\[ f(x) = |x| \quad -\pi < x < \pi \]

The coefficients are

\[ a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| \, dx = \frac{\pi}{2} \]

\[ a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \]

\[ = \frac{2}{\pi} \int_{0}^{\pi} x \cos(nx) \, dx \]

(By Integration By Parts) \[ = \frac{2}{\pi} \left[ \left. \frac{x \sin(nx)}{n} \right|_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin(nx)}{n} \, dx \right] \]

\[ = \frac{2}{\pi} \left[ (\pi/n) - \left. \frac{\cos(nx)}{n^2} \right|_{0}^{\pi} \right] \]

\[ = \frac{2}{\pi n} \left[ (-1)^n - 1 \right] \]

\[ \therefore a_n = 0 \quad \text{if } n \text{ even} \]

\[ = -\frac{4}{\pi n^2} \quad \text{if } n \text{ odd} \]
\[ b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) \, dx = 0 \]

as it's an integral of an odd fn.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos(nx) \]

(7) (a) \quad f(x) = \cos^2x

\[ = \frac{1}{2} + \frac{\cos(2x)}{2} \quad \text{The series} \]

(b) \quad f(x) = \sin(x - \pi/6)

\[ = \sin x \cos\left(\frac{\pi}{6}\right) - \cos x \sin\left(\frac{\pi}{6}\right) \]

\[ = \frac{\sqrt{3}}{2} \sin x - \frac{1}{2} \cos x \]

(c) \quad f(x) = \sin x \cos 2x

\[ = \frac{1}{2} \left[ \sin(x+2x) + \sin(x-2x) \right] \]

\[ = \frac{1}{2} \sin(3x) - \frac{1}{2} \sin x \]
$f(x) = |x| \quad -1 < x < 1$

\[ a_0 = \frac{1}{2a} \int_{-a}^{a} |x| \, dx = \frac{1}{2} \quad a = 1 \]

\[ b_n = \frac{1}{a} \int_{-a}^{a} |x| \sin \left( \frac{\pi x}{a} \right) \, dx = 0 \quad a = 1 \]

\[ a_n = \frac{1}{a} \int_{-a}^{a} |x| \cos \left( \frac{\pi x}{a} \right) \, dx \]

\[ = \int_{-1}^{1} 1 \cos (\pi x) \]

\[ = \frac{2}{\pi^2} \left[ (-1)^n - 1 \right] \text{ by earlier method} \]

\[ \therefore a_n = 0 \quad \text{if } n \text{ is even} \]

\[ = -\frac{2}{\pi^2} \]

\[ 1 \times 1 \sim \frac{1}{2} + \sum_{n=1}^{\infty} \left( -\frac{2}{\pi^2 n^2} \right) \cos(\pi n x) \text{ modd} \]
By Defn.

\[ \cosh(x) = \frac{1}{2} (e^x + e^{-x}) \quad (= \text{even}) \]

\[ \sinh(x) = \frac{1}{2} (e^x - e^{-x}) \quad (= \text{odd}) \]

So \( e^x = \cosh(x) + \sinh(x) \)

gives the desired decomposition

(8)

Say we have a periodic continuous function. Then yes. (If we assume the fn. has a convergent Fourier series)

In the case all the sine coefficients of a function vanish

\[ f(x) \equiv a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{\pi x}{a} \right) \quad -a < x < a \]

Then \( f(x) = f(-x) \)

But it may not happen that a function has a everywhere convergent Fourier series. In that case this is not true. For example—let us take any const fn. with finitely many discontinuities.
\( f(x) = \sin(x) \quad 0 < x < \pi \)
(11d) \[ f(x) = \sin(x) \quad 0 < x < \pi \]

The cosine series is

\[ f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) \quad 0 < x < \pi \]

\[ a_0 = \frac{1}{\pi} \int_0^\pi \sin x \, dx = \frac{2}{\pi} \]

\[ a_n = \frac{2}{\pi} \int_0^\pi \sin x \cos(nx) \, dx \]

\[ = \frac{2}{\pi} \int_0^\pi \frac{1}{2} \left[ \sin((n+1)x) - \sin((n-1)x) \right] \, dx \]

\[ = \frac{2}{\pi} \left[ \frac{\cos n \pi \cos n - 1}{n^2 - 1} \right] \]

\[ a_n = \begin{cases} \frac{4}{n(n-1)} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases} \]

\[ f(x) \sim \frac{2}{\pi} \left[ 1 - \sum_{n=1}^{\infty} \frac{2}{n\pi} \cos nx \right] \text{ is the even approx.} \]
Odd extension is just \( f(x) = \sin x \)

\[
f(x) = \begin{cases} 
0 & 1 < x < 3 \\
-1 & -1 < x < 1 \\
x & -3 < x < -1
\end{cases}
\]

The value taken by the series

\[
\alpha + x = 3, -3 \quad \frac{1}{2} \left( f(3^-) + f(3^+) \right) = \frac{-3}{2}
\]

\[
\alpha + x = 1 \quad \frac{1}{2} \left( f(1^-) + f(1^+) \right) = \frac{1}{2}
\]

\[
\alpha + x = -1 \quad \frac{1}{2} \left( f(-1^-) + f(-1^+) \right) = 0
\]
Suppose it coincides with \( p(x) \)

We know that \( \sin(n\pi) = 0 \)

So \( \sum \frac{1}{n^3} \sin(n\pi) = 0 \) at \( x = 0, \pi, 2\pi \)

\[ p \text{ has zeros } 0, \pi, 2\pi \]

\[ p(x) = a \times (x-\pi) (x-2\pi) \]

We also have

\[ \frac{1}{\pi} \int_{0}^{2\pi} p(x) \sin(n\pi) \, dx = \frac{1}{n^3} \]

\[ \text{L.H.S.} = 12k/\pi^3 \quad \text{(after calculating the integral)} \]

\[ \therefore k = \frac{1}{12} \]

\[ \therefore p(x) = \frac{1}{12} \times (x-\pi) (x-2\pi) \]