MAT 310 FALL 09 HOMEWORK 9

Due Wednesday, November 18

1. Let $P_2[0,1]$ be the vector space of real-valued polynomials on $[0,1]$ with L^2 inner product

$$\langle p, q \rangle = \int_0^1 p(t)q(t)dt.$$

Define the linear map $A : P_2[0,1] \to P_2[0,1]$ by

$$A(a_0 + a_1 x + a_2 x^2) = a_0 - a_2 x^2.$$

(a). Show that A is not self-adjoint.

(b) Show that the matrix of A with respect to the "standard" basis $\{1, x, x^2\}$ is

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This matrix is symmetric, (so equals its conjugate transpose). Why is this not a contradiction to (a).

2. Prove there does not exist a self-adjoint operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$T(1,1,0,1) = 2(1,1,0,1) \text{ and } T(2,3,-1,2) = -3(2,3,-1,2).$$

3. Find a basis of eigenvectors for the linear map of $\mathbb{R}^2$ into itself given by

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix},$$

in the standard basis of $\mathbb{R}^2$.

What is the matrix of A with respect to the eigenvector basis?

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a self-adjoint and positive definite linear map, with respect to the standard inner product, i.e. the dot product.

Prove that the expression

$$G(v, w) = T(v) \cdot w,$$

is an inner product on $\mathbb{R}^n$.

5. Prove the following statement is false, by exhibiting a counterexample. A linear map $T : V \to V$ which has an orthonormal basis $\{e_1, \cdots, e_n\}$ of V satisfying $||T(e_j)|| = ||e_j|| = 1$, for each j, is an isometry of V.

6. let $V = C^0([0,1])$ be the inner product space of complex valued continuous functions on $[0,1]$ with L^2 inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(\bar{t})dt.$$

Let $h \in V$ and define $T : V \to V$ by $T(f) = h \cdot f$. Show that T is an isometry if and only if $|h(t)| = 1$ for all $t \in [0,1]$.

7. Find a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, $A \neq Id$, such that $A^2 = Id$. Can you find one such that $A^3 = id$, or $A^n = Id$, for any n?