There are 5 problems worth 50 points total and a bonus problem worth up to 10 points.
Show all work. Always indicate carefully what you are doing in each step; otherwise it may not be possible to give you appropriate partial credit.

1. [6 points] Let \( W_1 \) and \( W_2 \) be linear subspaces of a vector space \( V \) such that \( W_1 + W_2 = V \) and \( W_1 \cap W_2 = \{0\} \). Prove that for each vector \( \alpha \in V \) there are unique vectors \( \alpha_1 \in W_1 \) and \( \alpha_2 \in W_2 \) such that \( \alpha = \alpha_1 + \alpha_2 \).
2. [12 points] Consider the vectors in $\mathbb{R}^4$ defined by

$$\alpha_1 = (-1, 0, 1, 2), \quad \alpha_2 = (3, 4, -2, 5), \quad \alpha_3 = (1, 4, 0, 9).$$

(a) [8 points] What is the dimension of the subspace $W$ of $\mathbb{R}^4$ spanned by the three given vectors? Find a basis for $W$ and extend it to a basis $B$ of $\mathbb{R}^4$.

(b) [4 points] Use a basis $B$ of $\mathbb{R}^4$ as in (a) to characterize all linear transformations $T : \mathbb{R}^4 \to \mathbb{R}^4$ that have the same null space $W$. What can you say about the rank of such a $T$? What is therefore the precise condition on the values of $T$ on $B$?
3. [10 points] Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the ordered basis for $\mathbb{R}^3$ consisting of 

$$
\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0).
$$

What are the coordinates of the vector $(a, b, c)$ in the ordered basis $\mathcal{B}$?

4. [10 points] Let $V$ be the vector space over $\mathbb{R}$ of all real polynomial functions $p$ of degree at most 2. For any fixed $a \in \mathbb{R}$ consider the shift operator $T : V \to V$ with $(T p)(x) = p(x + a)$.

Explain why $T$ is linear and find the range and null space of $T$. Is $T$ an isomorphism? Write down the matrix of $T$ with respect to the ordered basis $\mathcal{B} = \{1, x, x^2\}$. 


5. [12 points] Let $T$ be the linear operator on $\mathbb{R}^2$ defined by $T(x_1, x_2) = (-x_2, x_1)$.

(a) [3 points] What is the matrix of $T$ in the standard ordered basis for $\mathbb{R}^2$?

(b) [3 points] Interpret the operation of $T$ geometrically.

(c) [3 points] What is the matrix of $T$ in the ordered basis $B = \{\alpha_1, \alpha_2\}$, where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, -1)$?

(d) [3 points] Prove that for every real number $c$ the operator $(T - cI)$ is invertible.
**Bonus Problem** [up to 10 points] Let $T, U \in L(V, V)$ be linear operators on the finite dimensional vector space $V$. Prove that the rank of the composition $UT$ is less than or equal to the minimum of the ranks of $T$ and $U$. 