MAT 531 SPRING 16 HOMEWORK 6

Due Tuesday, March 8

1. Let T be a $(2,0)$ tensor on $\mathbb{R}^2$ given in global Cartesian coordinates by

$$T = \sum t_{ij} dx^i \otimes dx^j.$$

Find the expression for T in polar coordinates.

2.(a) Let again T be a bilinear form (i.e. $(2,0)$ tensor) on a manifold M. Let

$$detT = det(t_{ij})$$

where t_{ij} are the components of T in any local coordinate system x^i on M. Is $detT$ well-defined? (i.e. does it exist as a function on M?).

(b). Is the property $detT \neq 0$ well-defined on M?

3. Find the integral curves in $\mathbb{R}^2$ of the vector field $V = x^2 \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ and determine which integral curves are defined for all t.

4. The usual homogeneous charts on $\mathbb{RP}^2$ are given by

$$[x,y,z] \to (u_1,u_2) = (\frac{x}{z}, \frac{y}{z}) \quad on \quad U_3 = \{z \neq 0\},$$

$$[x,y,z] \to (v_1,v_2) = (\frac{x}{y}, \frac{z}{y}) \quad on \quad U_2 = \{y \neq 0\},$$

$$[x,y,z] \to (w_1,w_2) = (\frac{y}{x}, \frac{z}{x}) \quad on \quad U_1 = \{x \neq 0\}.$$

Show that there is a vector field on $\mathbb{RP}^2$ which in the U_1 chart has the form

$$V = w_1 \frac{\partial}{\partial w_1} - w_2 \frac{\partial}{\partial w_2}.$$

Find the expressions for V in the other two charts.

5. (a) Let

$$f : \mathbb{R}^n \to \mathbb{R}$$

be a smooth function with a finite number of isolated critical points q_i (where $\nabla f \neq 0$). Let $M_a = \{x \in \mathbb{R}^n : f(x) = a$ (the a-level set of f and similarly for $M_{(a,b)}$. Let

$$V = \frac{\nabla f}{|\nabla f|^2}.$$

Then V is a smooth vector field on $\mathbb{R}^n \setminus \{\cup q_i\}$. If φ_t denotes the flow of V, show that

$$f(\varphi_t(p)) = f(p) + t.$$

(b) Let $[a,b] \subset \mathbb{R}$ be an interval containing no critical values of f. Show that

$$\Phi : M_{(a,b)} \to (0, b-a) \times M_a,$$

$$\Phi(p) = (f(p) - a, \phi_{a-f(p)}(p)),$$

is a diffeomorphism. (It is actually a diffeomorphism of manifolds-with-boundary when closed intervals are used above in place of open intervals). Thus $M_{(a,b)}$ is topologically "simple" - just a product.