MAT 530 SPRING 16 HOMEWORK 1

Due Wednesday, Sept 7

Problems in Munkres Text:
Section 13: 4, 8(a)
Section 16: 8

1. Two metrics $d_1$ and $d_2$ on a set $X$ are called equivalent if there are constants $c, C > 0$ such that

$$cd_1(x, y) \leq d_2(x, y) \leq Cd_1(x, y),$$

for all $x, y \in X$. Prove that equivalent metrics induce the same topology on $X$.

2. Let $C^0[0, 1]$ be the space of continuous real-valued functions on $[0, 1]$. For $f, g \in C$, define

$$d_1(f, g) = \int_0^1 |f(x) - g(x)| dx, \quad d_{sup}(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|.$$

Show that $d_1$ and $d_{sup}$ are metrics on $C$.

Prove that the topologies on $C$ induced by these metrics are different. Is one of them finer than the other?

3. An arithmetic progression is a set of form

$$S(a, b) = \{an + b : n \in \mathbb{Z}\},$$

with $a, b \in \mathbb{Z}$. Define a subset $U \subset \mathbb{Z}$ to be open if it is either empty, or a union of arithmetic progressions.

(a). Show that this defines a topology on $\mathbb{Z}$ in which every non-empty open set is infinite.

(b) Prove the identity

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p \text{ prime}} S(p, 0).$$

Prove that each $S(p, 0)$ is a closed set while $\mathbb{Z} \setminus \{-1, 1\}$ is not closed. Show this leads to a contradiction if there are only finitely many prime numbers. (Hillel Furstenberg proof of Euclid’s theorem).