HOMEWORK 2, MAT 568, FALL 2010

Due: Thursday, Oct 28.

1. Let (M, g) be a Riemannian manifold and $f : M \to M$ a diffeomorphism. Let ∇^{f^*g} be the Levi-Civita connection for the metric f^*g and ∇^g the Levi-Civita connection for the metric g. Prove that

$$\nabla^{f^*g} = f^*\nabla^g,$$

i.e.

$$\nabla^{f^*g}_X Y = \nabla^g_{f_*X} f_*Y.$$

From this, deduce that the $(3, 1)$ Riemann curvature tensor transforms naturally under pullback:

$$R^{f^*g} = f^*R.$$

2. Let $\widetilde{g} = \lambda^2 g$, where λ is a positive constant, so that $\widetilde{g}$ is a rescaling of g. Show that

$$\widetilde{\nabla} = \nabla$$

and hence, if R denotes the $(3, 1)$ Riemann curvature tensor, then

$$\widetilde{R}(X, Y)Z = R(X, Y)Z.$$

On the other hand, if R is now the $(4, 0)$ curvature tensor, then show

$$\widetilde{R} = \lambda^{-2}R,$$

i.e.

$$\widetilde{g}(\widetilde{R}(X, Y)Z, W) = \lambda^{-2}g(R(X, Y)Z, W).$$

Deduce that the sectional, Ricci curvature on unit vectors, and scalar curvature transform under rescaling as λ^{-2}.

3. Now let $\widetilde{g} = u^2 g$ be a conformal change of the metric g. Prove that

$$\widetilde{\nabla}_X Y = \nabla_X Y + X(\log u)Y + Y(\log u)X - g(X, Y)\nabla \log u.$$

Extra Credit: Deduce the formula for the transformation of the curvature under conformal changes.

4. Let G be a Lie group with a bi-invariant Riemannian metric, i.e. the metric is invariant under right and left translations of the group. It follows that inner automorphisms $i_h(g) = h^{-1}gh$ are isometries of the metric and hence (verify this yourself) the adjoint action of the Lie algebra $\mathcal{L}(G)$ is skew-symmetric, i.e.

$$ad_U : \mathcal{L}(G) \to \mathcal{L}(G), \quad ad_U(X) = [X, U]$$

satisfies

$$\langle [X,U],Y\rangle = -\langle X,[Y,U]\rangle.$$

Use this to prove that (for left-invariant vector fields)

(a). $\nabla_X Y = \frac{1}{2}[X,Y]$.

(b). $R(X,Y)Z = \frac{1}{4}[Z,[X,Y]]$.

(c). $\langle R(X,Y)Z,W\rangle = -\frac{1}{4}\langle [X,Y],[Z,W]\rangle$.

Conclude that all the sectional curvatures are non-negative. Show that $Ric(X) = 0$ if and only if X commutes with all other left-invariant vector fields, i.e. X is in the center of $\mathcal{L}(G)$.

5. Let M^{n-1} be a hypersurface in $\mathbb{R}^n$ with the induced metric, and suppose a local patch of M is given as the graph of a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$. Suppose that $f(0) = 0$, so that the origin $0 \in M$ and $Df(0) = 0$, so that the tangent space to M at 0 is $\mathbb{R}^{n-1}$.

Show that the 2nd fundamental form of M at 0 is proportional to the Hessian of f:

$$A = \frac{1}{|\nabla f|} D^2 f.$$

6. Consider the hypersurface M in $\mathbb{R}^{n+1}$ given by

$$x^{n+1} = (x^n)^2,$$

with the induced Riemannian metric.

Prove that M is isometric to $\mathbb{R}^n$ with the flat metric, i.e. it is flat.

On the other hand, prove that the 2nd fundamental form A of M does not vanish anywhere.