REMARKS ON EVOLUTION OF SPACE-TIMES IN 3+1 AND 4+1 DIMENSIONS

MICHAEL T. ANDERSON

Abstract. A large class of vacuum space-times is constructed in dimension 4+1 from hyperboloidal initial data sets which are not small perturbations of empty space data. These space-times are future geodesically complete, smooth up to their future null infinity $\mathcal{I}^+$, and extend as vacuum space-times through their Cauchy horizon. Dimensional reduction gives non-vacuum space-times with the same properties in 3+1 dimensions.

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§1. Introduction

Let (Σ, g) be a complete Riemannian Einstein n-manifold with negative scalar curvature, normalized so that

$$Ric_g = -(n-1)g. \tag{1.1}$$

It is well-known that the Lorentzian cone over Σ, i.e. the metric

$$\mathbf{g} = -d\tau^2 + \tau^2 g \tag{1.2}$$

on $\mathbf{M} = \mathbb{R}^+ \times \Sigma$ is a vacuum solution to the Einstein equations in $n+1 = (1,n)$ dimensions. In the case of 3+1 dimensions, all Einstein metrics g on a 3-manifold Σ are of constant curvature and hence the 4-metric $\mathbf{g}$ is flat; in fact $(\mathbf{M}, \mathbf{g})$ is then just (a quotient of) the interior of the future light cone of a point in empty Minkowski space. In dimensions higher than 3, solutions of (1.1) are usually not of constant curvature and hence the space-times $(\mathbf{M}, \mathbf{g})$ are typically not flat.

For (Σ, g) as in (1.1), the space-time $(\mathbf{M}, \mathbf{g})$ in (1.2) is globally hyperbolic with Cauchy surface given by the n-manifold Σ. The Cauchy data on Σ are of the form

$$(g, K) = (g, g), \tag{1.3}$$

where K is the extrinsic curvature. This Cauchy data on Σ forms a hyperboloidal initial data set, c.f. [11], [3]. An important result of Friedrich [11] implies that on a 3-dimensional hyperboloidal initial data set Σ which is a small, suitable perturbation of standard hyperboloidal data in Minkowski space-time, one has long-time evolution to the future of Σ, with a smooth future null infinity $\mathcal{I}^+$. However, one does not expect long-time future evolution to hold for large perturbations off the standard hyperboloidal initial data in Minkowski space, in that one expects formation of black holes and singularities to the finite proper time future of Σ, at least generically.

The first purpose of this paper is to point out that such reasoning or expected behavior may not hold in higher, in particular $4+1$ dimensions. Thus, a large class of hyperboloidal initial data are constructed in $4+1$ dimensions for which the vacuum space-time $(\mathbf{M}, \mathbf{g})$ in (1.2) is future geodesically complete with smooth future null infinity $\mathcal{I}^+$, and which may be considered as *large* perturbations of the standard flat data, c.f. Theorem 3.1. Moreover, if Friedrich's theorem [11] generalizes to higher dimensions, one would then have *open sets* of such large initial data whose maximal Cauchy developments have the stated properties. This indicates there may be some strong differences between vacuum evolution of space-times in $3+1$ and higher dimensions.

A second purpose is to study the behavior of the space-times (1.2) at the Cauchy horizon Σ_0 where $\tau = 0$. In Theorem 4.2, we prove that the space-times $(\mathbf{M}, \mathbf{g})$ extend as $C^{1,1/2}$ weak solutions of the vacuum Einstein equations across the horizon Σ_0. This gives a large class of space-times in $4+1$ dimensions, depending in fact on 2 free functions of 3 variables, which have extensions past a Cauchy horizon. The behavior of the curvature tensor $\mathbf{R}$ of $(\mathbf{M}, \mathbf{g})$ at Σ_0 is singular, although the singularity is relatively mild. The curvature blows up at most as $x^{-1/2}$, where x is the Lorentzian distance to Σ_0. If one allows such singularities, then this class of space-times does not satisfy the strong cosmic censorship hypothesis in this dimension.

The results above point out some differences in the evolution of vacuum space-times in 3+1 and higher dimensions. However, if the initial data (1.1) on a 4-manifold Σ has an isometric S^1 action, then the vacuum space-time $(\mathbf{M}, \mathbf{g})$ can be dimensionally reduced to a 3+1 dimensional space-time, with Cauchy surface Σ/S^1. The resulting $3+1$ space-times have matter terms given by a massless scalar field or more generally by a wave map (non-linear σ-model) with target the hyperbolic plane $H^2(-1)$. The results above on long-time future evolution, smoothness of $\mathcal{I}^+$, and extension past Cauchy horizons also hold for such $3+1$ space-times. This situation is analysed in §5.

§2. Initial data set and space-time (M, g)

Let Σ be a compact manifold with non-empty boundary $\partial\Sigma$. A complete Riemannian metric g on Σ is conformally compact if there is a defining function ρ for $\partial\Sigma$ on Σ such that the conformally equivalent metric

$$\bar{g} = \rho^2 g \tag{2.1}$$

extends to a metric on the closure $\bar{\Sigma} = \Sigma \cup \partial\Sigma$. The compactification $\bar{g}$ is not uniquely determined by g, since there are many choices for the defining function. However, the conformal class $[\bar{g}]$ is well-defined. Given ρ, the metric $\gamma = \bar{g}|_{\partial\Sigma}$ is called the boundary metric associated to g and ρ and the conformal class $[\gamma]$ on $\partial\Sigma$ is the conformal infinity of (Σ, g) The metric g is called $C^{m,\alpha}$ conformally compact if there is a compactification $\bar{g}$ as in (2.1) which extends to a $C^{m,\alpha}$ metric on $\bar{\Sigma}$; here $C^{m,\alpha}$ is the class of metrics which are C^m with α-Hölder continuous mth derivatives in some (e.g. harmonic) coordinate system.

We begin by quoting the following result from [2]. Let $Met^0(S^3) = Met^0_{m,\alpha}(S^3)$ be the component of the space of metrics of non-negative scalar curvature on S^3 containing the round metric, and let $\mathcal{C}^0 = \mathcal{C}^0_{m,\alpha}$ be the space of conformal classes of $C^{m,\alpha}$ metrics on S^3 containing a representative in $Met^0(S^3)$. Let B^4 be the 4-ball, so that $\partial B^4 = S^3$.

Theorem 2.1. *Let γ be any $C^{m+1,\alpha}$ metric in $Met^0_{m+1,\alpha}(S^3)$, $m \geq 2$. Then there exists a complete $C^{m+1,\alpha}$ conformally compact Einstein metric $g = g_\gamma$ on $\Sigma = B^4$ with conformal infinity $[\gamma]$.*

This result also holds when $m = \infty$, and $m = \omega$, giving C^∞, respectively real-analytic, (C^ω), conformally compact Einstein metrics with prescribed conformal infinity $[\gamma] \in \mathcal{C}^0$.

Given a choice of boundary metric $\gamma \in [\gamma]$, within a neighborhood U of $\partial\Sigma = S^3$ the metric g may be written in the form

$$g = \frac{1}{1+s^2}ds^2 + s^2\gamma + h, \tag{2.2}$$

where the parameter $s \in [s_o, \infty)$, for some $s_o \in \mathbb{R}$ and the bilinear form h is bounded (w.r.t. g). Under the substitution $s = sinhr$, one has

$$(1+s^2)^{-1}ds^2 + s^2\gamma = dr^2 + sinh^2 r\gamma, \tag{2.3}$$

so that this part of the metric is a hyperbolic cone metric over the boundary metric γ. Of course the truncated cone metric in (2.3) is not Einstein unless $\gamma = \gamma_o$, the round metric on S^3. The term h is a perturbation term, possibly large into the interior, and Theorem 2.1 implies that h may chosen so that g in (2.2) satisfies the Einstein equations (1.1) globally. The function

$$\rho = 2e^{-r} = 2e^{-(arcsinh s)} \sim 2/s \tag{2.4}$$

is a geodesic defining function for $\partial\Sigma$ in (Σ, g), in that the integral curves of ∇r on (Σ, g) are geodesics. Similarly, the compactification

$$\bar{g} = \rho^2 g \tag{2.5}$$

is a geodesic compactification, in that $\rho(x) = dist_{\bar{g}}(x, \partial\Sigma)$, with boundary metric γ. This compactification may not have the optimal regularity, but for γ as in Theorem 2.1, $\bar{g}$ is at least a $C^{m,\alpha}$ conformal compactification. Combining (2.2) and (2.4) gives

$$\bar{g} = \rho^2 g = d\rho^2 + (1 - \frac{1}{4}\rho^2)\gamma + \rho^2 h, \tag{2.6}$$

so that

$$\rho^2 h \in C^{m,\alpha}(\bar{\Sigma}), \tag{2.7}$$

i.e. $\rho^2 h$ extends $C^{m,\alpha}$ smoothly up to $\partial\Sigma$. Of course since h is bounded, $\rho^2 h = 0$ on $\partial\Sigma$. The coordinate expressions (2.2) or (2.6) are valid up to the cut locus of $\partial\Sigma$ in $(\Sigma, \bar{g})$. One sees that the curvature tensor R_g of g in (2.2) approaches the curvature tensor R_{-1} of the hyperbolic metric at a rate

$$|R_g - R_{-1}| = O(\rho^2) = O(s^{-2}), \tag{2.8}$$

but not any faster in general. Thus, such metrics g are asymptotically hyperbolic.

As noted in the Introduction, given a conformally compact Einstein 4-manifold (Σ, g) as above, the space-time $(\mathbf{M}, \mathbf{g})$ in (1.2) is globally hyperbolic with Cauchy surface Σ and Cauchy data given by (1.3). Observe that Theorem 2.1 provides a large space of such initial data. Namely, the collection of such metrics g on the 4-ball $\Sigma = B^4$ are effectively parametrized by the space of their conformal infinities, i.e. the boundary conformal classes $[\gamma] \in \mathcal{C}^0$. The space $\mathcal{C}^0$ is a "large" space, in that for instance there are curves of unit volume metrics in $\mathcal{C}^0$ starting at the standard round metric γ_0 and diverging arbitrarily far away from γ_0 in terms of the size of the curvature or diameter. The resulting Einstein metrics on Σ are then also arbitrarily far away from the hyperbolic metric.

The time parameter τ varies over the domain $(0, \infty)$ and w.r.t. the initial Cauchy surface Σ the time evolution of the space-time is just given by rescalings of (Σ, g); thus, if Σ_τ is the τ-level set of the function τ, then $\mathbf{g}|_{\Sigma_\tau} = g_\tau = \tau^2 g$. Of course $\Sigma = \Sigma_1$. The mean curvature H of Σ_τ is given by $H = 4/\tau$, while the extrinsic curvature is pure trace.

Observe also that the space-time $(\mathbf{M}, \mathbf{g})$ is geodesically complete, (both time-like and null), to the future of Σ. This is not the case to the past of Σ; the surface $\Sigma_0 = \{\tau = 0\}$ forms the Cauchy horizon. This will be examined in more detail in §4 below. From (2.2), the space-time metric $\mathbf{g}$ may be written in the form

$$\mathbf{g} = -d\tau^2 + \tau^2(\frac{1}{1+s^2}ds^2 + s^2\gamma) + \tau^2 h, \tag{2.9}$$

in a neighborhood U of $\partial\Sigma$, with h a bilinear form on Σ.

§3. Asymptotic flatness

In this section, we prove that $(\mathbf{M}, \mathbf{g})$ is asymptotically flat in the sense of Penrose.

Theorem 3.1. *The space-time $(\mathbf{M}, \mathbf{g})$ is asymptotically flat to the future of Σ, in that $(\mathbf{M}, \mathbf{g})$ has a smooth conformal compactification*

$$\bar{\mathbf{g}} = \Omega^2 \mathbf{g}. \tag{3.1}$$

More precisely, if g is $C^{m+1,\alpha}$ conformally compact as in Theorem 2.1, then $\bar{\mathbf{g}}$ is (at least) $C^{m,\alpha}$, i.e. $\bar{\mathbf{g}}$ extends $C^{m,\alpha}$ to the future null infinity $\mathcal{I}^+$ of $(\mathbf{M}, \mathbf{g})$.

To prove this result, suppose first that $\gamma = \gamma_o$, so that the metric g on Σ is the Poincaré metric g_o on the 4-ball B^4 and $(\mathbf{M}, \mathbf{g})$ is the flat $(4+1$ dimensional) Minkowski metric $\mathbf{g_o}$ interior to a future light cone. Following the standard compactification of Minkowski space, c.f. [13], let

$$\Omega^2 = \frac{4}{4\tau^2 s^2 + (1+\tau^2)^2} = 4cos^2\frac{1}{2}(T+R)cos^2\frac{1}{2}(T-R), \tag{3.2}$$

where $\frac{1}{2}(T+R) = arctan\ v$, $\frac{1}{2}(T-R) = arctan\ u$ and (u,v) are advanced and retarded null coordinates, given in terms of (τ, s) by $v = \tau(s + (1+s^2)^{1/2}), u = -\tau(s - (1+s^2)^{1/2})$. Thus, the compactification $\bar{\mathbf{g}}_{\mathbf{o}}$ interior to the light cone in these coordinates becomes

$$\bar{\mathbf{g}}_{\mathbf{o}} = -dT^2 + dR^2 + sin^2 R\gamma_o, \tag{3.3}$$

with $R \geq 0$ and $T+R \in [0,\pi], T-R \in [0,\pi]$. The full metric (3.3), without restriction on the range of coordinates, is the Einstein static space-time on $\mathbb{R} \times S^4$.

Performing exactly the same computation for the conformal compactification (3.1) with Ω as in (3.2) gives

$$\bar{\mathbf{g}} = -dT^2 + dR^2 + sin^2 R\gamma + \Omega^2\tau^2 h, \tag{3.4}$$

where, as in (2.9), h is defined on Σ and so is independent of τ.

Future null infinity $\mathcal{I}^+$ of $(\mathbf{M}, \mathbf{g})$ is given by the locus

$$\mathcal{I}^+ = \{T + R = \pi\}, \text{ with } T \in (\frac{\pi}{2}, \pi). \tag{3.5}$$

It is clear by inspection that the conformal factor Ω and the first part $-dT^2 + dR^2 + sin^2 R\gamma$ of the metric $\bar{\mathbf{g}}$ are $C^{m,\alpha}$ up to $\mathcal{I}^+$. Further, in any region of $(\mathbf{M}, \mathbf{g})$ where τ is bounded away from 0 and ∞, corresponding to T in a compact subset of $(\frac{\pi}{2}, \pi)$, one has

$$\Omega^2\tau^2 h \sim \rho^2 h,$$

(c.f. (2.4)), which vanishes on $\mathcal{I}^+$ and extends $C^{m,\alpha}$ up to $\mathcal{I}^+$ by (2.7). The behavior at $\{T = R = \frac{\pi}{2}\}$, corresponding to the null infinity of the Cauchy horizon will be discussed in §4.

At future time-like infinity $\iota^+ = \{T = \pi, R = 0\}$, one also has $\Omega^2\tau^2 h \to 0$ so that the space part $dR^2 + sin^2 R\gamma + \Omega^2\tau^2 h$ of $\bar{\mathbf{g}}$ approximates the metric cone on γ near ι^+. This completes the proof.

Remark 3.2. As explained in the Introduction, Theorem 3.1 is in quite strong contrast to the situation in $3+1$ dimensions where singularities are expected to form to the finite future of "large" hyperboloidal initial data sets, i.e. initial data which are not close to that of the standard hyperboloid in Minkowski space-time.

Theorem 3.1 shows that such singularity formation does not occur, even though (Σ, g) may be far away from a hyperbolic metric. For example, the curvature of $\mathbf{g}$ near the origin of $\Sigma = \Sigma_1$ may become arbitrarily large, as $[\gamma]$ varies over $\mathcal{C}^0$. While the initial data (g, g) on Σ are not generic, in the sense of forming an open set in the space of all solutions to the constraint equations, the space of initial data is nevertheless large. Namely, it is parametrized by $\mathcal{C}^0$, and so includes a "large" open set of metrics on $\partial\Sigma = S^3$, as described following (2.8). Taking into account the equivalences from conformally related metrics and diffeomorphisms, the space of initial data formally corresponds locally to 2 free functions of 3 variables.

In fact, it does not seem unreasonable that Friedrich's result [11] can be extended to $4+1$ dimensions to include all suitable small perturbations of initial data as in (1.3). If so, then this thickening of the initial data would give large open sets in the space of initial data having long time evolution to the future and smooth $\mathcal{I}^+$.

Remark 3.3. The discussion above, in particular Theorems 2.1 and 3.1, also hold for other choices of the Cauchy surface Σ in place of B^4. For example, one may choose $\Sigma = \mathbb{R}^3 \times S^1$, with $\partial\Sigma = S^2 \times S^1$, or Σ any disc bundle over S^2 of degree $k \geq 2$, so that $\partial\Sigma = S^3/\mathbb{Z}_k$, c.f. [2]. In the latter case, Σ does not admit any hyperbolic metric.

Similarly, Theorem 3.1 and the discussion above hold for any conformally compact Einstein metric (1.1) on a 4-manifold, (or n-manifold), for example the Euclidean AdS black hole metrics with flat or hyperbolic horizons, c.f. [6] and references therein. However, a global existence result as in Theorem 2.1 is lacking when the boundary metrics are not in $\mathcal{C}^0$, (and lacking altogether in dimensions above 4).

§4. Cauchy horizon

Next, we discuss the behavior at the Cauchy horizon $\Sigma_0 = \{\tau = 0\}$. This was the initial setting of the work of Fefferman-Graham [10]. As usual, one must change coordinates near Σ_0, and so, following [10], write

$$\tau = v\rho, \tag{4.1}$$

in a neighborhood of $\tau = 0$. One could also use the more standard coordinate transformation $r = \tau s, t = \tau(1+s^2)^{1/2}$, but this and other coordinate changes lead to equivalent results. Since ρ is a geodesic defining function, the compactification $\bar{g}$ in (1.6) splits as

$$\bar{g} = d\rho^2 + g_\rho,$$

where g_ρ is a $C^{m,\alpha}$ curve of metrics on $\partial\Sigma$ with $g_0 = \gamma$. Hence

$$\mathbf{g} = -d\tau^2 + \frac{\tau^2}{\rho^2}(d\rho^2 + g_\rho). \tag{4.2}$$

Substituting the change of variables (4.1) in (4.2), this becomes

$$\mathbf{g} \equiv \mathbf{g}^+ = -\rho^2 dv^2 - 2\rho v dv d\rho + v^2 g_\rho.$$

The reason for the notation $\mathbf{g}^+$ will become clear later, but for now it signifies $\rho > 0$ or $\tau > 0$. This metric is degenerate at $\rho = 0$, but the degeneracy comes merely from the choice of coordinates. Thus, setting $\rho^2 = x$, the metric $\mathbf{g}$ has the form

$$\mathbf{g} = \mathbf{g}^+ = -x dv^2 - v dv dx + v^2 g_{\sqrt{x}}, \tag{4.3}$$

which is a (non-degenerate) Lorentz metric at $x = 0$ provided $v > 0$. The Cauchy horizon $\Sigma_0 = \{\tau = 0\} = \{x = 0\}$ topologically is the cone on S^3, with null geodesics parametrized by v and with (degenerate) metric

$$g_0 = \mathbf{g}|_{\Sigma_0} = v^2\gamma.$$

Now we examine the smoothness of the metric $\mathbf{g}$ up to the Cauchy horizon, and the possibility of extending the space-time $(\mathbf{M}, \mathbf{g})$ past the horizon. For simplicity, assume that $m = \omega$, i.e. the compactification $\bar{g}$ is real-analytic, (c.f. Theorem 2.1), so that the curve g_ρ is real-analytic in ρ.

As discussed in [10], the curve of metrics g_ρ has a Taylor expansion of the form

$$g_\rho = g_{(0)} + \rho^2 g_{(2)} + \rho^3 g_{(3)} + \rho^4 g_{(4)} + \rho^5 g_{(5)} + ..., \tag{4.4}$$

where the coefficients $g_{(j)}$ are bilinear forms on $\partial\Sigma$. Note that there is no term linear in ρ. The $g_{(2)}$ term is determined locally by the intrinsic geometry of the boundary metric $\gamma = g_{(0)}$, c.f. [10].

However, the term $g_{(3)}$ is global and not determined by the boundary metric γ. In fact only for very special Einstein metrics (Σ, g) does one have $g_{(3)} = 0$. By [1], Einstein metrics with $g_{(3)} = 0$ are critical points for the renormalized volume V or action in the sense of the AdS/CFT correspondence, on the moduli space of conformally compact Einstein metrics. Further, one has

$$g_{(3)} = 0 \tag{4.5}$$

if and only if the expansion (4.4) has only even powers of ρ.

Remark 4.1. Certain of the main results of [10] require that the expansion (4.4) is in even powers of ρ. While this can always be achieved locally, i.e. in a neighborhood of $\partial\Sigma$, c.f. [10, Thm. 2.3], there is at most a finite dimensional space of such conformally compact Einstein metrics globally on Σ. To see this, the evenness of the expansion, (and real-analyticity), implies the compactified metric $\bar{g}$ on Σ reflects across $\partial\Sigma$ to give a smooth metric $\widetilde{g}$ on the double $\widetilde{\Sigma} = \Sigma \cup_{\partial\Sigma} \Sigma$. The manifold $\widetilde{\Sigma}$ is closed, and $\widetilde{g}$ is a smooth solution of the conformally invariant Bach equation, (c.f. [4, Ch.4H]),

$$\delta d(Ric - \frac{s}{6}g) - 2W(Ric) = 0.$$

However, on a compact manifold, without boundary, there is at most a finite dimensional space of (conformal classes) of solutions to this equation, since it is essentially elliptic transverse to conformal classes.

Making the substitution $\rho = \sqrt{x}$ in (4.4) gives the expansion

$$g_{\sqrt{x}} = g_{(0)} + xg_{(2)} + x\sqrt{x}g_{(3)} + x^2 g_{(4)} + x^2\sqrt{x}g_{(5)} + ... \tag{4.6}$$

It follows that the metric $\mathbf{g}$ is, in general, at most $C^{1,1/2}$ smooth up to the Cauchy horizon Σ_0 in these coordinates. This degree of smoothness cannot be improved by changing coordinates. Of course if $g_{(3)} = 0$, then $\mathbf{g}$ is C^{ω} up to Σ_0 by the remark following (4.5).

On the other hand, we observe that the curvature $\mathbf{R}$ of $(\mathbf{M}, \mathbf{g})$ is bounded in many directions up to Σ_0, (away from the vertex $\{0\}$). To see this, let U, V, W be vectors tangent to Σ_τ at a point (τ, p), $p \in \Sigma_\tau$. Using the expression (1.2), standard formulas show the curvature of $(\mathbf{M}, \mathbf{g})$ at (τ, p) is given by

$$\mathbf{R}(V, W)U = \tau^{-2}[R_g(V, W)U - R_{-1}(V, W)U],$$

while any component of $\mathbf{R}$ containing a vector orthogonal to Σ_τ vanishes; here R_{-1} is the curvature of the hyperbolic metric g_{-1} on $\mathbb{R}^4$, so $R_{-1}(V, W)U = g_{-1}(V, U)W - g_{-1}(W, U)V$. Under the substitution (4.1), this gives

$$\mathbf{R}(V, W)U = v^{-2}\{\rho^{-2}[R_g(V, W)U - R_{-1}(V, W)U]\}. \tag{4.7}$$

On approach to any point in Σ_0 away from the vertex $\{0\}$ where $v = 0$ we have $\rho \to 0$. But by (2.8) the term $\rho^{-2}(R_g(V, W)U - R_{-1}(V, W)U)$ remains bounded as $\rho \to 0$. In fact from standard formulas relating the curvatures of g and $\bar{g}$, (c.f. [2]), one finds

$$lim_{\rho\to 0}\rho^{-2}[R_g - R_{-1}] = \bar{R} - \frac{1}{2}\bar{R}ic \wedge \bar{g} + \frac{\bar{s}}{3}\bar{g} \wedge \bar{g}, \tag{4.8}$$

where $\bar{s}, \bar{R}ic$ and $\bar{R}$ are the scalar, Ricci and full curvature of $\bar{g}$ in (2.5), evaluated at $\partial\Sigma$ and $\wedge$ denotes the Kulkarni-Nomizu product, c.f. [4, Ch.1]. These curvatures of $\bar{g}$ at $\partial\Sigma$ are in fact determined by the intrinsic geometry of the boundary metric γ at $\partial\Sigma$.

However, the curvature $\mathbf{R}$ is not bounded in all directions. Namely, a simple computation shows that in 2-planes spanned by the vector $\partial/\partial v$ and vectors tangent to S^3, (i.e. orthogonal to $\partial/\partial v$ and $\partial/\partial\rho$), the curvature blows up as $x^{-1/2}$, where x is the Lorentzian distance to Σ_0. (Of course, the curvature blows up at the rate of v^{-2} on approach to the singularity $\{0\}$, unless the space-time $(\mathbf{M}, \mathbf{g})$ is flat).

Nevertheless, the space-time $(\mathbf{M}, \mathbf{g})$ extends past the horizon Σ as a (weak) solution of the vacuum Einstein equations. To see this, just extend the variable $x \geq 0$ to allow negative values of x. It is immediate that the term $-xdv^2 - vdvdx$ in the expression (4.3) for $\mathbf{g}$ extends smoothly to $x < 0$. When $g_{(3)} = 0$, the term $g_{\sqrt{x}}$ in (4.3) is a power series in powers of x, so extends real-analytically across $\partial\Sigma$, giving a real-analytic extension of $\mathbf{g}$. When $g_{(3)} \neq 0$, the series for g_ρ

extends analytically in ρ to regions where $\rho < 0$. This corresponds to replacing $\sqrt{x}$, $x > 0$, by $-\sqrt{|x|}$, $x < 0$. Thus, the series for $g_{\sqrt{x}}$ becomes

$$g_{\sqrt{|x|}} = g_{(0)} + xg_{(2)} - x\sqrt{|x|}g_{(3)} + x^2 g_{(4)} - x^2\sqrt{|x|}g_{(5)} + x^3 g_{(6)} + ..., \tag{4.9}$$

when $x \leq 0$. In terms of $\rho < 0$, this has the form

$$g_\rho^- = g_{(0)} - \rho^2 g_{(2)} - \rho^3 g_{(3)} + \rho^4 g_{(4)} + \rho^5 g_{(5)} - \rho^6 g_{(6)} + ... \tag{4.10}$$

Reversing the substitution (4.1), i.e. in terms of (τ, ρ) coordinates, the metric $\mathbf{g} = \mathbf{g}^+$ extends across the Cauchy horizon as

$$\mathbf{g}^- = d\tau^2 + \frac{\tau^2}{\rho^2}(-d\rho^2 + g_\rho^-). \tag{4.11}$$

Referring to (4.2), one sees that $\mathbf{g}^-$ is obtained from $\mathbf{g}^+$ by interchanging the role of time and space, (in the direction ρ). Thus, (4.2) formally corresponds to (4.11) under the substitution $\tau \to i\tau, \rho \to i\rho$, with $g_\rho^+ \to g_\rho^-$. When $g_{(3)} = 0$, one sees also that $g_\rho^- = g_{i\rho}^+$.

The domain of $\mathbf{g}^-$ includes a neighborhood where $\tau < 0$, (or $\rho < 0$). In particular, it follows that this extension of $(\mathbf{M}, \mathbf{g})$ to a larger domain extends $\mathcal{I}^+$ to a larger domain; in the form (3.5), the parameter T varies over $(\frac{\pi}{2} - \delta, \pi)$, for some $\delta > 0$.

We summarize this analysis in the following result.

Theorem 4.2. *Suppose the boundary metric γ on $\partial\Sigma$ is real-analytic. Away from the vertex $\{0\}$, the space-time $(\mathbf{M}, \mathbf{g})$ extends past the Cauchy horizon Σ_0 as a vacuum space-time. The extended space-time is $C^{1,1/2}$ at the Cauchy horizon, but is C^ω elsewhere. Within its domain, the extended space-time is asymptotically flat, and gives a smooth extension of $\mathcal{I}^+$ to a larger null surface.*

We make several remarks on this construction.

Remark 4.3. Consider the construction above with regard to the strong cosmic censorship conjecture of Penrose, c.f. [8] and references therein. Roughly speaking, this states that a generic (vacuum) globally hyperbolic space-time does not admit a space-time extension past the Cauchy horizon, i.e. generically singularities form on approach to a Cauchy horizon.

Depending on the exact meaning of singularities, this is not the case for the space-times $(\mathbf{M}, \mathbf{g})$. While there is a curvature singularity at the Cauchy horizon, it is relatively mild. The curvature is integrable along curves terminating at Σ_0, and there are no strong curvature singularities at Σ_0, c.f. [9]. While the space-times $(\mathbf{M}, \mathbf{g})$ are not generic, in the sense of forming an open set, they do comprise a large family of vacuum solutions, c.f. Remark 3.2. Similarly, this situation occurs in $4+1$ dimensions, although see §5.

Remark 4.4. It would be interesting to understand what the maximal (vacuum) extension of $(\mathbf{M}, \mathbf{g})$ is, i.e. how large the domain of $\mathbf{g}^-$ is. Note that the hyperbolic cone metric (1.2) can be defined equally well in the interior of the past light cone in place of future cone. Is it possible that the metric $\mathbf{g}^-$ interpolates smoothly between the Cauchy horizons of these isometric past and future cones? If so, then of course the vertex $\{0\}$ is a naked singularity.

Remark 4.5. We comment briefly on generalizations of the results above to higher dimensions $n+1, n > 4$, and to space-times of negative cosmological constant. As in Remark 3.3, the results above hold for any $C^{m,\alpha}$ conformally compact Einstein metric, on any n-manifold Σ, although the part of Theorem 4.2 concerning the extension past Σ_0, i.e. the definition of $\mathbf{g}^-$ in (4.11), requires real-analyticity.

First, there is no known analogue of the global result of Theorem 2.1 in dimension $n > 4$. There are local results in all dimensions, c.f. [2], [5], [12], [14], showing that given any conformally compact Einstein metric g_1 on an n-manifold Σ with conformal infinity $[\gamma_1]$, there are conformally compact Einstein perturbations g of g_1, essentially parametrized by small perturbations $[\gamma]$ of $[\gamma_1]$. However, these lead to just small space-time perturbations, and may possibly be covered by the perturbation techniques of Friedrich [11], (or more precisely their generalization to higher dimensions). On the other hand, in higher dimensions one has stronger regularity of the compactification $\bar{g}$ at $\partial\Sigma$; in dimension $n+1$, the expansion (4.4) is even up to the $g_{(n-1)}$ term, c.f. [10], [14]. Thus, such metrics will be smoother, in particular of bounded curvature, at the Cauchy horizon.

Of course, one could take the $4+1$ dimensional space-times (1.2) and take the product with space-like $\mathbb{R}$ factors. However, this will lead to singularities, (of codimension 4), on $\mathcal{I}^+$.

Finally, starting with any Riemannian Einstein metric (Σ, g) as in Theorem 2.1, (or as above), one may also form "spherical cones", giving vacuum space-times with cosmological constant $\Lambda = -6$; (other values for $\Lambda < 0$ may be obtained by rescalings). Thus, set

$$\mathbf{g} = -d\tau^2 + cos^2\tau \cdot g,$$

defined on $\mathbf{M} = S^1 \times \Sigma$, or on the universal cover $\mathbb{R} \times \Sigma$. Of course, these coordinates break down at the Cauchy horizons where τ is a multiple of $\pm\frac{\pi}{2}$. However, the space-time $(\mathbf{M}, \mathbf{g})$ extends past the Cauchy horizon as a vacuum space-time exactly as before.

§5. Dimensional reduction

Let (Σ, g) be a $C^{m,\alpha}$ conformally compact Einstein metric, $m \geq 3$, dim $\Sigma = 4$, and suppose the conformal infinity $(\partial\Sigma, [\gamma])$ has a 1-parameter group of isometries, so that there is an S^1 action by isometries. By [2, Thm. 3.1], this action extends to an isometric S^1 action on Σ. This action of course extends to a space-like isometric S^1 action on $(\mathbf{M}, \mathbf{g})$. Hence, one may dimensionally reduce the 5-dimensional space-time $(\mathbf{M}, \mathbf{g})$ to obtain a 4-dimensional space-time $(\mathbf{M^4}, \mathbf{g^4})$, with Cauchy surface $\Sigma^3 = \Sigma^4/S^1$.

Following the standard dimensional reduction, the 5-metric may be written in the form

$$\mathbf{g} = e^{-2\lambda}\mathbf{g}^4 + e^{2\lambda}\theta \cdot \theta,$$

where θ is the unit form dual to the Killing field X and $\lambda = log|X|$. The metric $\mathbf{g^4}$ is a space-time metric on the manifold $\mathbf{M^4} = \mathbb{R}^+ \times \Sigma^3$. The field equations for $\mathbf{g^4}$ are

$$Ric = \frac{1}{2}(4d\lambda \otimes d\lambda + e^{-4\lambda}d\phi \otimes d\phi), \tag{5.1}$$

$$\Delta\lambda = -\frac{1}{2}e^{-4\lambda}|d\phi|^2, \tag{5.2}$$

$$divd\phi = 4 < d\lambda, d\phi >, \tag{5.3}$$

where ϕ is the twist potential, c.f. [15]. These are the vacuum equations coupled to wave-map matter, mapping to the hyperbolic plane $H^2(-1)$, i.e. to a non-linear σ-model with target $H^2(-1)$. In particular, if $\phi = const.$, then these are the vacuum equations coupled to a massless scalar field. Observe that $Ric \geq 0$, so that $(\mathbf{M^4}, \mathbf{g^4})$ satisfies the strong energy condition.

The metric $\mathbf{g^4}$ has singularities arising from the fixed points of the S^1 action. We describe one situation, (among many possibilities) where there are no singularities. Thus, suppose $\Sigma = \Sigma^4 = \mathbb{R}^3 \times S^1$, so that $\partial\Sigma = S^2 \times S^1$. Remarks 3.3 and 4.5 show that the results of §2-§4 hold for Σ. Choose any free S^1 action on $S^2 \times S^1$, (there are many), and let γ be any metric in $Met^0(S^2 \times S^1)$ invariant under the action. Hence any conformally compact Einstein metric g on $\mathbb{R}^3 \times S^1$ with conformal infinity $[\gamma]$ has an isometric S^1 action.

Now observe this action cannot have any fixed points. For all of the S^1 orbits are homotopic in Σ, since the orbit space is connected. The action on the boundary $S^2 \times S^1$ is free, so that the orbits are non-trivial in $\pi_1(\partial\Sigma)$, and hence non-trivial in $\pi_1(\Sigma)$. But at any fixed point, the orbit is trivial, giving a contradiction.

It follows that the quotient $\Sigma^3 = (\mathbb{R}^3 \times S^1)/S^1$ is a smooth 3-manifold, topologically $\mathbb{R}^3$. Carrying out the construction above gives a smooth space-time $(\mathbf{M^4}, \mathbf{g^4})$ with smooth hyperboloidal Cauchy surface $\Sigma_1 \equiv \Sigma^3$. As before, this space-time is future geodesically complete, with smooth $\mathcal{I}^+$ and with time evolution given by rescalings. This gives a constant mean curvature foliation Σ_τ, $\tau \in (0,\infty)$ of $(\mathbf{M^4}, \mathbf{g^4})$. Theorem 4.2 also holds for $(\mathbf{M^4}, \mathbf{g^4})$.

Since the time evolution of the metric $\mathbf{g^4}$ is given by rescalings, with time parameter τ as in (1.2), the potentials λ, ϕ evolve, (on the Cauchy surfaces Σ_τ), as $\lambda_\tau = \lambda_1 + log\tau$, $\phi_\tau = \tau^4\phi_1$, where $\lambda = \lambda_1, \phi = \phi_1$ are the potentials on $\Sigma = \Sigma_1$. We point out that if the S^1 action on the boundary $\partial\Sigma^4 = S^2 \times S^1$ is hypersurface orthogonal, i.e. $d\phi = 0$ on $S^2 \times S^1$, then the S^1 action on $\Sigma^4 = \mathbb{R}^3 \times S^1$ is also hypersurface orthogonal, i.e. $d\phi = 0$ on Σ. This may be seen by using the analogue of the field equation (5.3) on the Einstein 4-manifold (Σ, g), pairing it with ϕ and integrating by parts over (Σ, g). In fact, the data (λ, ϕ) are essentially determined by their boundary values on the conformal infinity $\partial\Sigma$ of Σ.

Thus again there is large collection of 3-dimensional initial data which have global evolution to the future, smooth $\mathcal{I}^+$, and which extend past the Cauchy horizon. In this situation, although the time evolution is simple, the spatial geometry of the Cauchy surfaces Σ_τ is complicated in general. By the Riemann mapping theorem the conformal infinity $(S^2, [\gamma])$ of Σ^3 is unique, but the matter fields (λ, ϕ) are freely specifiable on the boundary, within the constraint of $\mathcal{C}^0$. This gives a degree of freedom in the construction corresponding locally to 2 free functions of 2 variables.

If $\phi = 0$ on the boundary S^2, then $\phi \equiv 0$ on the $3+1$ space-time and the massless scalar field λ may be specified arbitrarily on the conformal infinity S^2, subject again only to the condition of $\gamma \in \mathcal{C}^0$. Thus, this situation is essentially opposite to that considered by Christodoulou in his study of the evolution of spherically symmetric $3+1$ gravity coupled to a massless scalar field, c.f. [7] and references therein.

§6. Conclusion

In conclusion, this paper presents some evidence that central issues in classical $3+1$ general relativity, such as singularity and black hole formation and the behavior at Cauchy horizons may be special to this dimension and analogues of these issues may not hold in higher dimensions. If this is the case, then of course the theoretical methods developed toward the resolution of these issues must depend on the special nature of $3+1$ dimensions.

This evidence comes from the analysis of the very simple time evolution of the much more complicated spatial geometries possible in higher dimensions. In this regard, it would be interesting to know if the long time existence results of Friedrich [10] generalize to higher dimension. This would strengthen the evidence for the differences in $3+1$ and higher dimensions considerably.

In the opposite direction, in the presence of a natural isometry group on the conformal infinity of the initial Cauchy surface, the results obtained on $4+1$ vacuum space-times are equivalent to those on the $3+1$ space-times with matter given by a reasonably general class of massless scalar fields or σ-models to $H^2(-1)$.

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Department of Mathematics
SUNY at Stony Brook
Stony Brook, N.Y. 11794-3651
anderson@math.sunysb.edu