Math 303: Midterm 2 Answers

November 14, 2007

1. The functions $y_1 = e^{2x}$ and $y_2 = e^{3x}$ are solutions to the differential equation

$$y'' - 5y' + 6y = 0.$$

a. (10 points) Verify that y_1 and y_2 are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

b. (10 points) Find a solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 4$.

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{3x}$$

$$\begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = 2c_1 + 3c_2 = 4 \end{cases}$$

$$c_1 = -1, c_2 = 2, \quad y = -e^{2x} + 2e^{3x}$$

2. (15 points) Find a specific solution to the equation

$$y'' - 2y' + 5y = \cos x$$

First, set up the general form of a particular solution

$$\begin{aligned} y_p &= A\cos x + B\sin x \quad \text{Notice no overlap with comp. solution} \\ y_p' &= -A\sin x + B\cos x \\ y_p'' &= -A\cos x - B\sin x \end{aligned}$$

Plugging in, we see

$$y'' - 2y' + 5y = \cos x(-A - 2B + 5A) + \sin x(-B + 2A + 5B) = \cos x$$

$$\begin{cases} 4A - 2B = 1 \\ 2A + 4B = 0 \end{cases} \implies \begin{cases} A = \frac{1}{5} \\ B = -\frac{1}{10} \end{cases}$$

$$y_p = \frac{1}{5}\cos x - \frac{1}{10}\sin x$$

Since the solution the complimentary homogeneous equation $y'' - 2y' + 5y = 0$ is of the form

$$y_c = e^x(c_1 \cos 2x + c_2 \sin 2x),$$

then any equation of the form

$$y = \frac{1}{5}\cos x - \frac{1}{10}\sin x + e^x(c_1 \cos 2x + c_2 \sin 2x)$$

for some values c_1, c_2 is also a specific solution.
3. Find general solutions to the following equations.

a. (10 points) $y^{(3)} - 6y'' + 9y' = 0$

$$\begin{aligned} r^3 - 6r^2 + 9r = r(r-3)^2 &= 0 \\ r &= 0, 3, 3 \\ y &= c_1 + (c_2 + c_3 x)e^{3x} \end{aligned}$$

b. (15 points) $y^{(4)} - y'' = 3x$.
Note that the complimentary solution y_c is not a solution to the above equation, but instead a solution to $y^{(4)} - y = 0$. A particular solution y_p must be added to it to have a solution of the above equation.

$$\begin{aligned} y &= y_c + y_p \\ y_c :& r^4 - r^2 = r^2(r+1)(r-1) = 0 \\ r &= 0, 0, -1, 1 \\ y_c &= c_1 + c_2 x + c_3 e^{-x} + c_4 e^x \end{aligned}$$

$$\begin{aligned} y_i &= Ax + B \quad \text{Initial guess} \\ y_p &= x^2(Ax + B) = Ax^3 + Bx^2 \quad \text{Remove overlap with } y_c \\ y_p^{(4)} - y'' &= -6Ax - B = 3x \\ A &= -\frac{1}{2}, B = 0 \\ y_p &= -\frac{1}{2}x^3 \end{aligned}$$

$$y = -\frac{1}{2}x^3 + c_1 + c_2 x + c_3 e^{-x} + c_4 e^x$$

4. A mass of $m = 1$ kg is attached to a large spring with spring constant $k = 9$ (N/m). It is set in motion with initial position $x_0 = 2$ (m) from the equilibrium position and initial velocity $v_0 = -6$ (m/s).

a. (15 points) If the mass-spring system is undamped (no resistance) and free (no external force), then what is the amplitude and period of the position function?

$$mx'' + kx = 0$$
$$x'' + 9x = 0$$

$$r^2 = 9 = 0$$
$$r = \pm 3i$$

$$x(t) = A\cos 3t + B\sin 3t$$
$$x'(t) = -3A\sin 3t + 3B\cos 3t$$
$$\begin{cases} A = x_0 = 2 \\ 3B = v_0 = -6 \end{cases}$$
$$x(t) = 2\cos 3t - 2\sin 3t$$

For $x(t)$ of the form above, the amplitude will be $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$.
The period is $\frac{2\pi}{3}$.
This can also be seen by transforming the solution to

$$x(t) = 2\sqrt{2}\cos(3t + \frac{\pi}{4}.)$$

b. (10 points) Suppose a dashpot with damping constant $c = 2$ (Ns/m) is connected to the spring (still with no external force). What is the pseudo-period of the position function?

$$x'' + 2x' + 9x = 0$$

$$r^2 + 2r + 9 = 0$$
$$r = -1 \pm i2\sqrt{2}$$

$$x(t) = e^{-t}(c_1\cos 2\sqrt{2}t + c_2\sin 2\sqrt{2}t)$$

The pseudo-period of the above solution is then $\frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$.

c. (10 points) Suppose that the mass-spring-dashpot system from part (b) is acted on by an external force of $F_0 \cos(\omega t)$ (Newtons) for some ω. Then, the steady-periodic solution will be of the form

$$x_{sp}(t) = C(\omega)\cos\left(\omega t - \alpha(\omega)\right),$$

where

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \qquad \alpha(\omega) = \begin{cases} \tan^{-1}\frac{c\omega}{k-m\omega^2} & k > m\omega^2 \\ \pi/2 & k = m\omega^2 \\ \pi + \tan^{-1}\frac{c\omega}{k-m\omega^2} & k < m\omega^2 \end{cases}$$

For what value(s) of $\omega > 0$, if any, will the system exhibit practical resonance?

To find practical resonance, we find what value of ω maximizes the amplitude function $C(\omega)$. To do this, we maximize $C(\omega)$ by taking the derivative and setting it equal to zero.

$$\begin{aligned} C(\omega) &= \frac{F_0}{\sqrt{(9-\omega^2)^2 + (2\omega)^2}} \\ &= \frac{F_0}{\sqrt{\omega^4 - 14\omega^2 + 81}} \end{aligned}$$

$$\begin{aligned} C'(\omega) &= \frac{-1/2 F_0}{(\omega^4 - 14\omega^2 + 81)^{3/2}}\left(4\omega^3 - 28\omega\right) \\ 0 &= 4\omega^3 - 28\omega = 4\omega(\omega^2 - 7) \\ \omega &= 0, -\sqrt{7}, \sqrt{7} \\ \omega &= \sqrt{7} \end{aligned}$$

The values $\omega = 0, \pm\sqrt{7}$ all give critical points of the function $C(\omega)$. However, we always consider $\omega \geq 0$, and by looking at a sign chart, we see that $\omega = \sqrt{7}$ gives us the global maximum for $C(\omega)$. Therefore, practical resonance occurs when $\omega = \sqrt{7}$.

5. (5 points) Transform the following 3rd-order equation into a system of linear equations

$$y^{(3)} - 5y'' + 14y' + 2y = 1.$$

$$\begin{cases} y' = y_1 \\ y_1' = y_2 \\ y_2' = -2y - 14y_1 + 5y_2 + 1 \end{cases}$$