1. The functions $y_1 = e^{2x}$ and $y_2 = e^{3x}$ are solutions to the differential equation $y'' - 5y' + 6y = 0$.

a. (10 points) Verify that $y_1$ and $y_2$ are linearly independent.

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x} \neq 0$$

b. (10 points) Find a solution satisfying the initial conditions $y(0) = 1$ and $y'(0) = 4$.

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{2x} + c_2 e^{3x}$$

$$\begin{cases} y(0) = c_1 + c_2 = 1 \\ y'(0) = 2c_1 + 3c_2 = 4 \end{cases}$$

$$c_1 = -1, c_2 = 2, \quad y = -e^{2x} + 2e^{3x}$$

2. (15 points) Find a specific solution to the equation $y'' - 2y' + 5y = \cos x$

First, set up the general form of a particular solution

$$y_p = A \cos x + B \sin x \quad \text{Notice no overlap with comp. solution}$$

$$y_p' = -A \sin x + B \cos x$$

$$y_p'' = -A \cos x - B \sin x$$

Plugging in, we see

$$y'' - 2y' + 5y = \cos x ( -4A - 2B + 5A ) + \sin x ( -B + 2A + 5B ) = \cos x$$

$$\begin{cases} 4A - 2B = 1 \\ 2A + 4B = 0 \end{cases} \implies \begin{cases} A = \frac{1}{5} \\ B = -\frac{1}{10} \end{cases}$$

$$y_p = \frac{1}{5} \cos x - \frac{1}{10} \sin x$$
Since the solution the complimentary homogeneous equation \( y'' - 2y' + 5y = 0 \) is of the form
\[
y_c = e^x(c_1 \cos 2x + c_2 \sin 2x),
\]
then any equation of the form
\[
y = \frac{1}{5} \cos x - \frac{1}{10} \sin x + e^x(c_1 \cos 2x + c_2 \sin 2x)
\]
for some values \( c_1, c_2 \) is also a specific solution.

3. Find general solutions to the following equations.

a. (10 points) \( y^{(3)} - 6y'' + 9y' = 0 \)

\[
r^3 - 6r^2 + 9r = r(r - 3)^2 = 0
\]
\[
r = 0, 3, 3
\]
\[
y = c_1 + (c_2 + c_3 x)e^{3x}
\]

b. (15 points) \( y^{(4)} - y'' = 3x \).

Note that the complimentary solution \( y_c \) is not a solution to the above equation, but instead a solution to \( y^{(4)} - y = 0 \). A particular solution \( y_p \) must be added to it to have a solution of the above equation.

\[
y = y_c + y_p
\]
\[
y_c: r^4 - r^2 = r^2(r + 1)(r - 1) = 0
\]
\[
r = 0, 0, -1, 1
\]
\[
y_c = c_1 + c_2 x + c_3 e^{-x} + c_4 e^x
\]
\[
y_i = Ax + B \quad \text{Initial guess}
\]
\[
y_p = x^2(Ax + B) = Ax^3 + Bx^2 \quad \text{Remove overlap with } y_c
\]
\[
y_p^{(4)} - y'' = -6Ax - B = 3x
\]
\[
A = -\frac{1}{2}, B = 0
\]
\[
y_p = -\frac{1}{2} x^3
\]
\[
y = -\frac{1}{2} x^3 + c_1 + c_2 x + c_3 e^{-x} + c_4 e^x
\]
4. A mass of $m = 1$ kg is attached to a large spring with spring constant $k = 9$ (N/m). It is set in motion with initial position $x_0 = 2$ (m) from the equilibrium position and initial velocity $v_0 = -6$ (m/s).

a. (15 points) If the mass-spring system is undamped (no resistance) and free (no external force), then what is the amplitude and period of the position function?

\[ m x'' + k x = 0 \]
\[ x'' + 9x = 0 \]

\[ r^2 = 9 = 0 \]
\[ r = \pm 3i \]

\[ x(t) = A \cos 3t + B \sin 3t \]
\[ x'(t) = -3A \sin 3t + 3B \cos 3t \]

\[ \begin{cases} A = x_0 = 2 \\ 3B = v_0 = -6 \end{cases} \]

\[ x(t) = 2 \cos 3t - 2 \sin 3t \]

For $x(t)$ of the form above, the amplitude will be $\sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$. The period is $\frac{2\pi}{3}$.

This can also be seen by transforming the solution to

\[ x(t) = 2\sqrt{2} \cos(3t + \frac{\pi}{4}). \]

b. (10 points) Suppose a dashpot with damping constant $c = 2$ (Ns/m) is connected to the spring (still with no external force). What is the pseudo-period of the position function?

\[ x'' + 2x' + 9x = 0 \]

\[ r^2 + 2r + 9 = 0 \]
\[ r = -1 \pm 2\sqrt{2} \]

\[ x(t) = e^{-t}(c_1 \cos 2\sqrt{2}t + c_2 \sin 2\sqrt{2}t) \]

The pseudo-period of the above solution is then $\frac{2\pi}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$. 
c. (10 points) Suppose that the mass-spring-dashpot system from part (b) is acted on by an external force of $F_0 \cos(\omega t)$ (Newtons) for some $\omega$. Then, the steady-periodic solution will be of the form

$$x_{sp}(t) = C(\omega) \cos(\omega t - \alpha(\omega)),$$

where

$$C(\omega) = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (\omega^2 \omega)^2}} \quad \alpha(\omega) = \begin{cases} \tan^{-1} \frac{\omega}{k - m\omega^2} & k > m\omega^2 \\ \pi/2 & k = m\omega^2 \\ \pi + \tan^{-1} \frac{\omega}{k - m\omega^2} & k < m\omega^2 \end{cases}$$

For what value(s) of $\omega > 0$, if any, will the system exhibit practical resonance?

To find practical resonance, we find what value of $\omega$ maximizes the amplitude function $C(\omega)$. To do this, we maximize $C(\omega)$ by taking the derivative and setting it equal to zero.

$$C(\omega) = \frac{F_0}{\sqrt{(9 - \omega^2)^2 + (2\omega^2 \omega)^2}}$$

$$= \frac{F_0}{\sqrt{\omega^4 - 14\omega^2 + 81}}$$

$$C'(\omega) = \frac{-1/2F_0}{(\omega^4 - 14\omega^2 + 81)^{3/2}} (4\omega^3 - 28\omega)$$

$$0 = 4\omega^3 - 28\omega = 4\omega(\omega^2 - 7)$$

$$\omega = 0, -\sqrt{7}, \sqrt{7}$$

The values $\omega = 0, \pm \sqrt{7}$ all give critical points of the function $C(\omega)$. However, we always consider $\omega \geq 0$, and by looking at a sign chart, we see that $\omega = \sqrt{7}$ gives us the global maximum for $C(\omega)$. Therefore, practical resonance occurs when $\omega = \sqrt{7}$.

5. (5 points) Transform the following 3rd-order equation into a system of linear equations

$$y^{(3)} - 5y'' + 14y' + 2y = 1.$$

$$\begin{align*}
  y' &= y_1 \\
  y_1' &= y_2 \\
  y_2' &= -2y - 14y_1 + 5y_2 + 1
\end{align*}$$