimension 3 all infinitesimal Einstein deformations are locally "pure-gauge", so that $k = \delta^*W$, where W is a multi-valued vector field on M, (well-defined on $\widetilde{M}$).

Now $K = Ker D\Pi$ consists of the forms κ such that $E'_\kappa = 0$ on M and

$$(3.2) \qquad \qquad \kappa_0^T = 0, \quad H'_\kappa = 0$$

at ∂M. Thus, $\kappa^T = \varphi\gamma$ is pure trace on ∂M. Suppose for example that $\kappa = \delta^*X$ for some global vector field X on M. At ∂M, writing $X = X^T + \nu N$ where X^T is the component of X tangent to ∂M and N is the outward unit normal, (3.2) is equivalent to

$$(\delta^*X)^T = \delta^*X^T + \nu A = \varphi\gamma,$$

$$-\Delta\nu - (|A|^2 + 2)\nu + X^T(H) = 0.$$

When $H = 0$, $J = \nu N$ is a Jacobi field of the minimal surface $\partial M \subset M$, and so this system describes Jacobi fields that preserve the conformal class, for some parametrization of ∂M. This is the "holonomy trivial" kernel $K_1 \subset K$:

$$(3.3) \qquad \qquad K_1 = \{X : \delta^*X = \varphi\gamma, \ H'_{\delta^*X} = 0 \text{ at } \partial M\};$$

here $X \in \chi^{m+1,\alpha}(M)$ is a $C^{m+1,\alpha}$ smooth vector field on M.

The first result of this section sets the stage for the proof of Theorem 1.1 and gives a key duality between the Dirichlet boundary data φ of $\kappa \in K$ and its Neumann data A'_κ or τ'_κ. We first prove the result for $\kappa \in K_1$; the proof in the general case $\kappa \in K$ follows later in Proposition 3.9.

Proposition 3.1. *For $\kappa = \delta^*X \in K_1$ with $\kappa^T = \varphi_X\gamma$ and $H = 0$ at ∂M, one has the duality*

$$(3.4) \qquad \int_{\partial M} \langle \tau'_\kappa, h^T \rangle = \int_{\partial M} E'_h(N, X) - \int_{\partial M} \varphi_X H'_h,$$

for any metric deformation h. In particular, if h is an infinitesimal Einstein deformation, then

$$(3.5) \qquad \int_{\partial M} \langle \tau'_\kappa, h^T \rangle = - \int_{\partial M} \varphi_X H'_h,$$

Proof: We begin with the identity (2.10), i.e.

$$(3.6) \qquad \int_{\partial M} \langle \tau'_k, h^T \rangle + \langle a(k), h^T \rangle + \int_M \langle E'_k, h \rangle = \int_{\partial M} \langle \tau'_h, k^T \rangle + \langle a(h), k^T \rangle + \int_M \langle E'_h, k \rangle,$$

for any deformations k, h, where $a(k) = -2\tau \circ k^T + \frac{1}{2}(trk^T)\tau$. Choose $k = \kappa \in Ker D\Pi$, so that κ is an infinitesimal Einstein deformation with $\kappa^T = \varphi_\kappa \gamma$, $H'_\kappa = 0$ on ∂M. Then

$$\langle -2\tau \circ \kappa^T, h^T \rangle = -2\varphi_\kappa \langle \tau, h^T \rangle = \langle -2\tau \circ h^T, \kappa^T \rangle,$$

so these terms in (3.6) cancel. Also $\langle \tau, \kappa^T \rangle = \varphi_\kappa tr\tau = -\varphi_\kappa H$, so (3.6) becomes

$$\int_{\partial M} \langle \tau'_\kappa, h^T \rangle + \varphi_\kappa \langle \tau, h^T \rangle = \int_{\partial M} \langle \tau'_h, \kappa^T \rangle - \tfrac{1}{2} trh^T \varphi_\kappa H.$$

Next $\langle \tau'_h, \kappa^T \rangle = \varphi_\kappa tr_\gamma \tau'_h$. One has $(tr_\gamma \tau)'_h = tr\tau'_h + tr'_\gamma \tau$ and $tr'_\gamma \tau = -\langle \tau, h^T \rangle$, so that $\langle \tau'_h, \kappa^T \rangle = -\varphi_\kappa H'_h + \varphi_\kappa \langle \tau, h^T \rangle$. Hence we obtain the formula

$$(3.7) \qquad \int_{\partial M} \langle \tau'_\kappa, h^T \rangle + \tfrac{1}{2} trh^T \varphi_\kappa H = \int_M \langle E'_h, \kappa \rangle - \int_{\partial M} \varphi_\kappa H'_h.$$

When $H = 0$, this becomes

$$(3.8) \qquad \int_{\partial M} \langle \tau'_\kappa, h^T \rangle = \int_M \langle E'_h, \kappa \rangle - \int_{\partial M} \varphi_\kappa H'_h.$$

Now assume $\kappa = \delta^* X$. Then integration by parts gives

$$\int_M \langle E'_h, \kappa \rangle = \int_M \langle E'_h, \delta^* X \rangle = \int_M \langle \delta(E'_h), X \rangle + \int_{\partial M} E'_h(N, X),$$

where N is the unit outward normal. By the Bianchi identity, $\delta(E'_h) = 0$ for all h, and (3.4) follows from (3.8). ∎

Observe that the left side of (3.4) depends only on the Dirichlet data of h^T on ∂M, while the right side depends on the (1st order) extension of h on ∂M to M.

Remark 3.2. We note that τ'_κ is transverse-traceless when $H = 0$ (although this will not be used in the actual proof):

$$(3.9) \qquad \delta\tau'_\kappa = 0, \quad tr\tau'_\kappa = 0,$$

so that τ'_κ is tangent to the Teichmüller space $\mathcal{T}(\partial M)$. The second equation in (3.9) is immediate; as above $tr(\tau'_\kappa) = -H'_\kappa + \langle \tau, \kappa^T \rangle = 0$, since $H = 0$. To prove the first equation, choose $h = \delta^* Y^T$ in (3.7), where Y^T is any (smooth) global vector field on M tangent to ∂M. Then h is clearly an infinitesimal Einstein deformation and $H'_h = Y^T(H) = 0$. Hence

$$\int_{\partial M} \langle \tau'_\kappa, \delta^* Y^T \rangle = \int_{\partial M} \langle \delta\tau'_\kappa, Y^T \rangle = 0,$$

where δ is the divergence on $(\partial M, \gamma)$. Since Y^T is arbitrary, it follows that $\delta\tau'_\kappa = 0$.

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To prove Theorem 1.1, it suffices to prove the kernel $K = KerD\Pi$ as in (3.2) is trivial:

$$(3.10) \qquad\qquad\qquad\qquad\qquad\qquad K = 0,$$

for $(M, g) \in \mathcal{E}$ with $H = 0$ at ∂M, i.e. any minimal surface boundary is a regular value of the boundary map Π, (since $indexD\Pi = 0$). To see this, let

$$\widetilde{\mathcal{M}} = \Pi^{-1}(\mathcal{C} \times \{0\}).$$

If the linearization of Π as in (1.2) is an isomorphism, then by the implicit function theorem for Banach spaces, $\widetilde{\mathcal{M}}$ is a smooth Banach manifold and the induced map

$$\Pi : \widetilde{\mathcal{M}} \to \mathcal{C},$$

is a local diffeomorphism at every $(M, g) \in \widetilde{\mathcal{M}}$. Passing to the quotient by the free action of $\mathrm{Diff}_0(M)$ on $\widetilde{\mathcal{M}}$ and $\mathcal{C}$ proves the claim.

The starting point of the proof of Theorem 1.1 is the main formula (3.4). The basic idea (somewhat oversimplified) is to show that for arbitrary boundary data $([h^T], H'_h)$ in $T\mathcal{C} \times C(\partial M)$, one can find an extension h (called a canonical extension below) such that

$$(3.11) \qquad\qquad\qquad\qquad\qquad \int_M \langle E'_h, \kappa \rangle = 0,$$

for any $\kappa \in K$. Equation (3.11) holds of course for h such that $E'_h = 0$, i.e. infinitesimal Einstein deformations. However, the presence of a kernel K shows exactly that $D\Pi$ is not surjective, i.e. not all boundary data $([h^T], H'_h)$ are realized as boundary data of infinitesimal Einstein deformations. The cokernel is essentially K itself, (cf. (3.23). A basic tool, cf. Propositions 3.5 and 3.10, is the construction of a suitable slice $Q \simeq K$ serving as a more effective cokernel for $ImD\Pi$. Further explanation for the construction of Q is given preceding Proposition 3.5.

Using Q, we construct an extension h of arbitrary boundary data $([h^T], H'_h)$ satisfying (3.11), so that (3.5) holds for arbitrary boundary data $([h^T], H'_h)$. Since h^T and H'_h are independent, this is only possible when $\tau'_\kappa = 0$ and $\varphi_\kappa = 0$. Via Corollary 2.3, this implies that $\kappa = 0$ (in divergence free gauge) giving $K = 0$.

This brief sketch of the method of proof is oversimplified, in that it is not quite true when $genus(\partial M) = 1$; in any case it requires considerable further work and details to implement this program.

We point out that most of the results of this section, namely from Proposition 3.1 up to and including Proposition 3.10, hold for arbitrary genus $genus(\partial M) \geq 1$ (or even $genus(\partial M) = 0$).

The next result, based on Proposition 3.1, is a major step in the proof of Theorem 1.1. Recall the definition of the superkernel $\mathbf{K}$ in (2.15).

Theorem 3.3. *Suppose $(M, g) \in \mathcal{E}$ has minimal surface boundary, so $H = 0$. Then*

$$(3.12) \qquad\qquad\qquad\qquad\qquad \mathbf{K} = 0 \Rightarrow K = 0.$$

The proof of Theorem 3.3 is rather long, and is broken down into a collection of Lemmas and Propositions. Overall, the method of proof is that used to prove the isometry extension theorem of [4], which states that any Killing vector field at $(\partial M, \gamma)$ which preserves the mean curvature extends to a Killing field of any Einstein filling manifold (M, g). We point out that Theorem 3.3 holds for constant curvature metrics in any dimension and for any λ; all of the proof except Proposition 3.9 below (which uses the fact that Einstein deformations in dimension 3 are locally pure gauge) holds for Einstein metrics in all dimensions.

For later purposes, we point out that the assumption $\mathbf{K} = 0$ is not used in the results below until Proposition 3.10; similarly the results below until Proposition 3.10 hold for $H = const$.

To begin, consider elliptic boundary data on ∂M for the elliptic operator L (the divergence-gauged linearized Einstein operator) in (2.12):

$$L : S^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

It was shown in [3] that if σ_1, σ_2 are any Riemannian metrics on ∂M, boundary data of the form

$$(3.13) \qquad \delta h = 0, \quad [h^T]_{\sigma_1} = h_1, \quad \langle \tau_h', \sigma_2 \rangle - \langle \tau, h \rangle = h_2 \quad \text{at} \quad \partial M,$$

form a well-posed elliptic boundary value system for L. Here $[h^T]_\sigma$ is the usual equivalence relation mod σ, i.e. $h_1^T \sim h_2^T$ if and only if $h_2^T = h_1^T + f\sigma$, for some smooth function f on ∂M. The most natural choice for σ in (3.13) is $\sigma = \gamma$, where the term $[h^T]_\gamma$ corresponds to the variation of the conformal class of γ. Also, $\langle \tau_h', \gamma \rangle - \langle \tau, h \rangle = tr_\gamma \tau_h' - \langle \tau, h \rangle = -H_h'$ gives the variation of the mean curvature H. This case $(\sigma_1, \sigma_2) = (\gamma, \gamma)$ will be the main case of interest, but we will also consider σ_i to be smooth Riemannian metrics close to γ.

Let $S^{m,\alpha}_{\sigma_1,\sigma_2}(M)$ be the space of $C^{m,\alpha}$ symmetric bilinear forms h on M such that at ∂M,

$$(3.14) \qquad \delta h = 0, \quad [h^T]_{\sigma_1} = 0, \quad \langle \tau_h', \sigma_2 \rangle - \langle \tau, h \rangle = 0.$$

The operator

$$(3.15) \qquad L : S^{m,\alpha}_{\sigma_1,\sigma_2}(M) \to S^{m-2,\alpha}(M)$$

is elliptic, so that in particular, $Im L$ is of finite codimension in $S^{m-2,\alpha}(M)$. Let $S^{m,\alpha}_\sigma(M) = S^{m,\alpha}_{\sigma,\sigma}(M)$.

Proposition 3.4. *On $S^{m,\alpha}_\sigma(M)$, the equation $L(h) = \ell$ with boundary data*

$$(3.16) \qquad \delta h = 0, \quad [h^T]_\sigma = 0, \quad \langle \tau_h', \sigma \rangle - \langle \tau, h \rangle = 0$$

forms an elliptic formally self-adjoint boundary value problem.

More generally the pair of boundary data

$$(3.17) \qquad [h_1^T]_{\sigma_1} = 0, \quad \langle \tau_{h_1}', \sigma_2 \rangle - \langle \tau, h_1 \rangle = 0, \quad \text{and} \quad [h_2^T]_{\sigma_2} = 0, \quad \langle \tau_{h_2}', \sigma_1 \rangle - \langle \tau, h_2 \rangle = 0,$$

with $\delta h_i = 0$ at ∂M are adjoint elliptic boundary value problems provided

$$(3.18) \qquad [\sigma_1, \sigma_2] = 0,$$

when σ_i are viewed as linear maps via the metric γ, and

$$(3.19) \qquad \langle \tau, \sigma_1 \rangle [1 - \tfrac{1}{2}\langle \sigma_2, \gamma \rangle] = \langle \tau, \sigma_2 \rangle [1 - \tfrac{1}{2}\langle \sigma_1, \gamma \rangle].$$

Proof: We prove (3.17) which then implies (3.16). We claim that

$$(3.20) \qquad \int_M \langle L(h_1), h_2 \rangle = \int_M \langle h_1, L(h_2) \rangle,$$

for $h_1 \in S^{m,\alpha}_{\sigma_1,\sigma_2}(M)$ and $h_2 \in S^{m,\alpha}_{\sigma_2,\sigma_1}(M)$. Consider the formula (3.6). For the main boundary terms one has

$$\langle \tau_{h_1}', h_2 \rangle = \varphi_{h_2} \langle \tau_{h_1}', \sigma_2 \rangle = \varphi_{h_2} \varphi_{h_1} \langle \tau, \sigma_1 \rangle,$$

$$\langle \tau_{h_2}', h_1 \rangle = \varphi_{h_1} \langle \tau_{h_2}', \sigma_1 \rangle = \varphi_{h_1} \varphi_{h_2} \langle \tau, \sigma_2 \rangle.$$

Also for the a term, $\langle \tau \circ h_1, h_2 \rangle = \varphi_{h_1} \varphi_{h_2} \langle \tau \circ \sigma_1, \sigma_2 \rangle$ and $\langle \tau \circ h_2, h_1 \rangle = \varphi_{h_2} \varphi_{h_1} \langle \tau \circ \sigma_2, \sigma_1 \rangle$. These terms cancel when (3.18) holds. Next

$$\tfrac{1}{2} tr h_1^T \langle \tau, h_2 \rangle = \tfrac{1}{2} \varphi_{h_1} \varphi_{h_2} \langle \sigma_1, \gamma \rangle \langle \tau, \sigma_2 \rangle,$$

$$\tfrac{1}{2} tr h_2^T \langle \tau, h_1 \rangle = \tfrac{1}{2} \varphi_{h_2} \varphi_{h_1} \langle \sigma_2, \gamma \rangle \langle \tau, \sigma_1 \rangle,$$

Combining the equations above shows that (3.20) holds if

$$\langle \tau, \sigma_1 \rangle + \tfrac{1}{2}\langle \sigma_1, \gamma \rangle \langle \tau, \sigma_2 \rangle = \langle \tau, \sigma_2 \rangle + \tfrac{1}{2}\langle \sigma_2, \gamma \rangle \langle \tau, \sigma_1 \rangle,$$

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which is the same as (3.19).

It follows then from (3.6) that

$$\int_M \langle E'(k), h \rangle = \int_M \langle k, E'(h) \rangle.$$

By (2.13), the operator L differs from E' by the operator $\delta^*\delta$. One has

$$\int_M \langle \delta^*\delta h, k \rangle = \int_M \langle \delta h, \delta k \rangle + \int_{\partial M} k(\delta h, N).$$

The first term is symmetric in h and k, and the last (boundary) term vanishes, since $\delta h = 0$ at ∂M.

This proves the claim and the result follows.

∎

In the following, until Proposition 3.10, we will assume $(\sigma_1, \sigma_2) = (\sigma, \sigma)$. Later in Proposition 3.10, we assume $(\sigma_1, \sigma_2) = (\sigma, \gamma)$; note that (3.18) holds in this case and (3.19) is equivalent to

(3.21) $$H[\langle \sigma, \gamma \rangle - 2] = 0.$$

Since the boundary value problem (3.16) is elliptic and formally self-adjoint, it follows from ellliptic regularity that the operator

$$L_\sigma = L|_{S_\sigma} : S_\sigma^{m,\alpha}(M) \to S^{m-2,\alpha}(M)$$

satisfies

(3.22) $$K_\sigma = (Im L_\sigma)^\perp,$$

i.e. the kernel K_σ of L_σ equals the annihilator of $Im L_\sigma$ on L^2. Thus

(3.23) $$Im L_\sigma \oplus K_\sigma = S^{m-2,\alpha}(M).$$

Note that $dim K_\sigma$ may depend on σ but for σ sufficiently close to γ,

$$dim K_\sigma \leq dim K_\gamma.$$

The kernel consists K_σ forms k satisfying $L(k) = 0$ with boundary conditions

$$L(k) = 0, \quad \delta k = 0, \quad [k^T]_\sigma = 0, \quad \langle \tau'_k, \sigma \rangle - \langle \tau, k \rangle = 0.$$

Observe that for $\sigma = \gamma$, $K_\sigma = K_\gamma = K$ is the kernel of $D\Pi$ in (3.2), in divergence-free gauge.

More generally, again by Proposition 3.4, the operator $L_{\sigma,\gamma} : S_{\sigma,\gamma}^{m,\alpha}(M) \to S^{m-2,\alpha}(M)$ satisfies

(3.24) $$Im L_{\sigma,\gamma} \oplus K_{\gamma,\sigma} = S^{m-2,\alpha}(M),$$

where $K_{\gamma,\sigma}$ is the space of forms k satisfying $L(k) = 0$ and the boundary conditions

$$\delta k = 0, \quad [k^T]_\gamma = 0, \quad \langle \tau'_k, \sigma \rangle - \langle \tau, k \rangle = 0.$$

The splittings in (3.23) and (3.24) are L^2 orthogonal.

To motivate the next result, on $S_\sigma^{m,\alpha}(M)$ form the operator

$$\widetilde{L}(h) = L(h) + \pi_{K_\sigma}(h),$$

where π_{K_σ} is the L^2 orthogonal projection onto K_σ along $Im L_\sigma$. By (3.23), $\widetilde{L}$ is an isomorphism

$$\widetilde{L} : S_\sigma^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

By a standard subtraction procedure (cf. (3.42) below for details) it follows that any boundary data $\delta h = 0$, $[h^T]_\sigma = h_1$, $\langle \tau'_h, \sigma \rangle - \langle \tau, h \rangle = h_2$ on ∂M has an extension h on M such that

$$L(h) = k_\sigma \in K_\sigma, \quad \widetilde{L}(h) = 0.$$

One easily verifies that $\delta h = 0$ on M (cf. Lemma 3.6) so that

$$(3.25) \qquad \int_M \langle E_h', \kappa \rangle = \int_M \langle k_\sigma, \kappa \rangle.$$

However, at this point, it is difficult to evaluate or understand the right side of (3.25), since k_σ and κ are global terms on M. One has no reason to believe that such an extension h satisfies (3.11).

The purpose of the next result is to construct a different slice Q to ImL in place of K, for which the argument above can be carried out and for which the left side of (3.25) can be effectively computed (first for $\kappa \in K_1$). A similar slice construction to $ImL_{\sigma,\gamma}$ is given in Proposition 3.10 below.

Proposition 3.5. *There exists an open set $\mathcal{S}$ of smooth Riemannian metrics σ (arbitrarily) near γ and, for each $\sigma \in \mathcal{S}$, a finite dimensional space Q_σ with $dimQ_\sigma = dimK_\sigma$, consisting of smooth forms of the type*

$$(3.26) \qquad q = D^2 f - (\Delta f + 2f)g,$$

and satisfying the conditions:

$$(3.27) \qquad \delta q = 0,$$

$$(3.28) \qquad ImL_\sigma \oplus Q_\sigma = S^{m-2,\alpha}(M).$$

Proof: A standard computation gives

$$\delta D^2 f = -d\Delta f - Ric(df) = \delta[(\Delta f + 2f)g],$$

since $Ric = 2g$, so that (3.27) follows immediately. For the moment, let f be arbitrary in $C^{m,\alpha}(M)$.

To establish the slice property (3.28), by the orthogonal direct sum decomposition (3.23) it suffices to show that for each $q \in Q_\sigma$ there exists $k \in K_\sigma$ such that

$$(3.29) \qquad \int_M \langle q, k \rangle \neq 0,$$

so that Q_σ has no elements orthogonal to K_σ.

Now computing (3.29) one has

$$\int_M \langle D^2 f, k \rangle = \int_M \langle df, \delta k \rangle + \int_{\partial M} \langle k(N), df \rangle = \int_{\partial M} \delta_\gamma(k(N)^T)f + k_{00}N(f),$$

since $\delta k = 0$ by Lemma 2.1. Set

$$\alpha = \int_{\partial M} \delta_\gamma(k(N)^T)f + k_{00}N(f),$$

so

$$(3.30) \qquad \int_M \langle q, k \rangle = \alpha - \int_M (\Delta f + 2f)trk.$$

On the other hand, since $L(k) = 0$ and $\delta k = 0$, taking the trace of (2.12) gives

$$(3.31) \qquad \Delta trk + 2trk = 0.$$

Integrating by parts and using the fact that $k^T = \varphi\sigma$ on ∂M $(\varphi = \varphi_k)$ gives

$$\int_M (\Delta f + 2f)trk = \int_{\partial M} N(f)trk - N(trk)f$$

$$= \int_{\partial M} N(f)k_{00} + N(f)\varphi tr_\gamma\sigma - N(trk)f = \alpha + \int_{\partial M} -\delta_\gamma(k(N)^T)f + N(f)\varphi tr_\gamma\sigma - N(trk)f,$$

$$= \alpha + \int_{\partial M} N(f)\varphi tr_\gamma\sigma - 2H_k'f - f\langle A, k \rangle,$$

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where the last equality follows from (2.17). Substituting this in (3.30) gives then the basic formula

$$(3.32) \qquad \int_M \langle q, k \rangle = - \int_{\partial M} \varphi_k N(f) tr_\gamma \sigma - f[2H'_k + \langle A, k \rangle].$$

This holds for all $k = k_\sigma \in K_\sigma$, for any σ. From this, we need to establish (3.29).

Observe that if (3.32) vanishes for all choices of f, then necessarily

$$\varphi = 0 \quad \text{and} \quad H'_k = 0.$$

Namely one can set $f = 0$ and $N(f)$ arbitrary on ∂M to obtain $\varphi = 0$; given this one can then choose f arbitrary to obtain $H'_k = 0$.

The discussion above holds for each choice of smooth symmetric form σ and each $k_\sigma \in K_\sigma$. In particular, it applies to the "original" case $\sigma = \gamma$. For any σ as above, consider the "reduced kernel" $\widetilde{K}_\sigma \subset K_\sigma$ consisting of those $k_\sigma \in K_\sigma$ with $\varphi_{k_\sigma} = 0$,

$$\widetilde{K}_\sigma = \{ k \in K_\sigma : k^T = 0 \text{ on } \partial M \}.$$

Let P_σ be the L^2 orthogonal complement of $\widetilde{K}_\sigma$ in K_σ so that

$$K_\sigma = \widetilde{K}_\sigma \oplus P_\sigma.$$

If π_j is a basis for P_σ, then the boundary values φ_j ($\pi_j^T = \varphi_j \sigma$ on ∂M) are linearly independent. For convenience, assume $\{\varphi_j\}$ are orthonormal in $L^2(\partial M)$.

To begin, we choose a slice for $P_\gamma \subset K_\gamma$. Thus choose $f_i \in C^{m,\alpha}(M)$, $1 \leq i \leq dim P_\sigma$, such that $f_j = 0$ on ∂M and $N(f_j) = \varphi_j$ on ∂M. Then

$$(3.33) \qquad \int_{\partial M} \varphi_j N(f_i) = \delta_{ij}.$$

Define then forms q_i as in (3.26). This gives a space Q_{P_γ} with $dim Q_{P_\gamma} = dim P_\gamma$ for which the slice property holds, i.e. for any $q = \sum a_i q_i \in Q_{P_\gamma}$, there exists $\pi \in P_\gamma$ such that

$$(3.34) \qquad \int_M \langle q, \pi \rangle \neq 0.$$

For σ close to γ, with the same choice of $f_i = 0$ and of $N(f_i)$ on ∂M, (3.33) gives

$$(3.35) \qquad \int_{\partial M} \varphi_j N(f_i) tr_\gamma \sigma \sim \delta_{ij},$$

so the slice condition (3.34) still holds. This gives a slice Q_{P_σ} for all σ (close to γ).

To obtain a slice for $\widetilde{K}_\sigma$, choose σ as follows. For the central choice $\sigma = \gamma$, the kernel K_γ consists of forms satisfying

$$L(k) = 0, \ \delta k = 0, \ [k^T]_\gamma = 0 \text{ and } -H'_k = \langle \tau'_k, \gamma \rangle - \langle \tau, k \rangle = 0.$$

while the reduced kernel $\widetilde{K}_\gamma$ consists of forms satisfying the further requirement $k^T = 0$ on ∂M. Now choose σ such that (as functions on ∂M)

$$(3.36) \qquad \langle \tau'_k, \sigma \rangle - \langle \tau, k \rangle \neq 0,$$

for all non-zero $k \in \widetilde{K}_\gamma$. (If $\widetilde{K}_\gamma = 0$, then $K_\gamma = P_\gamma$, so that (3.34) gives the required slice property for K_σ with $\sigma = \gamma$). If some $k_\sigma \in K_\sigma$ satisfies $k_\sigma = k \in \widetilde{K}_\gamma$, then one has of course $\langle \tau'_k, \sigma \rangle - \langle \tau, k \rangle \neq 0$ by (3.36) but by definition of K_σ, $\langle \tau'_{k_\sigma}, \sigma \rangle - \langle \tau, k_\sigma \rangle = 0$, a contradiction. Thus $k_\sigma \notin \widetilde{K}_\gamma$ for all k_σ, i.e.

$$K_\sigma \cap \widetilde{K}_\gamma = 0,$$

for all σ satisfying (3.36).

Now by definition $k \in \widetilde{K}_\gamma$ if and only if $H'_k = 0$ and $\varphi = \varphi_k = 0$. Thus $k_\sigma \notin \widetilde{K}_\gamma$ exactly when either $H'_k \neq 0$ or $\varphi_{k_\sigma} \neq 0$. In the second case, $k_\sigma \in P_\sigma$ and so (3.35) gives the slice property. If $\varphi_{k_\sigma} = 0$, $k_\sigma \in \widetilde{K}_\sigma$ implies $H'_{k_\sigma} \neq 0$. Moreover if k_j is a basis of $\widetilde{K}_\sigma$, then the functions $\{H'_{k_j}\}$ are linearly independent, and hence so are the functions $\{2H'_{k_j} + \langle A, k_j \rangle\}$ (since $k_j^T = 0$). Thus again from the basic formula (3.32), there is a choice of basis functions $\{f_i\}$ (with $N(f_i) = 0$ for instance) which gives the slice property as in (3.33)-(3.34) on $\widetilde{K}_\sigma$. This together with (3.34) itself, gives the slice property for all of K_σ.

To complete the proof, it suffices then to prove there exists an open set of σ near γ such that (3.36) holds. Observe first that for $k \in \widetilde{K}_\gamma$, (so $\varphi_k = 0$), $A'_k \neq 0$ on ∂M. For if $k^T = (A'_k)^T = 0$ on ∂M, then by Corollary 2.3, $k = 0$ on M. It follows that if k_j is a basis for $\widetilde{K}_\gamma$ then the symmetric forms A'_{k_j}, and hence τ'_{k_j}, are linearly independent on ∂M.

Note that (3.36) may be reformulated as: find a linear map B, close to the identity, such that

$$(3.37) \qquad tr(B\tau'_k) \equiv tr_\gamma B\tau'_k \neq 0,$$

for all $0 \neq k \in \widetilde{K}_\gamma$.

Each τ'_k is trace-free with respect to γ, since $\varphi_k = 0$ so that $tr_\gamma \tau'_k = -H'_k = 0$. Thus each τ'_k has a non-trivial positive part $(\tau'_k)^+$ given by composing τ'_k with the projection onto the positive eigenspaces of τ'_k. In particular, on any basis k_j of $\widetilde{K}_\tau$, the forms $(\tau'_{k_j})^+$ are linearly independent on ∂M. Hence they are linearly independent pointwise on some open set $\Omega \subset \partial M$. To simplify the notation, set $\tau'_{k_j} = T_j$ and $(\tau_{k_j})^+ = (T_j)^+$.

Choose points $p_i \in \Omega$, $1 \leq i \leq dim\widetilde{K}_\gamma$ with disjoint neighborhoods $U_i \subset \Omega$ and positive bump functions η_i supported in U_i, with $\eta_i(p_i) = 1$. For the moment, set $B = \sum_j \eta_j T_j^+$, where for each i, the basis forms $\{T_i^+\}$ satisfy

$$(3.38) \qquad \langle T_i^+, T_j \rangle(p_i) = 0, \quad \text{for all} \ \ j > i.$$

Such a basis may be constructed inductively as follows. At p_1, choose any basis k_1 of $\widetilde{K}_\gamma$. Fix k_1 and $T_1 = T'_{k_1}$; via the standard Gram-Schmidt process, construct then the basis forms k_j, $j \geq 2$ satisfying (3.38) at p_1. Next in the space spanned by $\{k_j\}$, $j \geq 2$, repeat the process at p_2, starting with T_2 and constructing forms k_j, $j \geq 3$ satisfying (3.38) at p_2. One continues inductively in this way through to the last point. Note that a different basis of $\widetilde{K}_\gamma$ is thus used at each point p_i. At any given p_r one has

$$(3.39) \qquad tr(BT_k)(p_r) = \langle B, T_k \rangle(p_r) = \sum_{i,j} \eta_i c_j \langle T_i^+, T_j \rangle(p_r),$$

where $k = \sum c_j k_j$ in the basis associated to p_r.

Now suppose that there exists $k \in \widetilde{K}_\gamma$ such that $tr(B\tau'_k) = 0$. Evaluating (3.39) at p_1 gives, by (3.38),

$$tr(B\tau'_k)(p_1) = c_1 |T_1^+|^2(p_1) = 0,$$

so that $c_1 = 0$. Using this, and by the construction of the basis at p_2, one has similarly

$$tr(BT'_k)(p_2) = c_2 |T_2^+|^2(p_2) = 0,$$

so that $c_2 = 0$. Continuing in this way, it follows that $c_r = 0$ for all r, and hence by the construction of the bases at $\{p_r\}$, $k = 0$. Thus $tr(BT'_k) \neq 0$ for all non-zero $k \in \widetilde{K}_\gamma$. This establishes (3.37) for this choice of B.

Finally, note that for $B' = Id$, $tr_\gamma(B'\tau'_k) = 0$, for all $k \in \widetilde{K}_\gamma$. Also, on the unit sphere in $\widetilde{K}_\gamma$ the space of functions $tr_\gamma(B\tau'_k)$ is compact, and so bounded away from the zero function. Hence,

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choosing ε sufficiently small and replacing B by $Id + \varepsilon B$ gives a smooth metric $\sigma > 0$, close to γ on ∂M, satisfying (3.36).
∎

Proposition 3.5 gives the existence of a "good" slice Q_σ to ImL_σ as in (3.28) consisting of forms q of the form (3.26). Of course it is possible that $K_\sigma = 0$, for some or all $\sigma \neq \gamma$, while $K_\gamma \neq 0$.

Now, as discussed prior to Proposition 3.5, form the operator

$$(3.40) \qquad \widetilde{L}(h) = L(h) + \pi_{Q_\sigma}(h),$$

where π_{Q_σ} is the L^2 orthogonal projection of K_σ onto Q_σ along ImL_σ. Proposition 3.5 implies that $\widetilde{L}$ is an isomorphism

$$(3.41) \qquad \widetilde{L} : S_\sigma^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

Given any boundary data $\delta h = 0$, $[h^T]_\sigma = h_1$, $\langle \tau'_h, \sigma \rangle - \langle \tau, h \rangle = h_2$ on ∂M, let h_e be a smooth extension of the boundary data to M. Assume without loss of generality that h_e depends smoothly on the boundary data (σ, h_1, h_2). Let $\widetilde{L}(h_e) = z$. By the isomorphism property above, there is a unique $h_0 = t + k_\sigma$, with $t \in K_\sigma^\perp \subset S_\sigma^{m,\alpha}(M)$ and $k_\sigma \in K_\sigma$, such that $\widetilde{L}(h_0) = z$. Hence, setting

$$(3.42) \qquad h = h_e - h_0 = h_e - t - k_\sigma,$$

gives

$$(3.43) \qquad L(h) = q, \quad \widetilde{L}(h) = 0,$$

where $q = \pi_{Q_\sigma}(k_\sigma - h_e)$.

We will refer to h in (3.42) as the "canonical" extension of the boundary data (σ, h_1, h_2). The next two results give some basic properties of this extension.

Lemma 3.6. *For any boundary data h_1 and h_2 as above, the solution h of (3.43) satisfies*

$$(3.44) \qquad \delta h = 0,$$

on M.

Proof: By (3.43) and (3.27), it follows that $\delta L(h) = 0$. Also, by (2.13), for any h, one has $\delta L(h) = \delta\delta^*(\delta(h))$ (since $\delta E' = 0$ by the Bianchi identity). Hence,

$$\delta\delta^*(\delta(h)) = 0,$$

on M. The result then follows by Lemma 2.1.
∎

Proposition 3.7. *For any boundary data (σ, h_1, h_2) the canonical extension h in (3.42) satisfies*

$$(3.45) \qquad \int_{\partial M} E'_h(N, X) = \int_{\partial M} q(N, X) = \int_{\partial M} \varphi_X[-2N(f) + Hf],$$

where f is associated to q as in (3.26) and $\kappa = \delta^* X \in K_1$.

Proof: We have $L(h) = E'(h) + \delta^*\delta(h) = E'(h)$ by (2.13). Since $L(h) = q$,

$$E'_h = q,$$

so that for such h, $\int_{\partial M} E'_h(N, X) = \int_{\partial M} q(N, X)$. Thus we need to show that

$$(3.46) \qquad \int_{\partial M} q(N, X) = \int_{\partial M} D^2 f(N, X) - (\Delta f + 2f)\nu = \int_{\partial M} \varphi_X[-2\varphi_X N(f) + Hf],$$

where $\nu = \langle N, X \rangle$.

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Computing the first $D^2(N, X)$ term in (3.46) gives

$$\int_{\partial M} \langle \nabla_X \nabla f, N \rangle = \int_{\partial M} \nu NN(f) + X^T \langle \nabla f, N \rangle - \langle \nabla f, \nabla_{X^T} N \rangle$$

$$= \int_{\partial M} \nu NN(f) - div(X^T)N(f) - A(X^T, \nabla f) = \int_{\partial M} \nu NN(f) - div(X^T)N(f) - f\delta(A(X^T))$$

$$= \int_{\partial M} \nu NN(f) - div(X^T)N(f) + f\langle A, \delta^*(X^T) \rangle + f dH(X^T),$$

where we have used the fact that $(\delta A)(X^T) = -dH(X^T)$. Since $(\delta^* X)^T = \varphi_X \gamma$, one has $\delta^*(X^T) + \nu A = \varphi_X \gamma$, so that $div(X^T) = -\nu H + 2\varphi_X$. It follows that

$$(3.47) \qquad \int_{\partial M} D^2 f(N, X) = \int_{\partial M} \nu[NN(f) + HN(f) - f|A|^2] - 2\varphi_X N(f) + f\varphi_X H + f X^T(H).$$

On the other hand, for the second term in (3.46) one has $\Delta f = \Delta_{\partial M} f + HN(f) + NN(f)$ so that

$$(3.48) \qquad \int_{\partial M} (\Delta f + 2f)\nu = \int_{\partial M} f\Delta_{\partial M}\nu + HN(f)\nu + NN(f)\nu + 2f\nu.$$

Subtracting (3.48) from (3.47) gives

$$\int_{\partial M} E_h'(N, X) = -\int_{\partial M} f[\Delta\nu + (|A|^2 + 2)\nu - X^T(H)] - 2N(f)\varphi_X - \varphi_X H f$$

$$= \int_{\partial M} f H_\kappa' - 2N(f)\varphi_X + \varphi_X H f,$$

where $H_\kappa' = 2H_{\delta^* X}'$. Since $H_\kappa' = 0$, the result follows. $\blacksquare$

The next two results give a partial proof of Theorem 3.3. The full proof of Theorem 3.3 is then completed after the proof of Proposition 3.10 below.

Let $\widetilde{K}_1$ be the reduced kernel in K_1, i.e. $\kappa \in \widetilde{K}_1$ if and only if $\kappa = \delta^* X$ for some X with $\kappa^T = 0$ and $H_\kappa' = 0$ at ∂M.

Proposition 3.8. *One has*

$$(3.49) \qquad\qquad\qquad \widetilde{K}_1 = 0.$$

Proof: Recall the formula (3.4)

$$(3.50) \qquad\qquad \int_{\partial M} \langle \tau_\kappa', h^T \rangle = \int_{\partial M} E_h'(N, X) - \int_{\partial M} \varphi_X H_h',$$

valid for any smooth h on M.

Suppose $\kappa \in \widetilde{K}_1$, $\kappa = \delta^* X$. The last term in (3.50) then vanishes since $\varphi_X = 0$. Choose then σ as in Proposition 3.5 and let h be the canonical extension of any boundary data (h_1, h_2). By Proposition 3.7, it follows that

$$\int_{\partial M} E_h'(N, X) = \int_{\partial M} \varphi_X[-2N(f) + Hf] = 0,$$

(again since $\varphi_X = 0$). Hence, for any boundary data $[h^T]_\sigma = h_1$,

$$\int_{\partial M} \langle \tau_\kappa', h^T \rangle = 0.$$

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Let $\mathcal{D}$ be the space of Dirichlet boundary data $\{h^T\}$, for which there is an extension h of h^T to M such that

$$(3.51) \qquad \int_{\partial M} E'_h(N, X) = 0.$$

This includes of course Dirichlet boundary values of all Einstein deformations, but in general, is considerably larger. The space $\mathcal{D}$ is linear subpace of $S^{m,\alpha}(\partial M)$, of finite codimension. To see this, the conformal classes $[h^T]_\gamma$ of $\mathcal{D}$ are of finite codimension in the space $T(\mathcal{C})$ of all infinitesimal deformations of conformal classes, (by the Fredholm property of L_γ). Moreover, by (3.50) and Remark 3.2, any pure-trace deformation $h^T = f\gamma$ is in $\mathcal{D}$, since $\varphi_X = 0$. Hence $\mathcal{D}$ is of finite codimension in $S^{m,\alpha}(\partial M)$.

Given σ and given a slice Q_σ as in (3.28), for any boundary data (h_1, h_2), the canonical extension h satisfies (3.51). Thus the "Dirichlet data" $h_1 = [h^T]_\sigma$ may be arbitrarily prescribed (as may the "Neumann data" h_2); for any given equivalence class h_1, there is an $h^T \in \mathcal{D}$ with $[h^T]_\sigma = h_1$.

Let h_0^σ be the trace-free part of h with respect to σ, i.e.

$$h_0^\sigma = h - \frac{\langle h, \sigma \rangle}{\langle \sigma, \sigma \rangle} \sigma,$$

so that $\langle h_0^\sigma, \sigma \rangle = 0$ and let V_σ be the space of all such forms. One has a natural embedding $V_\sigma \subset S^{m,\alpha}(\partial M)$ and a natural projection map $\pi_\sigma : S^{m,\alpha}(\partial M) \to V_\sigma$. Any class $[h^T]_\sigma$ is uniquely represented by an $h_0^\sigma \in V_\sigma$ and hence

$$\pi_\sigma(\mathcal{D}) = V_\sigma.$$

The same construction holds for $\sigma = \gamma$, with $V_\gamma \subset S^{m,\alpha}(\partial M)$ the usual space of trace-free forms on ∂M with respect to γ. For σ close to γ, the subspaces V_σ and V_γ are close, as are the projection maps π_σ, π_γ. In particular, there is a natural isomorphism $I : V_\sigma \to V_\gamma$. It follows that

$$\pi_\gamma(\mathcal{D}) = V_\gamma,$$

so that boundary values h^T of deformations h^T satisfying (3.51) surject onto all conformal classes. Since τ'_κ is trace-free, so in fact $\tau'_\kappa \in V_\gamma$, it follows that

$$\tau'_\kappa = 0,$$

which together with $H = 0$ gives $A'_\kappa = 0$. Since $\kappa^T = 0$, the result then follows from Corollary 2.3. ∎

Next we extend Proposition 3.8 to the full reduced kernel $\widetilde{K}$. General deformations $\kappa \in Ker D\Pi$ are not necessarily of the form

$$(3.52) \qquad \kappa = \delta^* X.$$

However, in dimension 3, all infinitesimal Einstein deformations are constant curvature deformations and hence all deformations are locally of the form (3.52). In particular (3.52) holds on the universal cover $\widetilde{M}$. This leads to the full version of Proposition 3.8.

Proposition 3.9. *One has*

$$(3.53) \qquad \widetilde{K} = 0,$$

and hence $K = P = P_\gamma$. Consequently, $K_\sigma = P_\sigma$ for any σ sufficiently close to γ.

Proof: As above, consider the universal cover $\widetilde{M}$ of M with boundary $\widetilde{\partial M}$ the lift of ∂M to $\widetilde{M}$. Recall the developing map gives an isometric immersion $D : \widetilde{M} \to S^3$. The group $\pi_1(M) = \mathbb{Z} * \cdots * \mathbb{Z}$ acts isometrically on $\widetilde{M}$. A fundamental domain $F \subset \widetilde{M}$ for the action of $\pi_1(M)$ is topologically a thickening of a "g-cross", i.e. $g = genus(\partial M)$ linearly independent line segments intersecting at a common midpoint. The boundary ∂F consists of two parts; first the "intrinsic" boundary

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$\partial_i F = \widetilde{\partial M} \cap \bar{F}$ coming from the lift of ∂M, and second, $2g$ discs D_j in $\widetilde{M}$ meeting $\partial_i F$ in $2g$ circles. Pairs of discs $D_j \cup D'_j$ are glued by an isometry to obtain the quotient manifold M, and give rise to g discs in the interior of M. Equivalently, M is cut along g discs to obtain F.

Any form $\kappa \in Ker D\Pi$ lifts to a form, also called κ on F, with κ of the form (3.52). Under the gluing of the paired discs, the vector field X is transformed to $X + Z$, for some Killing field Z on S^3, depending on j.

On F, the formula (3.6) holds and gives

$$(3.54) \qquad \int_{\partial F} \langle \tau'_\kappa, h^T \rangle + \langle a(\kappa), h^T \rangle = \int_{\partial F} \langle \tau'_h, \kappa \rangle + \langle a(h), \kappa^T \rangle + \int_F \langle E'_h, \delta^* X \rangle.$$

The terms on the boundary discs $D_j \cup D'_j$ involving only κ cancel when glued, since κ is well-defined on M. Hence,

$$\int_{\partial_i F} \langle \tau'_\kappa, h^T \rangle + \langle a(\kappa), h^T \rangle = \int_{\partial_i F} \langle \tau'_h, \kappa \rangle + \langle a(h), \kappa^T \rangle + \int_F \langle E'_h, \delta^* X \rangle,$$

which as in (3.8) gives

$$\int_{\partial_i F} \langle \tau'_\kappa, h^T \rangle = -\int_{\partial_i F} \varphi_X H'_h + \int_F \langle E'_h, \delta^* X \rangle.$$

As before,

$$(3.55) \qquad \int_F \langle E'_h, \delta^* X \rangle = \int_{\partial F} E'_h(N, X).$$

The construction and properties of the slices Q (and the associated operator $\widetilde{L}$) hold for the full kernel $K = K_\sigma$; they do not require holonomy trivial deformations. Thus we carry out the construction with L, $\widetilde{L}$, Q, canonical extension and such as before on M, and lift up to the fundamental domain F. One thus has prescribed boundary data (h_1, h_2) for h along the intrinsic boundary $\partial_i F$. Note one has no such exact boundary control along the gluing discs $D_j \cup D'_j$.

As in Proposition 3.7, one has $E'_h = q$ for some $q \in Q_\sigma$ and lifting this data up to F gives

$$\int_{\partial F} E'_h(N, X) = \int_{\partial F} q(N, X).$$

We claim that

$$(3.56) \qquad \int_{D_j \cup D'_j} q(N, X) = 0,$$

for each pair of discs, $1 \leq j \leq g$. Observe that $\delta^* X = \delta^*(X + Z)$ for any Killing field Z on $\widetilde{M}$ while $\nu_{X+Z} = \nu_X + \nu_Z$. To prove (3.56), it suffices to prove

$$(3.57) \qquad \int_{D_j} q(N, X) = \int_{D_j} q(N, X + Z),$$

since when the discs D_j and D'_j are glued, the normal vectors N point in the opposite directions, producing a cancelation which gives (3.56).

To prove (3.57), one calculates exactly as in (3.47)-(3.48), replacing X by $X + Z$. Preceding (3.47), $\delta^*(X^T) = -\nu A + \varphi_X \gamma$ is replaced by $\delta^*(X^T + Z^T) = -\nu_{X+Z} A + \kappa^T$, while $div(X^T) = -\nu H + 2\varphi_X$ is replaced by $div(X^T + Z^T) = -\nu_{X+Z} H + tr\kappa^T$. Of course $\kappa = \delta^* X = \delta^*(X + Z)$. It follows then easily that

$$\int_{D_j} q(N, X + Z) - \int_{D_j} q(N, X) = -\int_{D_j} f[\Delta \nu_Z + (|A|^2 + 2)\nu_Z - Z^T(H)] = 0,$$

since $H'_{\delta^* Z} = 0$.

Equation (3.56) implies that

$$\int_{\partial F} E_h'(N, X) = \int_{\partial_i F} q(N, X),$$

and rest of the proof of Proposition 3.9 is the same as that of Proposition 3.8.

∎

To understand the remaining part P of the kernel K, we consider now the boundary conditions

(3.58) $$[h^T]_\sigma = h_1, \quad -H_h' = \langle \tau_h', \gamma \rangle - \langle \tau, h \rangle = h_2.$$

As in (3.14), let $S_{\sigma,\gamma}^{m,\alpha}(M)$ be the space of $C^{m,\alpha}$ smooth forms on M satisfying $(h_1, h_2) = (0, 0)$ for h_1 and h_2 as in (3.58). These boundary conditions are no longer self-adjoint, but for σ close to γ, are close to being self-adjoint. Let

$$L_{\sigma,\gamma} = L|_{S_{\sigma,\gamma}^{m,\alpha}(M)} : S_{\sigma,\gamma}^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

The operator $L_{\sigma,\gamma}$ is elliptic and close to L_σ, and so is Fredholm, of Fredholm index zero. The kernel $K_{\sigma,\gamma} = Ker L_{\sigma,\gamma}$ is given by forms k with $L(k) = 0$, $\delta k = 0$ with $(h_1, h_2) = (0, 0)$ in (3.58) (so that in particular $H_k' = 0$) and

$$Im L_{\sigma,\gamma} \oplus K_{\sigma,\gamma} = S^{m-2,\alpha}(M).$$

This is a direct sum decomposition, which is almost, but not exactly, L^2 orthogonal. As in (3.22),

(3.59) $$(Im L_{\sigma,\gamma})^\perp = K_{\gamma,\sigma},$$

where $K_{\gamma,\sigma}$ consists of forms k satisfying $\delta k = 0$, $[k^T]_\gamma = 0$, $\langle \tau_k', \sigma \rangle - \langle \tau, k \rangle = 0$.

Now the analog of Proposition 3.5 in this setting is:

Proposition 3.10. *Under the assumptions* $\mathbf{K} = 0$ *and* $H_{\partial M} = 0$, *there exist smooth Riemannian metrics* σ *on* ∂M *arbitrarily close to* γ, *and a slice* $Q_{\sigma,\gamma}$, *consisting of smooth forms* $q = D^2 f - (\Delta f + 2f)g$ *as in (3.26), such that*

(3.60) $$Im L_{\sigma,\gamma} \oplus Q_{\sigma,\gamma} = S^{m-2,\alpha},$$

and such that, (cf. (3.45))

(3.61) $$\int_{\partial M} \varphi_X[-2N(f) + Hf] = 0.$$

Proof: The computations following (3.29) remain valid as before and (3.32) still holds. In this setting, we now choose $2N(f) - Hf = 0$ at ∂M, (e.g. $N(f) = 0$ when $H = 0$), so that (3.61) holds automatically and (3.32) becomes

$$\int_M \langle q, k \rangle = \int_{\partial M} f[2H_k' + \langle A, k \rangle] = \int_{\partial M} f \langle A, k \rangle.$$

Note that if $k \in K_{\sigma,\gamma}$ has $\varphi_k = 0$, then $k \in \widetilde{K}_\gamma$ and by (3.53), $\widetilde{K}_\gamma = 0$. Hence $\varphi_k \neq 0$ for all non-zero $k \in K_{\sigma,\gamma}$. First choose then σ so that

(3.62) $$\langle A, k \rangle \neq 0,$$

(as functions on ∂M) for all non-zero $k = k_\sigma \in K_{\sigma,\gamma}$. Of course $k^T = \varphi_k \sigma$ on ∂M; note also that if $\varphi_k = 0$ on any open set in ∂M, then $\varphi_k = 0$ on ∂M, since φ_k is analytic.

We then choose f_i such that

(3.63) $$\int_{\partial M} f_i \langle A, k_j \rangle = \delta_{ij},$$

where k_j is a basis of $K_{\sigma,\gamma}$ with $k_j^T = \varphi_j \sigma$ and $\{\varphi_j\}$ orthonormal in $L^2(\partial M)$.

Next we claim that there is a further choice of σ (satisfying (3.62)) verifying the slice property (3.60). As in (3.59), $(ImL_{\sigma,\gamma})^\perp = K_{\gamma,\sigma}$ and so to show that $q \notin ImL_{\sigma,\gamma}$, it suffices to show that, for any non-zero q,

$$\int_M \langle q, \widetilde{k} \rangle \neq 0,$$

for some $\widetilde{k} \in K_{\gamma,\sigma}$. Computing this in same way as before gives

(3.64) $$\int_M \langle q, \widetilde{k} \rangle = \int_{\partial M} f(2H'_{\widetilde{k}} + \langle A, \widetilde{k} \rangle) = \int_{\partial M} 2fH'_{\widetilde{k}},$$

since $\widetilde{\kappa} = \varphi\gamma$ and $H = 0$. By assumption, $\langle \tau'_{\widetilde{k}}, \sigma \rangle - \langle \tau, \widetilde{k} \rangle = 0$, and hence (again since $H = 0$) $\langle \tau'_{\widetilde{k}}, \sigma \rangle = 0$. Note that if $H'_{\widetilde{k}} = 0$ then $\widetilde{k} \in K = K_\gamma$, since $\widetilde{k}^T = \varphi\gamma$. Now by assumption $\mathbf{K} = \mathbf{0}$. Hence there is an open set of σ near γ such that, for any $\widetilde{k} \in K_\gamma$, $\langle \tau'_{\widetilde{k}}, \sigma \rangle \neq 0$. This is equivalent to the statement that there is an open set of σ near γ such that for $\widetilde{k} \in K_{\gamma,\sigma}$, $H'_{\widetilde{k}} \neq 0$. Clearly, by the openness property for instance, one may choose σ in addition so that (3.62) holds.

As in (3.63), one may then choose linearly independent boundary functions f_i and form the slice Q_σ. ∎

Propositions 3.9 and 3.10 essentially complete the proof of Theorem 3.3. Namely, given $Q_{\sigma,\gamma}$ as in (3.60), one defines the operator $\widetilde{L}$ as in (3.40) to obtain the canonical extension h of boundary values $[h^T]_\sigma = h_1$ and $-H'_h = h_2 = 0$. By (3.45) and (3.61),

(3.65) $$\int_M \langle E'_h, \kappa \rangle = \int_{\partial M} E'_h(N, X) = 0,$$

for $\kappa \in K_1$ with $[h^T]_\sigma = h_1$ arbitrary. Similarly, by Proposition 3.9, (3.65) holds for arbitrary $\kappa \in K$. Thus (3.5) simplifies to

$$\int_{\partial M} \langle \tau'_\kappa, h^T \rangle = 0.$$

As in the proof of Proposition 3.7, it follows that $(\tau'_\kappa)^T = 0$, so $(A'_\kappa)^T = 0$. Hence, since $\mathbf{K} = 0$, $\kappa = 0$.

This completes the proof of Theorem 3.3. ∎

We recall that all the work above holds for ∂M of arbitrary genus.

We now proceed toward proving

$$\mathbf{K} = 0,$$

which, via Theorem 3.3, will prove Theorem 1.1. Assume in the following that $genus(\partial M) \geq 2$; the case $genus(\partial M) = 1$ will be discussed afterwards.

As discussed in the Introduction, let $\mathrm{Diff}_0(M) = \mathrm{Diff}_0^{m+1,\alpha}(M)$ be the group of $C^{m+1,\alpha}$ diffeomorphisms of M isotopic to the identity and mapping $\partial M \to \partial M$. When $genus(\partial M) \geq 2$, $\mathrm{Diff}_0(M)$ acts freely on $\mathcal{C}$ and so on the target space $\mathcal{C} \times C$ of Π. When ∂M is minimal (or of constant mean curvature) the linearization $D\Pi$ of Π descends to a smooth Fredholm map

(3.66) $$D\Pi : T(\mathcal{E}/\mathrm{Diff}_0(M)) \to T\mathcal{T}(\partial M) \times C(\partial M),$$

$$D\Pi(h) = ([h^T], H'_h),$$

where $[h^T] = [h^T + f\gamma + \delta^* Y^T]$, for any function f and vector field Y^T tangent to ∂M.

Next, we proceed one step further and divide out by the full diffeomorphism group $\mathrm{Diff}(M) = \mathrm{Diff}^{m+1,\alpha}(M)$ of M, at the linearized level. At the boundary, this corresponds to dividing out (setting to zero) the metric variations which are induced by vector fields normal to ∂M.

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Thus, let the functions $\nu \in C^{m+1,\alpha}(\partial M)$, identified with normal vector fields νN, act on the boundary space $T\mathcal{T}(\partial M) \times C^{m-1,\alpha}((\partial M)$ as the normal variation of the data:

$$(3.67) \qquad \nu(([h], H_h')) = ([h + \delta^*(\nu N)], H_h' + H_{\delta^*(\nu N)}'),$$

with $\delta^*(\nu N) = \nu A$. This gives an action of $C^{m+1,\alpha}(\partial M)$ on $T\mathcal{T}(\partial M) \times C^{m-1,\alpha}(\partial M)$. The stabilizer J_1 of the action at any point $([h], H_h')$ consists of ν such that $\nu(([h], H_h')) = ([h], H_h')$, i.e. $[\nu A] = 0$ and $H_{\nu A}' = 0$. Thus

$$\delta^* Y^T + \nu A = \varphi \gamma, \quad H_{\nu A}' = 0,$$

for some Y^T, φ. Since ∂M has no tangential conformal vector fields, Y^T is uniquely determined by ν. The vector field $Y = Y^T + \nu N$ gives an element $\delta^* Y$ in the holonomy trivial kernel K_1 as in (3.3). Let $\mathcal{Z}$ be the space Killing fields Z on (M, g), so $\delta^* Z = 0$. It follows that

$$(3.68) \qquad dim\, J_1 = dim\, \mathcal{Z} + dim\, K_1.$$

The stabilizer is the same at all points, so the quotient group $C^{m+1,\alpha}(\partial M)/J_1$ acts freely on $\mathcal{T}(\partial M) \times C(\partial M)$.

Let $\{H_{\nu A}'\}$ denote the space of variations of H, as ν ranges over $C^{m+1,\alpha}(\partial M)$ and let

$$(3.69) \qquad C^{m-1,\alpha}(\partial M)/\{H_{\nu A}'\} = \widehat{J}.$$

Since the Jacobi operator $\mathcal{H} : C^{m+1,\alpha}(\partial M) \to C^{m-1,\alpha}(\partial M)$, $\mathcal{H}(f) = H_{fA}' = -\Delta f - (|A|^2 + 2)f$ is self-adjoint, there is a natural isomorphism between the kernel J of $\mathcal{H}$ (the space of Jacobi fields) and the cokernel $\widehat{J}$ of $\mathcal{H}$.

Note that for any prescribed function η on ∂M, there is a metric variation ℓ of (M, g) (not necessarily Einstein) such that $H_\ell' = \eta$; thus

$$C^{m-1,\alpha}(\partial M) = \{H_\ell'\}.$$

Lemma 3.11. *The orbit space of the action of $C^{m+1,\alpha}(\partial M)$ on $T(\mathcal{T}(\partial M)) \times C^{m-1,\alpha}(\partial M)$ is given by*

$$(3.70) \qquad [T(\mathcal{T}(\partial M)/(J/J_1)] \times \widehat{J}.$$

Proof: If two orbits

$$([h_1], \chi_1) = ([h_2], \chi_2),$$

are equal, then

$$h_2 = h_1 + \nu A,$$

(mod $\delta^* Y^T$) and

$$\chi_2 = \chi_1 + H_{\nu A}'.$$

Without loss of generality we may assume that χ_1 and χ_2 are orthogonal to the space of variations $\{H_{\nu A}'\}$ as in (3.69). It follows that $\chi_1 = \chi_2$ and so $H_{\nu A}' = 0$. Hence

$$\nu \in J,$$

i.e. νN is a Jacobi field. The subspace $J_1 \subset J$ acts trivially. Moreover, distinct elements χ_1, $\chi_2 \in \widehat{J} \simeq J$ give rise to distinct orbits. This gives (3.70).
∎

It follows that the map $D\Pi$ in (3.66) descends further to a map

$$(3.71) \qquad \widehat{D\Pi} : T(\mathcal{E}/\mathrm{Diff}(M)) \to [T(\mathcal{T}(\partial M)/(J/J_1)] \times \widehat{J},$$

$$\widehat{D\Pi}(h) = ([h^T], [H_h']),$$

where now $([h^T], [H_h']) = ([h^T + \varphi \gamma + \delta^* Y^T + \nu A], [H_h' + H_{\nu A}'])$, for any functions φ, ν and vector field Y^T tangent to ∂M. Set $Y = Y^T + \nu N$.

Note that

$$(3.72) \qquad T(\mathcal{E}/\mathrm{Diff}(M)) = \{h\}/\{\delta^* Y\} \simeq T_\rho \mathcal{R}(M),$$

is naturally isomorphic to the space of infinitesimal holonomy deformations of (M, g).

Now consider the analog of the operator L in (3.15) in this setting, (where $\sigma_1 = \sigma_2 = \gamma$). We have $T(Met(M)/\mathrm{Diff}(M)) = S^{m,\alpha}(M)/\delta^*(\chi^{m+1,\alpha}) \simeq \{\ell\}/\{\delta^* Y\}$. As in (3.14), the zero boundary values in this space are given as

$$([h^T, [H'_h]) = (0, 0),$$

i.e.

$$(3.73) \qquad h^T = \varphi\gamma + \delta^* Y, \ Y = Y^T + \nu A, \ \text{with } H'_h = H'_{\nu A}.$$

Let then $\widehat{\mathcal{S}}_0^{m,\alpha}(M)$ be the subspace of $\widehat{\mathcal{S}}^{m,\alpha}(M) = S^{m,\alpha}(M)/\delta^*(\chi^{m+1,\alpha})$ satisfying (3.73) and consider the induced Einstein operator

$$(3.74) \qquad E' : \widehat{\mathcal{S}}_0^{m,\alpha}(M) \to \widehat{\mathcal{S}}^{m-2,\alpha}(M).$$

Since E' is diffeomorphism invariant, i.e. $E'(\delta^* Y) = 0$, (3.74) is well-defined. Alternately, since we are dividing out by the full diffeomorphism group, there is no longer a need for a gauge choice, such as the divergence-free gauge. Namely, dividing out by all diffeomorphisms or vector fields at ∂M constitutes dividing out by 3 degrees of freedom. The gauge condition $\delta k = 0$ at ∂M, or equivalently the constraint equations (2.3)-(2.4), constitute 3 scalar conditions. Consequently, the divergence-free gauge condition in (3.14) is now eliminated.

Elements $\widehat{k}$ in the kernel of E', or equivalently $\widehat{D\Pi}$, are equivalence classes of infinitesimal Einstein deformations which at ∂M have the form

$$(3.75) \qquad \widehat{k}^T = \varphi\gamma + \delta^* Y, \ \ Y = Y^T + \nu N \ \text{with } H'_{\widehat{k}} = H'_{\nu A}.$$

We claim that

$$(3.76) \qquad index\, \widehat{D\Pi} = 0.$$

Recall first that $index\, D\Pi = 0$. This follows from the self-adjoint property of the boundary conditions (3.16), (Proposition 3.4) which gives the splitting (3.23), together with the standard extension/subtraction argument as discussed in (3.40)-(3.43).

Now $K = Ker\, D\Pi$ consists of all infinitesimal Einstein deformations κ such that $\kappa^T = \varphi\gamma$, $H'_\kappa = 0$. The kernel $\widehat{K}$ of $\widehat{D\Pi}$ consists of equivalence classes $[\kappa]$ of κ under the equivalence relation generated by $\delta^* Y$, as in (3.75). Thus,

$$dim\, \widehat{K} = dim\, K - dim\, K_1.$$

Similarly, from (3.23),

$$ImL_\gamma \oplus K = S^{m-2,\alpha}(M).$$

so that $Coker\, L_\gamma = K$. Again one divides out by the space of forms $\delta^* Y$, so that $dimCoker E' = dimK - dimK_1$. This proves the claim.

Now recall from (2.37)

$$(3.77) \qquad dim\, T_e \mathcal{R}(M) = dim\, H^1(\pi_1(M), Ad\, e) = dim\, T(\mathcal{E}/\mathrm{Diff}(M)) = 6g,$$

while for (M, g) of holonomy ρ, we let

$$(3.78) \qquad dim\, T_\rho \mathcal{R}(M) = dim\, H^1(\pi_1(M), Ad\, \rho) = dimT(\mathcal{E}/\mathrm{Diff}(M)) \equiv D_\rho \leq 6g.$$

Generically, $D_\rho = 6g - 6$.

Corollary 3.12. *For genus$(\partial M) \geq 2$, one has*

$$(3.79) \qquad\qquad dim\, J_1 = dim\, \mathcal{Z} + dim\, K_1 = 6,$$

when $\rho = e$, so that

$$(3.80) \qquad\qquad K_1 = 0.$$

The holonomy trivial kernel consists only of Killing fields, i.e. the only solutions to

$$\delta^* Y^T + \nu A = \varphi\gamma, \quad H'_{\nu A} = 0,$$

are Killing fields Z. For general holonomy representation ρ, one has

$$(3.81) \qquad\qquad dim\, J_1 = D_\rho - (6g - 6),$$

so that $dim\, J_1 = dim\, \mathcal{Z}$ and $K_1 = 0$. Generically, $J_1 = \mathcal{Z} = 0$.

Proof: The first equality in (3.79) is of course just (3.68). Referring to (3.71)-(3.72), one has

$$dim\, T(\mathcal{E}/\mathrm{Diff}(M) = D_\rho \leq 6g,$$

while

$$dim[T(\mathcal{T}(\partial M)/(J/J_1)] \times \widehat{J} = 6g - 6 + j_1,$$

where $j_1 = dim J_1$. By the Fredholm alternative one then has

$$D_\rho = 6g - 6 + j_1 + index\widehat{D\Pi},$$

which via (3.76) gives (3.79) and (3.81). The last statement then follows from (2.38). ∎

We are now (finally) in position to complete the proof of Theorem 1.1.

Proof of Theorem 1.1.

The idea at this point is to apply the slice construction as in Propositions 3.5 and 3.10 for the map $\widehat{D\Pi}$ in place of the map $D\Pi$, and with the kernel $\widehat{K}$ in place of Q; more precisely, we choose specific representatives for the slice $\widehat{K}$.

To begin, by Corollary 3.12, ($K_1 = 0$), any non-zero $k \in K$ induces a non-zero $\widehat{k} \in \widehat{K}$, and hence

$$(3.82) \qquad\qquad \widehat{K} \simeq K.$$

In particular, $\widehat{K}$ consists of forms with non-trivial holonomy.

We now choose (unique) representatives $\widehat{k}$ of $[\widehat{k}] \in \widehat{K}$ such that

$$\widehat{k} \perp Im(\delta^* Y).$$

Such representatives can be constructed as minimizers of the L^2 norm $||\widehat{k} + \delta^* Y||_{L^2}$ as Y ranges over the space $\chi^{m+1,\alpha}(M)$ of $C^{m+1,\alpha}$ vector fields on M.

For such $\widehat{k}$, one has

$$0 = \int_M \langle \widehat{k}, \delta^* Y \rangle = \int_M \langle \delta\widehat{k}, Y \rangle + \int_{\partial M} \widehat{k}(N, Y),$$

for all Y, and hence

$$(3.83) \qquad\qquad \widehat{k}(N, \cdot) = 0, \quad \delta\widehat{k} = 0.$$

These are exactly the properties needed of the slice $Q = Q_\sigma$ used in (3.27) (for Lemma 3.6) and (3.45) used for showing $\int_{\partial M} E'_h(N, X) = 0$. Thus we choose $\widehat{K}$ to be the slice for $Im\widehat{D\Pi}$ with representatives satisfying (3.83). In exactly the same way as before, (forming the operator $\widetilde{L}$ by

addition of the projection operator to $\widehat{K}$ to L), it follows then that for boundary data $([h^T], [H'_h])$ arbitrarily prescribed, one has a canonical extension h such that as in (3.65)

$$(3.84) \qquad \int_M \langle E'_h, \kappa \rangle = 0,$$

for any $\kappa \in K = KerD\Pi$. Here the boundary data h^T is prescribed modulo addition of terms $\varphi\gamma + \delta^*Y^T + \nu A$ with H'_h prescribed mod $H'_{\nu A}$.

Now return to main formula (3.50); we choose $\kappa \in \mathbf{K}$, so $\kappa^T = \varphi\gamma$, $\tau'_\kappa = 0$ and $H'_\kappa = 0$. One then has

$$0 = \int_{\partial M} \langle \tau'_\kappa, h^T \rangle = -\int_{\partial M} \varphi_\kappa H'_h.$$

Now H'_h may be arbitrarily prescribed modulo $H'_{\nu A}$. On the other hand, since $\delta^*(\nu N)$ is an infinitesimal Einstein deformation,

$$\int_{\partial M} \varphi_\kappa H'_{\nu A} = -\int_{\partial M} \langle \tau'_\kappa, \nu A \rangle = 0,$$

exactly since $\tau'_\kappa = 0$. It follows that

$$\int_{\partial M} \varphi_\kappa H'_h = 0,$$

with H'_h arbitrarily prescribed. Hence

$$\varphi_\kappa = 0$$

and hence $\kappa = 0$ by Corollary 2.3. This shows

$$\mathbf{K} = 0,$$

and thus by Theorem 3.3,

$$K = KerD\Pi = 0.$$

This completes the proof of Theorem 1.1.

∎

We now turn to the proofs of Theorems 1.2 and 1.3. The main aspects of the proofs of these results are treated concurrently; further the proofs follow closely the proof of Theorem 1.1 above. We recall that Propositions 3.1 through to Proposition 3.10 hold for $genus(\partial M) = 1$.

For $\partial M = T^2$, let $\text{Diff}_0(M) = \text{Diff}_0^{m+1,\alpha}(M)$ denote the group of diffeomorphisms of M isotopic to the identity, mapping $\partial M \to \partial M$, and fixing a given point $p_0 \in \partial M$. This differs from the diffeomorphism group used for higher genus by the action of T^2 on itself by the translation group T.

The vector fields $T \in \mathcal{T}$ are conformal vector fields on $(\partial M, \gamma)$ so that

$$(3.85) \qquad dim\{\delta^*T\} = 2 - i,$$

where as in (2.22), i is the number of tangential Killing fields on $(\partial M, \gamma)$.

As before, we form the quotient by dividing out by the tangential diffeomorphisms $\text{Diff}_0(M)$ (corresponding to vector fields Y^T) and then dividing out by normal vector fields $Y = Y^T + \nu N$. The action (3.67) is defined the same way here for $\partial M = T^2$, with stabilizer J_1. The analog of (3.68) is

$$(3.86) \qquad dim\, J_1 + (2 - i) = dim\, \mathcal{Z} - i + dim\, K_1.$$

Namely, an element $\nu \in J_1$ determines a vector field Y^T uniquely up to the space of conformal Killing fields of dimension $2 - i$. The pair (Y^T, ν) give either a non-tangential Killing field $Y = Z$ at ∂M, or an element of K_1. This gives (3.86).

Lemma 3.11 holds here in the same way and the induced map $\widehat{D\Pi}$ is defined as in (3.71). As before, one has

$$E' : \widehat{\mathcal{S}}_{0,v}^{m,\alpha}(M) \to \widehat{\mathcal{S}}^{m-2,\alpha}(M),$$

where now the domain $\widehat{\mathcal{S}}_{0,v}^{m,\alpha}(M)$ consists of zero boundary values in the space $S^{m,\alpha}(M)/\delta^*(\chi_v^{m+1,\alpha}) = T(Met(M)/\mathrm{Diff}_{0,v}(M))$, where the subscript v denotes vector fields vanishing at a fixed point. One has

(3.87) $$\widehat{\mathcal{S}}_{0,v}^{m,\alpha}(M) = \widehat{\mathcal{S}}_0^{m,\alpha}(M) \oplus \{\delta^*T\}.$$

The proof that

(3.88) $$index\ \widehat{D\Pi} = 0$$

is the same as in (3.76).

In analogy to (3.77) and (3.78), for $genus(\partial M) = 1$ one has

(3.89) $$dim\,T_e\mathcal{R}(M) = dim\,H^1(\pi_1(M), Ad\,e) = dim\,T(\mathcal{E}/\mathrm{Diff}(M)) = 6,$$

while for general ρ,

(3.90) $$dim\,T_\rho\mathcal{R}(M) = dim\,H^1(\pi_1(M), Ad\,rho) = dim\,T(\mathcal{E}/\mathrm{Diff}(M)) = D_\rho \leq 6,$$

(and generically $D_\rho = 2$).

Corollary 3.13. *When $genus(\partial M) = 1$,*

(3.91) $$dim\,J_1 = 6 - i, \quad dim\,K_1 = 2 - i,$$

when $\rho = e$, where i is the number of tangential Killing fields. Thus all elements in K_1 are given by tangential conformal Killing fields and

$$K_1 \subset \mathbf{K}.$$

For general ρ, one has

(3.92) $$dim\,J_1 = D_\rho - i, \quad dim\,K_1 = 2 - i.$$

Each $\nu \in J_1$ is the normal component of some Killing field on (M, g).

Proof: The same argument as the proof of Corollary 3.12 gives

$$D_\rho + 2 - i = 2 + j_1 + index(D\Pi),$$

where for the right side we use (3.71)-(3.72), (3.85) and (3.87). Thus the first equality in (3.91) follows from (3.88), while the second then follows from (3.86). The same argument of course gives (3.92). The last statement follows t hen again from (3.86).

∎

The analog of (3.82) in this setting is

$$\widehat{K} \simeq K/K_1,$$

and one has $K/\mathbf{K} \subset K/K_1$. As before, we use the slice $\widehat{K}$ for the mapping $\widehat{D\Pi}$, with representatives $\widehat{k}$ as in (3.83). However in this case, there is no slice for K_1, of dimension $2 - i$, so that there is only a partial slice to ImL.

Nevertheless, it follows by the canonical extension process as before that for all boundary data $([h^T], [H'_h])$ in a space $\mathcal{G}$ of codimension $2 - i = dim\,K_1$ in $\mathcal{C}^{m,\alpha} \times C^{m-1,\alpha}(\partial M)$, there is a canonical extension h such that (3.84) holds for any $\kappa \in Ker D\Pi$ and thus

(3.93) $$\int_{\partial M} \langle \tau'_\kappa, h^T \rangle = -\int_{\partial M} \varphi_\kappa H'_h.$$

Hence, for $\kappa \in \mathbf{K}$ (where $\tau'_\kappa = 0$) the associated trace terms φ_κ on ∂M are constrained to lie in a space of dimension $dim\, K_1 = 2 - i$ in $C^{m-1,\alpha}(\partial M)$, namely the space orthogonal to the space of variations $\{H'_h\}$ for canonical extensions h with boundary data in $\mathcal{G}$. Together with (2.24), it follows that

$$(3.94) \qquad\qquad\qquad\qquad dim\, \mathbf{K} = 2 - i,$$

and hence by Corollary 3.13

$$\mathbf{K} = K_1.$$

It also follows that

$$(3.95) \qquad\qquad\qquad\qquad K = K_1 = \mathbf{K}.$$

To see this, one sets $h^T = 0$ in (3.93) and lets $\{H'_h\}$ vary over the space (corresponding to $\mathcal{G}$ above) of codimension $2 - i$. Since, as above, the boundary values φ_κ can thus range only over a space of dimension $2 - i$, the claim follows by Proposition 3.9.

The relations (3.94) and (3.95) are the genus 1 analog of Theorem 1.1, i.e. $K = 0$ when $genus(\partial M) \geq 2$.

We thus have three cases to consider;

$$i = 0,\ 1 \text{ or } 2.$$

We first rule out the case $i = 0$ corresponding to $dim\, K = 2$.

Proposition 3.14. *For $\partial M = T^2$ minimal, one has*

$$dim\, K \leq 1.$$

Proof: Suppose instead $dim\, K = 2$, (so $i = 0$). Then $K = \mathbf{K} = K_1$ is generated by 2 tangential conformal Killing fields on T^2, the translations $T_i = \partial_{x_i}$:

$$(3.96) \qquad\qquad\qquad\qquad \mathcal{L}_{T_i}\gamma = \varphi_i \gamma.$$

It is well-known, cf. [22], that the metric $\widetilde{\gamma} = \lambda \gamma$ is flat, where $\lambda = |A|$. Since the translations T_i are Killing vector fields for $\widetilde{\gamma}$, $\mathcal{L}_{T_i}\widetilde{\gamma} = 0$, one has

$$(3.97) \qquad\qquad\qquad\qquad \varphi_i = -\partial_{x_i}(\log \lambda).$$

We may assume that φ_i are not identically zero on any open set in T^2. For if this were so for φ_1, then by analyticity $\varphi_1 \equiv 0$ on ∂M and hence T_1 is a non-zero Killing field on (T^2, γ) giving a contradiction, i.e. we are in the cases $i = 1$ or $i = 2$.

Since the second fundamental form A is a holomorphic quadratic differential, it has constant coefficients in the basis T_i; by a change of basis if necessary we may assume that $T_i = \partial_{x_i}$ are eigenvectors of A, so that

$$(3.98) \qquad\qquad\qquad\qquad A = dx_1^2 - dx_2^2.$$

By Corollary 3.13, the conformal factors φ_i are normal components of Killing fields $Z_i = V_i + \varphi_i N$, $Z_i^T = V_i$, so that

$$(3.99) \qquad\qquad\qquad\qquad \tfrac{1}{2}\mathcal{L}_{V_i}\gamma + \varphi_i A = 0.$$

The proof is now essentially a computation using (3.96)-(3.99).

Let $\mathcal{Z}$ be the Lie algebra of Killing fields on (M, g) with tangential and normal projections $Z = V + \nu N$ as above. Observe that the induced spaces $\mathcal{V}$ and $\mathcal{N}$ of tangential and normal components are also Lie algebras. This follows from the expansion

$$[Z_1, Z_2] = [V_1, V_2] + (V_1(\nu_2) - V_2(\nu_1))N + (\nu_1 N(\nu_2) - \nu_2 N(\nu_1))N,$$

and the fact that the Killing property implies $N(\nu_i) = 0$.

Define a linear map $F : \mathcal{T} \to \mathcal{V}$ by $F(T) = V$, where $Z = V + \varphi_T N$. Here φ_T is the conformal factor given by (3.96); φ_T determines uniquely a Killing field $Z \in \mathcal{Z}$ since there are no tangential Killing fields. We next claim that the bracket

$$\{T_1, T_2\} = [T_1, F(T_2)] - [T_2, F(T_1)]$$

is a Lie bracket on $\mathcal{T}$.

To see this, from (3.99), one has

$$\tfrac{1}{2}\mathcal{L}_{T_2}\mathcal{L}_{V_1}\gamma + T_2(\varphi_1)A = 0,$$

$$\tfrac{1}{2}\mathcal{L}_{T_1}\mathcal{L}_{V_2}\gamma + T_1(\varphi_2)A = 0,$$

since $\mathcal{L}_{T_i}A = 0$. Since also $T_2(\varphi_1) = T_1(\varphi_2)$ by (3.97), we obtain

$$\mathcal{L}_{T_2}\mathcal{L}_{V_1}\gamma = \mathcal{L}_{T_1}\mathcal{L}_{V_2}\gamma.$$

On the other hand,

$$\mathcal{L}_{V_1}\mathcal{L}_{T_2}\gamma = \mathcal{L}_{V_1}\varphi_2\gamma = V_1(\varphi_2)\gamma - 2\varphi_1\varphi_2 A,$$

$$\mathcal{L}_{V_2}\mathcal{L}_{T_1}\gamma = \mathcal{L}_{V_2}\varphi_1\gamma = V_2(\varphi_1)\gamma - 2\varphi_2\varphi_1 A,$$

so that

$$\mathcal{L}_{V_1}\mathcal{L}_{T_2}\gamma - \mathcal{L}_{V_2}\mathcal{L}_{T_1}\gamma = (V_1(\varphi_2) - V_2(\varphi_1))\gamma.$$

Hence

$$\mathcal{L}_{[T_2, V_1]}\gamma - \mathcal{L}_{[T_1, V_2]}\gamma = \chi\gamma,$$

$\chi = V_1(\varphi_2) - V_2(\varphi_1)$, so that $[T_2, V_1] - [T_1, V_2]$ is a tangential conformal Killing field on (T^2, γ), and so in $\mathcal{T}$. It is easily verified that the Jacobi identity holds.

Thus $\{,\}$ is a Lie bracket on $\mathcal{T}$. Since the only two-dimensional Lie algebra is abelian, it follows that $\{,\} = 0$. In particular,

(3.100) $$[T_2, V_1] = [T_1, V_2].$$

It follows also that $\chi = 0$, i.e. the normal component of $[Z_1, Z_2]$ vanishes. Again since there are no tangential Killing fields, one has $[Z_1, Z_2] = 0$, i.e.

(3.101) $$[V_1, V_2] = 0 \text{ and } V_1(\varphi_2) - V_2(\varphi_1) = 0.$$

Write $V_i = a_i T_1 + b_i T_2 = a_i \partial_{x_1} + b_i \partial_{x_2}$. Expanding (3.100) out gives

(3.102) $$\partial_{x_2}a_1 = \partial_{x_1}a_2, \quad \partial_{x_2}b_1 = \partial_{x_1}b_2.$$

Next one also has from (3.99)

$$\mathcal{L}_{V_2}\mathcal{L}_{V_1}\gamma + V_2(\varphi_1)A + \varphi_1\mathcal{L}_{V_2}A = 0,$$

$$\mathcal{L}_{V_1}\mathcal{L}_{V_2}\gamma + V_1(\varphi_2)A + \varphi_2\mathcal{L}_{V_1}A = 0,$$

and since $[Z_1, Z_2] = 0$, it follows that

$$\mathcal{L}_{V_2}A = \frac{\varphi_2}{\varphi_1}\mathcal{L}_{V_1}A,$$

so that these two forms are proportional. By (3.98)

$$\mathcal{L}_{V_1}A = 2(dx_1 \cdot da_1 - dx_2 \cdot db_1),$$

$$\mathcal{L}_{V_2}A = 2(dx_1 \cdot da_2 - dx_2 \cdot db_2),$$

and hence

(3.103) $$\partial_{x_1}a_2 = \frac{\varphi_2}{\varphi_1}\partial_{x_1}a_1, \quad \partial_{x_2}b_2 = \frac{\varphi_2}{\varphi_1}\partial_{x_2}b_1, \quad \partial_{x_2}a_2 - \partial_{x_1}b_2 = \frac{\varphi_2}{\varphi_1}(\partial_{x_2}a_1 - \partial_{x_1}b_1).$$

On other hand, expanding out (3.99) and using (3.98) gives $\frac{1}{2}(a_i\varphi_i+b_i\varphi_i)\gamma+da_i\cdot dx_1+db_i\cdot dx_2 = -\varphi_i(dx_1^2-dx_2^2)$. Since there are no cross terms in γ or A, it follows that

$$\partial_{x_2}a_i + \partial_{x_1}b_i = 0.$$

Setting here $i = 1$ gives $\partial_{x_2}a_1 + \partial_{x_1}b_1 = 0$ and using (3.102) gives $\partial_{x_1}a_2 + \partial_{x_1}b_1 = 0$. Similarly setting $i = 2$ gives $\partial_{x_2}a_2 + \partial_{x_1}b_2 = 0$ and using (3.102) again gives $\partial_{x_2}a_2 + \partial_{x_2}b_1 = 0$. Combining these, it follows that

(3.104) $$a_2 + b_1 = const.$$

Returning to (3.103) and using (3.104), (3.102) gives then

$$\partial_{x_1}a_2 = \frac{\varphi_2}{\varphi_1}\partial_{x_1}a_1, \ \ \partial_{x_2}b_2 = \frac{\varphi_2}{\varphi_1}\partial_{x_2}b_1, \ \ \partial_{x_2}a_2 = \frac{\varphi_2}{\varphi_1}\partial_{x_2}a_1,$$

so that

(3.105) $$\frac{\partial_{x_1}a_2}{\partial_{x_1}a_1} = \frac{\partial_{x_2}a_2}{\partial_{x_2}a_1} = \frac{\partial_{x_2}b_2}{\partial_{x_2}b_1} = \frac{\partial_{x_1}b_2}{\partial_{x_1}b_1}.$$

Now by (3.102), the 1-form $W^a = a_1dx_1 + a_2dx_2$ is closed, and similarly for W^b, so they are locally exact. All harmonic 1-forms on $(T^2,\widetilde{\gamma})$ are constant (parallel). Thus by subtracting off constants, we may assume that W_a and W_b are globally exact 1-forms, so that $a_i = \partial_i\alpha$, $b_i = \partial_i\beta$.

It follows then from (3.105) that

$$\partial_{x_1x_1}\alpha \cdot \partial_{x_2x_2}\alpha = (\partial_{x_1x_2}\alpha)^2 \geq 0.$$

The function $\alpha : T^2 \to \mathbb{R}$ is smooth and the second partials $\partial_{x_1x_1}\alpha$, $\partial_{x_2x_2}\alpha$ have the same sign everywhere. Hence $\Delta\alpha = \partial_{x_1x_1}\alpha + \partial_{x_2x_2}\alpha \geq 0$ or $\Delta\alpha \leq 0$ everywhere on $(T^2,\widetilde{\gamma})$. By the maximum principle, $\alpha = const$ and, for the same reasons, $\beta = const$ as well.

It follows that the vector fields V_i are harmonic on $(T^2,\widetilde{\gamma})$, hence parallel, and so linear combinations of T_1, T_2. They both thus satisfy (3.96) and (3.99), which implies that $A = -\gamma$, a contradiction.

∎

Proof of Theorem 1.2.

Proposition 3.14 implies that for $(M,g) \in \mathcal{M}$ with toral boundary, either

$$dim\, K = 0 \ \ or \ \ dim\, K = 1.$$

In the first case ($i = 2$), the boundary (T^2, γ) has two linearly independent Killing fields T_i which are restrictions of ambient Killing fields on (M, g). For (M, g) immersed in S^3, $\rho = e$, it follows immediately that T^2 is a Clifford torus, (up to rigid motion). In the second case ($i = 1$), the boundary (T^2, γ) has a non-zero Killing field T which is the restriction of an ambient Killing field on (M, g). Hence for (M, g) immersed in S^3, M and ∂M are invariant under an isometric S^1 action, with $S^1 \subset SO(4)$. By a classical result of Hsiang-Lawson [17], the only such cohomogeneity one minimal torus embedded in S^3 is the Clifford torus.

∎

We conclude this section with the proof of Theorem 1.3, formulated more precisely in the next two results.

Theorem 3.15. *Suppose $genus(\partial M) = 1$. If some $(M,g) \in \mathcal{M}$ has*

(3.106) $$K = KerD\Pi = 0,$$

then (3.106) holds at all points in the connected component $\mathcal{M}_0$ of $\mathcal{M}$ containing (M,g). The map

(3.107) $$\Pi : \mathcal{M}_0 \to \mathcal{T}(\partial M),$$

is a global diffeomorphism, equivariant with respect to the action of the mapping class group $SL(2, \mathbb{Z})$. The holonomy map

$$\chi : \mathcal{M}_0 \to \mathcal{R}(M),$$

is surjective and for generic ρ, $\chi^{-1}(\rho)$ consists of 4 minimal boundaries in a space M_ρ of fixed holonomy.

The boundary ∂M of each element $(M, g) \in \mathcal{M}_0$ is a "Clifford torus", invariant under the action of two Killing fields on (M, g), tangent to ∂M. In particular $(\partial M, \gamma)$ is a flat metric on a torus, and the second fundamental form A is parallel with respect to γ.

Proof: Since $i = 2$, any $(M, g) \in \mathcal{M}$ with $K = 0$ has two linearly independent Killing fields on $(\partial M, \gamma)$, both restrictions of ambient Killing fields on (M, g). Hence $(\partial M, \gamma)$ is flat and A is parallel with respect to γ. This structure is unique up to global isometries of (M, g). In particular, any Jacobi field preserving this structure is a Killing field. Note that generically, there are no non-trivial rigid motions ($D_\rho = 2$ generically).

Recall from Section 2.2 that $\mathcal{R}(T^2) = T^2/W$, where the Weyl group $W = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The holonomy representation ρ is determined by a single element $g = \rho(1)$ in the maximal torus $T^2_{max} \subset SO(4)$, modulo conjugation by an element in the Weyl group. We may assume that g is rotation $(e^{i\theta_1}, e^{i\theta_2})$ in $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ by angles θ_1, θ_2. The standard Clifford torus in S^3 is just these two circles $S^1 \times S^1$, i.e. is naturally identified with the maximal torus of $SO(4)$.

View $\theta_i \in \mathbb{R} = \widetilde{S}^1$ and recall $\widetilde{M} = D^2 \times \mathbb{R}$. Suppose the second S^1 is essential in $\pi_1(M)$ and so lifts to the line $\mathbb{R}$ with coordinate θ_2. The holonomy element $\rho(1) = g_1 = (e^{i\theta_1}, 1)$ describes a twist by θ_1 of the disc D^2 and translation by $2\pi = 2\pi e^0$ along the line $\mathbb{R}$; the space $M = M_\rho$ is the quotient of $\widetilde{M}$ by this action. The holonomy element $\rho(1) = g_2 = (1, e^{i\theta_2})$ gives no twist of D^2 and a translation by $2\pi e^{\theta_2}$ along $\mathbb{R}$, again with M_ρ the corresponding quotient. These are the Fenchel-Nielsen coordinates on the Teichmüller space $(\mathbb{R}^2)^+ = (\theta_1, 2\pi e^{\theta_2})$ of T^2.

Each quotient of $\widetilde{M}$ by $\rho(1) = g \in SO(4)$ gives $M = M_\rho$ with "Clifford torus" boundary, so ∂M is minimal, with flat metric and parallel second fundamental form. This shows that Π in (3.107) is a global diffeomorphism.

Recall that $(M, g) \in \mathcal{M}$ is determined up to isometry only by tangential diffeomorphisms of ∂M isotopic to the identity and not translations. In this case, the translations are isometries (generated by tangential Killing fields) and so act trivially on $\mathcal{M}_0$. However the diffeomorphisms of ∂M not isotopic to the identity (and extended to M) act non-trivially on $\mathcal{M}$. This is the action of the mapping class group $SL(2, \mathbb{Z})$ and it is clear that Π descends to a diffeomorphism of moduli spaces

$$\Pi : \mathcal{M}_0/SL(2, \mathbb{Z}) \to \mathcal{T}(\partial M)/SL(2, \mathbb{Z}).$$

Regarding the map to the representation variety $\mathcal{R}(M)$, in Fenchel-Nielsen coordinates on $\mathcal{M}_0$ one has

$$\widehat{\chi} : \mathcal{M}_0 \to T^2,$$
$$\widehat{\chi}(\theta_1, 2\pi e^{\theta_2}) = (e^{i\theta_1}, e^{i\theta_2}),$$

where T^2 is viewed as the maximal torus of $SO(4)$. Thus

$$\chi = \pi \circ \widehat{\chi},$$

where $\pi : T^2 \to T^2/W = \mathcal{R}(M)$ is the quotient map.

For a generic point in $\mathcal{R}(M)$, the orbit of W has order 4. Thus, for a generic ρ, the space M_ρ has 4 isometrically distinct minimally embedded Clifford tori Σ_j, $1 \leq j \leq 4$; each of these forms a distinct boundary for M_ρ and are obtained from each other by deformation within the fixed holonomy space M_ρ. Thus, the unique (up to rigid motion) Clifford torus in S^3 bifurcates into 2 or 4 Clifford tori in M_ρ, depending on whether ρ is a singular or regular point in $\mathcal{R}(M)$.

$\blacksquare$

Theorem 3.16. *Suppose genus*$(\partial M) = 1$. *If some* $(M, g) \in \mathcal{M}$ *has*

(3.108)
$$dim\, K = dim\, Ker D\Pi = 1,$$

then (3.108) holds at all points in the connected component $\mathcal{M}_1$ *of* $\mathcal{M}$ *containing* (M, g). *The space* $\mathcal{M}_1$ *is a smooth 2-dimensional manifold and has a foliation* $\mathcal{F}$ *by curves* σ *with tangent vectors in* K. *The boundary map*

$$\Pi : \mathcal{M}_1 \to \mathcal{T}(\partial M)$$

is of index 0, with image in $\mathcal{T}(\partial M)$ *a curve* $\widetilde{\zeta} = \Pi(\zeta)$, *where* ζ *is a curve in* $\mathcal{M}_1$ *transverse to the foliation* $\mathcal{F}$.

The boundary ∂M *of each element* $(M, g) \in \mathcal{M}$ *is a surface of revolution, i.e. invariant under an isometric* S^1 *action* $S^1 \subset Isom(M, g)$.

Proof: Since $i = 1$ in this case, there is one tangential Killing field on ∂M extending to a Killing field on (M, g) and one conformal Killing (but not Killing) field T on ∂M. Hence (M, g) is a surface of revolution. The form $\delta^* T$ generates the kernel $K = K_1 = \mathbf{K}$. By Corollary 3.13, the conformal factor φ for T generates a Killing Jacobi field φN, so that $Z = Z^T + \varphi N$ is Killing, as in (3.99) and (3.96).

Observe that by (3.93) the element $(0, \varphi)$ lies in the cokernel of $D\Pi$ (its orthogonal to $ImD\Pi$). Since $D\Pi$ has index 0, $(0, \varphi) = Coker(D\Pi)$. It follows that $\pi \circ D\Pi : \mathcal{E} \to T\mathcal{C}$ is surjective, where $\pi : T\mathcal{C} \times C \to T\mathcal{C}$ is projection on the first factor. Hence by the implicit function theorem for Banach manifolds, the component $\widetilde{\mathcal{M}_1} = \Pi^{-1}(\mathcal{C} \times \{0\})$ containing (M, g) is locally a smooth Banach manifold and $\Pi : \widetilde{\mathcal{M}_1} \to \mathcal{C}$ is Fredholm, of index 0. Dividing out by the free action of the diffeomorphism group $\mathrm{Diff}_0(M)$ shows that $\mathcal{M}_1$ is a smooth 2-dimensional manifold with $\Pi : \mathcal{M}_1 \to \mathcal{T}(\partial M)$ of index 0.

It is clear from Proposition 3.14 and Theorem 3.15 that (3.108) then holds on the full component $\mathcal{M}_1$ of $\mathcal{M}$.

The conformal translation field T generates a flow of diffeomorphisms, which in turn generates a non-trivial curve σ through each point in $\mathcal{M}_1$. Clearly Π maps each σ to a point in $\mathcal{T}(\partial M)$. This completes the proof.

$\blacksquare$

4. Further Results

In this section, we discuss generalizations of the results above to constant mean curvature (CMC) surfaces, and surfaces in flat and hyperbolic space forms.

We begin with the proof of Theorem 1.4.

Proof of Theorem 1.4.

The general framework presented in the Introduction holds without change for 3-manifolds of constant positive curvature (M, g) with constant mean curvature (CMC) boundary ∂M, so $H = H_0$, for some constant $H_0 \in \mathbb{R}$.

All of the results and discussion of Section 2 also hold for $H = H_0 \in \mathbb{R}$ with the only modification that the Frankel-Lawson result (2.27) holds provided

$$H = H_0 \geq 0;$$

this is assumed henceforth.

Most all of the results and discussion of Section 3 also hold, without change, in the CMC case. We list below the only modifications that need to be made to pass from $H = 0$ to $H = H_0 \in \mathbb{R}^+$.

1. Proposition 3.1: the main formula (3.4) is valid, when τ'_κ is replaced by $\tau'_\kappa + \frac{1}{2}\varphi_\kappa H_0 \gamma = A'_\kappa - \frac{tr A'_\kappa}{2}\gamma$, cf. (3.7). This change should be made throughout Section 3. Note that this term is trace-free and that $\tau'_\kappa = A'_\kappa$ when $\varphi_\kappa = 0$ on ∂M.

2. Proposition 3.8: This deals with case $\varphi_\kappa = 0$ in which case $\tau'_\kappa + \frac{1}{2}\varphi_\kappa H_0 \gamma = \tau'_\kappa = A'_\kappa$, so the proof is exactly the same when $H = H_0$.

3. Proposition 3.10: The proof is exactly the same, modulo the following minor modifications. First choose f such that $2N(f) - H_0 f = 0$ so that (3.61) holds automatically and (3.32) becomes

$$\int_M \langle q, k \rangle = \int_{\partial M} f[2H'_k + \langle A, k \rangle - \tfrac{1}{2}\varphi_k H_0 tr_\gamma \sigma].$$

Then (3.62) is replaced by $\langle A, k \rangle - \frac{1}{2}\varphi_k H_0 tr_\gamma \sigma \neq 0$ with similar modification to (3.63) while (3.64) becomes

$$\int_M \langle q, \widetilde{k} \rangle = \int_{\partial M} f[2H'_{\widetilde{k}} + \langle A, \widetilde{k} \rangle - \varphi_{\widetilde{k}} H_0] = \int_{dm} 2f H'_{\widetilde{k}},$$

since $\widetilde{k} = \varphi\gamma$.

For $\widetilde{k} \in K_{\gamma,\sigma}$ one has $\langle \tau'_{\widetilde{\kappa}}, \sigma \rangle - \langle \tau, \widetilde{\kappa} \rangle = 0$. As following (3.64), if $H'_{\widetilde{k}} = 0$, then $\widetilde{k} \in K_\gamma$. As before, since $\mathbf{K} = 0$, there is an open set of σ near γ in the space orthogonal to $\langle \gamma \rangle$ such that $\langle (\tau'_{\widetilde{k}})_0, \sigma \rangle \neq 0$, for all $\widetilde{k} \in K_\gamma$; here $(\tau'_{\widetilde{k}})_0$ is the trace-free part of $\tau'_{\widetilde{k}}$. Write then $\sigma = \alpha\gamma + \sigma^\perp$ where $\alpha \in \mathbb{R}$ and $tr_\gamma \sigma^\perp = 0$. By (3.21), we must impose $\alpha = 1$. However, $\langle \tau'_{\widetilde{\kappa}}, \gamma \rangle - \langle \tau, \widetilde{\kappa} \rangle = H'_{\widetilde{k}} = 0$, and so $\sigma^\perp$ may be arbitrarily prescribed. Hence, again as before, there is an open set of σ in the space orthogonal to $\langle \gamma \rangle$ such that for $\widetilde{k} \in K_{\gamma,\sigma}$, $H'_{\widetilde{k}} \neq 0$. The proof then proceeds as in the proof of Proposition 3.10.

With the modifications above, the proof of Theorem 3.3 applies verbatim to CMC boundaries, and it follows that Theorem 3.3 holds for such configurations. Moreover, besides the use of 1 above, there are no further changes needed to the proof of Theorem 1.1. This proves Theorem 1.1 for CMC boundaries.

In addition, Theorems 1.2 and 1.3 also hold for CMC boundaries. The only change to the proofs is that in the proof of Proposition 3.14, A should be replaced by its trace-free part A_0, which is a holomorphic quadratic differential as in (2.23). The proof runs exactly the same with this modification. ∎

To conclude, we consider other constant curvature metrics. Suppose then the compact 3-manifold (M, g) is of constant curvature Λ with $\Lambda = 0, -1$, so flat or hyperbolic. Again, all of the methods and results in Sections 2 and 3 hold in these cases also, with two modifications.

First, as above, (2.27) does not hold in general anymore. Thus, in the following, we simply assume that M is a handlebody. Second, via the developing map, the universal cover $\widetilde{M}$ immerses isometrically into $\mathbb{R}^3$ or $\mathbb{H}^3$ respectively. In analogy to $Isom(S^3) = SO(4)$, one has here

$$Isom^+(\mathbb{R}^3) \simeq \mathbb{R}^3 \times_\alpha SO(3), \quad Isom^+(\mathbb{H}^3) = PSL(2, \mathbb{C}).$$

Note that both groups are 6-dimensional, as is $Isom(S^3)$. In contrast to the positive case, M itself never immerses in $\mathbb{R}^3$ or $\mathbb{H}^3$, since there are no compact minimal surfaces in $\mathbb{R}^3$ or $\mathbb{H}^3$. Instead, the handlebody M may immerse in a flat space-form $\mathbb{R}^3/\Gamma$ or hyperbolic space-form $\mathbb{H}^3/\Gamma$. In particular, the holonomy representation

$$\rho : \pi_1(M) \to PSL(2, \mathbb{C}) \text{ or } \mathbb{R}^3 \times_\alpha SO(3),$$

is always non-trivial.

By the Gauss equation (2.4) there are no minimal boundaries of genus 0 or 1 when $\Lambda = -1$. When $\Lambda = 0$, there are no minimal boundaries of genus 0 and the only minimal boundaries of genus 1 are totally geodesic and flat. Thus, in the following we assume $genus(\partial M) \geq 2$ and consider only

Theorem 1.1, i.e. the moduli space of flat or hyperbolic metrics on a handlebody M with minimal (or CMC) boundary ∂M.

An especially interesting case is a Heegaard decomposition of a compact hyperbolic 3-manifold $N = M_1 \cup M_2$, with common boundary $\partial M_i = \Sigma$ a minimal surface embedded in N; such boundaries exist in quite general circumstances, cf. [25]. The hyperbolic structure on N is rigid, but one may consider hyperbolic deformations of each of the Heegaard components M_i. Note that the two components M_1 and M_2 in N are glued together by an element of the mapping class group $\Gamma(\Sigma)$ of Σ.

It is easily verified that the proof of Theorem 1.1 carries over without further changes to flat and hyperbolic handlebodies with minimal or CMC boundary. We state the result in the hyperbolic case; the same result holds in the flat case.

Theorem 4.1. *The space $\mathcal{M}$ of hyperbolic metrics on a handlebody M with boundary $\partial M = \Sigma$ a minimal (or CMC) surface with genus$(\Sigma) \geq 2$, is a smooth manifold, and the boundary map*

$$\text{(4.1)} \qquad\qquad \Pi : \mathcal{M} \to \mathcal{T}(\partial M),$$

is everywhere a local diffeomorphism. The map Π is equivariant with respect to the action of the mapping class group on each factor.

Again the full boundary map Π in (1.2) is a local diffeomorphism at any $(M, g) \in \mathcal{M}$.

It is worth considering this result in the context of work of Uhlenbeck [28] and Taubes [26], who show that moduli spaces of minimal surfaces Σ in local hyperbolic 3-manifolds of the form $N = \Sigma \times I$ are parametrized by the cotangent bundle of Teichmüller space $\mathcal{T}(\Sigma)$. Here the central fiber $\Sigma \times \{0\}$ is minimal, but does not bound in N. In addition, in most situations the hyperbolic metric on N is incomplete.

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