Math 310: Midterm 1

October 10, 2006

Name: **ID number:**

There are 5 questions worth a total of 100 points, plus one small bonus question worth 10 points. Please justify all your statements, and write neatly so that we can read and follow your answers. Continue your answers on the back of the pages. Also, please turn off cell phones.

Question 1. (20 points) (i) Define a subspace of a vector space.

(ii) Suppose that V is a finite dimensional vector space and that $W \subseteq V$ is a subspace such that $\dim W = \dim V$. Prove carefully that $W = V$.

1	20pt	
2	20pt	
3	20pt	
4	30pt	
5	10pt	
Total	100pt	
bonus	10pt	

Question 2. (20 points) (i) Let $(v_1, \ldots, v_n)$ be a list of vectors in V. What does it mean to say that this list is linearly independent? Give the formal definition.

(ii) Give an example of a list of three vectors in $\mathbb{R}^4$ that is linearly independent and another that is linearly dependent. (You only need to give brief explanations.)

(iii) Suppose that the list v_1, v_2, v_3 is linearly independent. Show that the list $v_1 + 2v_2 + v_3, v_2 + v_3, v_3$ is also linearly independent.

Question 3. (20 points) (i) Find a basis for the subspace

$$U := \{(y_1, y_2, y_3, y_4) \in \mathbb{F}^4 : y_1 = y_2 = y_2 + y_3 + y_4\},$$

and prove that the elements you give do form a basis.

(ii) Suppose that W is another subspace of $\mathbb{F}^4$ such that $U + W = \mathbb{F}^4$. What can you say about $\dim W$?

Question 4. (30 points) (i) Show that a linear map $T : V \to W$ is injective if and only if $\operatorname{Null} T = \{0\}$.

(ii) Let $V := \mathcal{P}(3)$ the polynomials of degree ≤ 3 and coefficients in $\mathbb{F}$. Define $T : V \to V$ by $T(f) = (z^2 + z)f''$, where f'' denotes the second derivative of f. Describe $\operatorname{Null} T$ and $\operatorname{Range} T$. What are their dimensions?

(iii) Find the matrix $\mathcal{M}(T)$ that represents T with respect to the standard basis $f_0 := 1,\ f_i = z^i, i = 1, 2, 3.$

Question 5. (10 points, plus 10 points bonus) Let $L : V \to W$ be a linear map.

(i) Suppose that $w_1, \ldots, w_n$ is a linearly independent list in V and that L is injective. Show that the list $(Lw_1, \ldots, Lw_n)$ is linearly independent.

(ii) **Bonus:** Is it possible for the list $w_1, \ldots, w_n$ to be linearly dependent while $(Lw_1, \ldots, Lw_n)$ is linearly independent?