

EXISTENCE AND STABILITY OF EVEN DIMENSIONAL ASYMPTOTICALLY DE SITTER SPACES

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ABSTRACT. A new proof of Friedrich's theorem on the existence and stability of asymptotically de Sitter spaces in 3+1 dimensions is given, which extends to all even dimensions. In addition we characterize the possible limits of spaces which are globally asymptotically de Sitter, to the past and future.

1. INTRODUCTION.

Consider globally hyperbolic vacuum solutions (M^{n+1}, g) to the Einstein equations with cosmological constant $\Lambda > 0$, so that

$$(1.1) \quad Ric_g - \frac{R_g}{2}g + \Lambda g = 0.$$

The simplest solution is (pure) de Sitter space on $M^{n+1} = \mathbb{R} \times S^n$, with metric

$$(1.2) \quad g_{dS} = -dt^2 + \cosh^2(t)g_{S^n(1)}.$$

More generally, let (N^n, g_N) be any compact Riemannian manifold with metric g_N satisfying the Einstein equation $Ric_{g_N} = (n-1)g_N$. Then the (generalized) de Sitter metric

$$(1.3) \quad g_{dS}^N = -dt^2 + \cosh^2(t)g_N,$$

on $\mathbb{R} \times N$ is also a solution of (1.1), with $\Lambda = n(n-1)/2$.

Let dS^+ be the space of all globally hyperbolic spacetimes (M^{n+1}, g) satisfying (1.1), with a spatially compact Cauchy surface, which are asymptotically de Sitter (dS) to the future, i.e. future conformally compact in the sense of Penrose; the terminology asymptotically simple is also used in this context. Thus there is a smooth function Ω such that the conformally compactified metric

$$(1.4) \quad \bar{g} = \Omega^2 g,$$

extends to the compactified spacetime $\bar{M} = M \cup \mathcal{I}^+$, where \mathcal{I}^+ is a compact n -manifold without boundary. The function Ω is smooth on \bar{M} , with $\Omega > 0$, $\mathcal{I}^+ = \Omega^{-1}(0)$ and $d\Omega \neq 0$ on \mathcal{I}^+ . The boundary metric $\gamma = \bar{g}|_{\mathcal{I}^+}$ depends on the choice of Ω ; however the conformal class $[\gamma]$ of γ is independent of Ω and is called future conformal infinity. Such spacetimes are geodesically complete to the future of an initial compact Cauchy surface Σ diffeomorphic to \mathcal{I}^+ . There are no restrictions on the class $[\gamma]$ or the topology of \mathcal{I}^+ , and so such spacetimes are sometimes also called asymptotically locally de Sitter. Changing the time orientation gives the same notion for dS^- , with future conformal infinity \mathcal{I}^+ replaced by past conformal infinity \mathcal{I}^- . The smoothness of \bar{g} in (1.4) up to \bar{M} may be measured in Hölder spaces $C^{m,\alpha}$, but we will mostly use Sobolev spaces H^s which are more natural in this context.

In addition, let dS^\pm be the space of such globally hyperbolic spacetimes which are in both dS^+ and dS^- ; thus such spacetimes are (completely) global, in the sense that they are geodesically complete and asymptotically simple both to the past and to the future.

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Mathematically, the most significant result on the structure of such spacetimes is Friedrich's theorem [8], [9] that in $3 + 1$ dimensions, the Cauchy problem with data on \mathcal{I}^+ , (or \mathcal{I}^-) is well-posed, cf. also [10] for recent discussions. Thus, for arbitrary Cauchy data on \mathcal{I}^+ , there is a unique spacetime (M^4, g) which realizes this data at future infinity. Moreover, small but arbitrary variations of the Cauchy data give rise to small perturbations of the solution. It follows in particular that the space dS^\pm of global solutions is open; thus spaces in dS^\pm , in particular pure de Sitter space (M^4, g_{dS}) , are stable under small perturbations of the Cauchy data at \mathcal{I}^+ , (or \mathcal{I}^-). The same statement holds for perturbations of the data on a compact Cauchy surface Σ for (M^4, g) .

The purpose of this paper is to extend Friedrich's theorem to arbitrary even dimensions. Let \mathcal{I}^+ be any closed n -manifold, n odd, and let γ be any H^{s+n} smooth Riemannian metric on \mathcal{I}^+ , $s > \frac{n}{2} + 1$. Next, let τ be any H^s symmetric bilinear form on \mathcal{I}^+ satisfying the constraints

$$(1.5) \quad tr_\gamma \tau = 0, \quad \delta_\gamma \tau = 0,$$

i.e. τ is transverse-traceless with respect to γ . Define $\gamma_1 \sim \gamma_2$ and $\tau_1 \sim \tau_2$ if these data are conformally related, i.e. there exists $\lambda : \mathcal{I}^+ \rightarrow \mathbb{R}^+$ such that $\gamma_2 = \lambda^2 \gamma_1$ and $\tau_2 = f(\lambda) \tau_1$, where f is chosen so that (1.5) holds for τ_2 , cf. [5] for the exact transformation formula. Let $([\gamma], [\tau])$ be the equivalence class of (γ, τ) . Then Cauchy data for the Einstein equations (1.1) with $\Lambda > 0$ consist of triples $(\mathcal{I}^+, [\gamma], [\tau])$. The form τ corresponds to the order n behavior of the metric; roughly for \bar{g} as in (1.3), $\tau = (\partial_\Omega)^n \bar{g}|_{\mathcal{I}^+}$; see §2 for further details.

Theorem 1.1. *The Cauchy problem for the Einstein equations with Cauchy data $(\mathcal{I}^+, [\gamma], [\tau])$ at future conformal infinity is well-posed in $H^{s+n} \times H^s$, for any $s > \frac{n}{2} + 2$.*

Thus, given any Cauchy data $([\gamma], [\tau]) \in H^{s+n}(\mathcal{I}^+) \times H^s(\mathcal{I}^+)$ satisfying (1.5), up to isometry there is a unique Einstein metric $(M^{n+1}, g) \in dS^+$ whose conformal compactification as in (1.4) induces the given data $([\gamma], [\tau])$ on \mathcal{I}^+ .

This result has the following simple consequence:

Theorem 1.2. *The space dS^\pm is open with respect to the $H^{s+n} \times H^s$ topology on \mathcal{I}^+ , $s > \frac{n}{2} + 2$. Thus, given any dS solution $(M^{n+1}, g_0) \in dS^\pm$, any $H^{s+n} \times H^s$ small perturbation of the Cauchy data $([\gamma], [\tau])$ on \mathcal{I}^+ (or \mathcal{I}^-) gives rise to complete solution $(M^{n+1}, g) \in dS^\pm$ globally close to (M^{n+1}, g_0) . In particular, the even-dimensional pure de Sitter spaces g_{dS}^N in (1.3) are globally stable.*

Here, globally close is taken with respect to a natural H^s topology on the conformal compactification $\bar{M} = M \cup \mathcal{I}^+ \cup \mathcal{I}^-$, see the proof for details. The complete solution (M^{n+1}, g) induces $H^{s+n} \times H^s$ Cauchy data at both past and future conformal infinity $\mathcal{I}^-, \mathcal{I}^+$. Of course the size of the allowable perturbations in Theorem 1.2 depends on (M^{n+1}, g_0) .

We describe briefly the main ideas in the proof of Theorem 1.1; full details are given in §2. The Einstein equations (1.1) induce a 2nd order system of equations for a compactified metric \bar{g} in (1.4). However, this system is degenerate at $\mathcal{I}^+ = \{\Omega = 0\}$ and this degeneracy causes severe problems in trying to prove the well-posedness of the system. In $3 + 1$ dimensions, Friedrich [8] has developed a larger and more complicated system of evolution equations, the conformal Einstein equations, for the (unphysical) metric \bar{g} together with other variables. This expanded system is non-degenerate and shown to be symmetric hyperbolic; then standard results on such systems lead to the well-posedness of the conformal field equations. However, it seems very unlikely that this method could succeed in higher dimensions, cf. [10], due at least in part to the special form of the Bianchi equations in $3 + 1$ dimensions.

The approach taken here is to replace the Einstein equation by a more complicated but conformally invariant higher order equation for the metric alone, whose solutions include the vacuum Einstein metrics (with Λ term). In $3 + 1$ dimensions, this system is the system of 4th order Bach

equations, cf. (2.12) below. The Bach equations have been used in a number of contexts in connection with issues related to conformal infinity, cf. [14], [15], [16], for example.

In higher even dimensions, in place of the Bach tensor, we use the ambient obstruction tensor \mathcal{H} of Fefferman-Graham [6], which agrees with the Bach tensor in 3 + 1 dimensions; this tensor is also characterized as the stress-energy tensor of the conformal anomaly, cf. [5]. The tensor \mathcal{H} is a symmetric bilinear form, depending on a given metric g on M^{n+1} and its derivatives up to order $n + 1$. The equation

$$(1.6) \quad \mathcal{H} = 0$$

is conformally invariant, and includes all Einstein metrics (of arbitrary signature and Λ -term). It is a system of $(n + 1)^{\text{st}}$ -order equations in the metric, whose leading order term in suitable coordinates is of the form $\square^{\frac{n+1}{2}}$, where \square is the wave operator of the metric g . Conformal invariance implies that the system (1.6) is non-degenerate at $\mathcal{I}^+ = \{\Omega = 0\}$. Theorem 1.1 is then proved by showing that natural gauge choices for the diffeomorphism and conformal invariance of (1.6) lead again to a symmetrizable system of evolution equations.

In the context of Theorem 1.2, it is of interest to understand the closure $\overline{dS^\pm}$ of the space dS^\pm , i.e. the structure of spacetimes which are limits of spacetimes in dS^\pm but not themselves in dS^\pm . A first step in this direction was taken in [2] in 3 + 1 dimensions, and Theorem 1.1 allows one to extend this to any even dimension. Let $\overline{dS^\pm}$ be the closure of dS^\pm with respect to the $H^{s+n} \times H^s$ topology on the Cauchy data on either \mathcal{I}^+ or \mathcal{I}^- , i.e. the union of the closures with respect to data on \mathcal{I}^- and \mathcal{I}^+ . Let $\partial dS^\pm = \overline{dS^\pm} \setminus dS^\pm$ be the resulting boundary consisting of limits of spaces in dS^\pm which are not in dS^\pm .

Theorem 1.3. *For $(n + 1)$ even, a space in the boundary ∂dS^\pm of dS^\pm , is described by one of the following three configurations:*

I. A pair of solutions $(M, g^+) \in dS^+$ and $(M, g^-) \in dS^-$, each geodesically complete and globally hyperbolic. One has $\mathcal{I}^- = \emptyset$ for (M, g^+) and $\mathcal{I}^+ = \emptyset$ for (M, g^-) . Both solutions (M, g^+) and (M, g^-) are “infinitely far apart”.

II. A single geodesically complete and globally hyperbolic solution $(M, g) \in dS^+$, either with a partial compactification at \mathcal{I}^- , or $\mathcal{I}^- = \emptyset$.

III. A single geodesically complete and globally hyperbolic solution $(M, g) \in dS^-$, either with a partial compactification at \mathcal{I}^+ , or $\mathcal{I}^+ = \emptyset$.

Cases II and III have been distinguished here, but these behaviors become identical under a switch of time orientation.

One of the main points here is that singularities do not form on spaces within $\overline{dS^\pm}$. One does expect singularities to form “past” the boundary ∂dS^\pm . The most natural limits are those of type I; this behavior occurs very clearly and explicitly in the family of dS Taub-NUT metrics on $\mathbb{R} \times S^n$, cf. [2] for further discussion. It would be very interesting to know more about the structure of $\overline{dS^\pm}$; for instance, is it compact and connected?

Theorems 1.1-1.3 are proved in §2, and we close the paper with some remarks on extending these results to vacuum equations with $\Lambda \leq 0$ and to the Einstein equations coupled to matter fields.

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2. PROOFS OF THE RESULTS.

Throughout the paper, we consider globally hyperbolic vacuum spacetimes (M, g) with $\Lambda > 0$ in $(n + 1)$ dimensions. By rescaling if necessary, it is assumed that Λ is normalized to $\Lambda = n(n - 1)/2$, so that the Einstein equations read

$$(2.1) \quad Ric_g = ng.$$

The simplest solution of (2.1) is (pure) deSitter space on $M = \mathbb{R} \times S^n$, with metric (1.2), or its generalization in (1.3). These de Sitter metrics g_{dS} are geodesically complete and globally conformally compact, i.e. in dS^\pm . In fact, defining $s \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by $\cosh(t) = \frac{1}{\cos(s)}$ and letting

$$(2.2) \quad \bar{g} = \cos^2(s)g,$$

one has

$$\bar{g}_{dS} = -ds^2 + g_{S^n(1)},$$

which is the metric on the Einstein static spacetime in the region $s \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. The metric \bar{g}_{dS} is real analytic on the closure $\bar{M} = M \cup \mathcal{I}^+ \cup \mathcal{I}^-$ and the loci $\mathcal{I}^+ = \{s = \frac{\pi}{2}\} = \{t = \infty\}$, $\mathcal{I}^- = \{s = -\frac{\pi}{2}\} = \{t = -\infty\}$ represent future and past conformal infinity. The induced metric on \mathcal{I}^\pm is of course the unit round metric $g_{S^n(1)}$ on S^n . The same discussion holds for (N, g_N) as in (1.3) in place of $g_{S^n(1)}$.

Consider Einstein metrics (M^{n+1}, g) in dS^+ , so that there is a compactification

$$(2.3) \quad \bar{g} = \rho^2 g$$

as in (1.4) to future conformal infinity \mathcal{I}^+ , with $\mathcal{I}^+ = \{\rho = 0\}$; all of the analysis below works equally well for spaces in dS^- .

A compactification $\bar{g} = \rho^2 g$ as in (2.3) is called geodesic if $\rho(x) = \text{dist}_{\bar{g}}(x, \mathcal{I}^+)$. These are often the simplest compactifications to work with for computational purposes. Each choice of boundary metric $\gamma \in [\gamma]$ on \mathcal{I}^+ determines a unique geodesic defining function ρ , (and vice versa). The Gauss Lemma gives the splitting

$$(2.4) \quad \bar{g} = -d\rho^2 + g_\rho, \quad g = \rho^{-2}(-d\rho^2 + g_\rho),$$

where g_ρ is a curve of metrics on \mathcal{I}^+ . The asymptotic behavior of g at \mathcal{I}^+ is thus determined by the behavior of g_ρ as $\rho \rightarrow 0$. For example, the geodesic compactification of the de Sitter metric (1.2) with respect to the unit round metric at \mathcal{I}^+ is

$$\bar{g}_{dS} = -d\rho^2 + (1 + (\frac{\rho}{2})^2)g_{S^n(1)},$$

for $\rho \in [0, \infty)$.

Now consider a Taylor series type expansion for the curve g_ρ on \mathcal{I}^+ . This was analysed in case of asymptotically hyperbolic or AdS metrics with $\Lambda < 0$ by Fefferman-Graham [6], and for dS metrics by Starobinsky [19] when $n = 3$. This idea of course has further antecedents in the Bondi-Sachs expansion and peeling properties of the Weyl tensor when $\Lambda = 0$. In any case, the FG expansion holds equally well for metrics in dS^+ (or dS^-) in place of asymptotically AdS metrics; in fact the two expansions are very closely related, cf. [2], [18] and further references therein.

The exact form of the expansion depends on whether n is odd or even. If n is odd, then

$$(2.5) \quad g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \dots + \rho^{n-1} g_{(n-1)} + \rho^n g_{(n)} + \rho^{n+1} g_{(n+1)} + \dots,$$

with $g_{(0)} = \gamma$.

This expansion is even in powers of ρ up to order $n - 1$. The coefficients $g_{(2k)}$, $0 < k < n/2$ are locally determined by the boundary metric $\gamma = g_{(0)}$; they are explicitly computable expressions in the curvature of γ and its covariant derivatives. For example for $n \geq 3$,

$$(2.6) \quad g_{(2)} = \frac{1}{n-2} \left(\text{Ric}_\gamma - \frac{R_\gamma}{2(n-1)} \gamma \right),$$

cf. also [5], [2] for formulas for $g_{(k)}$ for $k > 2$.

The term $g_{(n)}$ is transverse-traceless, i.e.

$$(2.7) \quad \text{tr}_\gamma g_{(n)} = 0, \quad \delta_\gamma g_{(n)} = 0,$$

but is otherwise undetermined by γ and the Einstein equations (2.1); thus, at least formally, it is freely specifiable. For $k > n$, terms $g_{(k)}$ occur for k both even and odd; the term $g_{(k)}$ depends on two boundary derivatives of $g_{(k-2)}$. The main point is that all coefficients $g_{(k)}$ are locally computable expressions in $g_{(0)}$ and $g_{(n)}$.

Mathematically, the expansion (2.5) is formal, obtained by compactifying the Einstein equations and taking iterated Lie derivatives of \bar{g} at $\rho = 0$. If the geodesic compactification \bar{g} is in $C^{m,\alpha}(M)$, then the expansion holds up to order $m + \alpha$, in the sense that

$$(2.8) \quad g_\rho = g_{(0)} + \rho^2 g_{(2)} + \dots + \rho^m g_{(m)} + O(\rho^{m+\alpha}).$$

Suppose instead n is even. Then the expansion reads

$$(2.9) \quad g_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \dots + \rho^{n-2} g_{(n-2)} + \rho^n g_{(n)} + \rho^n (\log \rho) \mathcal{H} + \dots$$

Again the terms $g_{(2k)}$ up to order $n - 2$ are explicitly computable from the boundary metric γ , as is the coefficient \mathcal{H} of the $\rho^n (\log \rho)$ term. The term $g_{(n)}$ satisfies

$$tr_\gamma g_{(n)} = a, \quad \delta_\gamma g_{(n)} = b,$$

where a and b are explicitly determined by the boundary metric γ and its derivatives, but $g_{(n)}$ is otherwise undetermined by γ and the Einstein equations; as before, it is formally freely specifiable. The series (2.9) is even in powers of ρ , (at all orders) and terms of the form $\rho^{2k} (\log \rho)^j$ appear at order $> n$. Again the coefficients $g_{(k)}$ and $\mathcal{H}_{(k)}$ depend on two derivatives of $g_{(k-2)}$ and $\mathcal{H}_{(k-2)}$.

Although the expressions (2.5) and (2.9) are only formal in general, Fefferman-Graham [6] showed that if the undetermined terms $(g_{(0)}, g_{(n)})$ are analytic on the boundary \mathcal{I}^+ , then the expansion (2.5) converges, (for n odd), cf. also [2]. Thus g_ρ is analytic in ρ for ρ small and one has a dS Einstein metric in this region given by (2.4). A similar result has recently been proved by Kichenassamy [13], (cf. also [17]), for n even; in this case the polyhomogeneous expansion (2.9) converges to g_ρ for ρ small.

The term \mathcal{H} , which appears only when n is even, has a number of important interpretations. First Fefferman-Graham [6] observed that this tensor, locally computable in terms of the boundary metric γ , is a conformal invariant of γ and is (by definition) an obstruction to the existence of a formal power series expansion of the compactified Einstein metric; in fact it is the only obstruction. The tensor \mathcal{H} is also important in the (A)dS/CFT correspondence, in that (up a constant) it equals the stress-energy tensor (i.e. the metric variation) of the conformal anomaly of the corresponding CFT, cf. [5]. It also arises as the stress-energy or metric variation of the Q -curvature of the boundary metric γ , cf. [7].

The tensor \mathcal{H} is transverse-traceless

$$(2.10) \quad \delta \mathcal{H} = tr \mathcal{H} = 0,$$

and a conformal invariant of weight $2 - n$, i.e. if $\tilde{g} = \lambda^2 g$, then $\tilde{\mathcal{H}} = \lambda^{2-n} \mathcal{H}$. Further, if g is conformal to an Einstein metric, with any value of Λ , then

$$(2.11) \quad \mathcal{H} = 0.$$

In addition, as observed in [6], these properties hold for metrics of any signature and so the equation (2.11) can be viewed as a conformally invariant version of the Einstein equations with an arbitrary Λ term and arbitrary signature. We are not aware of any analogue of such a tensor in odd dimensions.

We will use the tensor \mathcal{H} to study de Sitter type solutions of the Einstein equations (2.1). Although the derivation of the obstruction tensor \mathcal{H} arises from the structure at infinity of conformally compactified odd-dimensional Einstein metrics, once it is given, one can use it to study the Einstein equations themselves in even dimensions. Thus, replacing n by $n + 1$, a vacuum solution of the Einstein equations (M^{n+1}, g) with $\Lambda > 0$, (or any Λ), in even dimensions is a solution of (2.11). For $(M^{n+1}, g) \in dS^+$, the equation (2.11), being conformally invariant, also holds for the compactified

Einstein metric \bar{g} in (2.2); moreover it has the important advantage of being a non-degenerate system of equations in \bar{g} . As is well-known [8], the translation of the Einstein equations for (M, g) to the compactified setting \bar{g} leads to a degenerate system of equations for \bar{g} .

When $n = 3$, so $\dim M = 4$, up to a constant factor \mathcal{H} is the Bach tensor B , given by

$$(2.12) \quad B = D^*D(\text{Ric} - \frac{R}{6}g) + D^2(\text{tr}(\text{Ric} - \frac{R}{6}g)) + \mathcal{R},$$

where \mathcal{R} is a term quadratic in the full curvature of g . (The specific form of \mathcal{R} will not be of concern here). In general, for $n \geq 3$ odd, one has, again up to a constant factor,

$$(2.13) \quad \mathcal{H} = (D^*D)^{\frac{n+1}{2}-2}[D^*D(P) + D^2(\text{tr}P)] + L(D^n\gamma),$$

where

$$(2.14) \quad P = P(\gamma) = \text{Ric}_g - \frac{R_g}{2n}g,$$

cf. [7] for example. This is a system of PDE's in the metric g , of order $n+1$; $L(D^n g)$ denotes lower order terms involving the metric up to order n .

The Bach equation $B = 0$ was originally developed by Bach as a conformally invariant version of the Einstein equations (with $\Lambda = 0$), and has been extensively studied in this context, cf. [14], [15], [16] for some recent work and references therein. It was also used in [3] to study regularity properties of conformally compact Riemannian Einstein metrics.

While Einstein metrics, (of any signature and Λ), are solutions of (2.11), of course not all solutions of (2.11) are Einstein. In addition, for Lorentzian metrics, \mathcal{H} is not a hyperbolic system of PDE's in any of the usual senses; the equation (2.11) is invariant under diffeomorphisms and conformal changes of the metric, and so requires at least a choice of diffeomorphism and conformal gauge to obtain a hyperbolic system.

To describe these gauge choices, suppose $(M, g) \in dS^+$, so that g is an Einstein metric, satisfying (2.1), and so (2.11), which is asymptotically dS to the future. Assume that (M, g) has a geodesic compactification which is at least C^n ; then

$$(2.15) \quad \bar{g} = \rho^2 g = -d\rho^2 + g_\rho,$$

and g_ρ has the expansion (2.8), with $m = n$, $\alpha = 0$. One has $\mathcal{I}^+ = \{\rho = 0\}$ and we set $\gamma = g_{(0)}$. By the solution to the Yamabe problem, one may assume without loss of generality that the representative $\gamma \in [\gamma]$ has constant scalar curvature, i.e.

$$(2.16) \quad R_\gamma = \text{const},$$

on \mathcal{I}^+ . However, closer study shows that the operator P in (2.14) is not well-behaved in the coordinates adapted to (2.15), i.e. the natural geodesic coordinates (ρ, y_i) , where y_i are local coordinates on \mathcal{I}^+ extended to coordinate functions on M to be invariant under the flow of $\nabla\rho$. Further, with this choice of conformal gauge, it is difficult to control the scalar curvature \bar{R} of \bar{g} .

It is simplest and most natural to choose a conformal gauge of constant scalar curvature, (although other choices are possible). Thus, set

$$(2.17) \quad \tilde{g} = \sigma^2 \bar{g},$$

where σ is chosen to make $\tilde{R} = \text{const}$. In this gauge, the equation (2.13) for \tilde{g} simplifies to

$$(2.18) \quad \mathcal{H} = (\widetilde{D^*D})^{\frac{n+1}{2}-1} \tilde{R}ic + L(D^n \tilde{g}) = 0.$$

The choice of constant for \tilde{R} is not important, but it simplifies matters if one chooses

$$(2.19) \quad \tilde{R} = \bar{R}|_{\mathcal{I}^+} = -\frac{n(n-2)}{n-1} R_\gamma \equiv c_0.$$

The middle equality follows by taking the trace of (2.6), and combining this with the Raychaudhuri equation on \bar{g} and (2.28) below.

For the diffeomorphism gauge, we choose, as usual, harmonic coordinates x_α with respect to \tilde{g} ;

$$\tilde{\square}x_\alpha = 0.$$

It is assumed that the Cauchy data for x_0 are such that x_0 is a defining function for \mathcal{I}^+ near \mathcal{I}^+ , and we relabel $x_0 = t$ so that the coordinates are (t, x_i) , $i = 1, \dots, n$. As usual, Greek letters are used for spacetime indices, while Latin is used for spatial indices. Equivalently, but from a slightly different point of view, given arbitrary local coordinates x_α , with x_0 a defining function for the boundary, the condition that x_α is harmonic with respect to \tilde{g} is

$$(2.20) \quad \tilde{\square}x_\alpha = \partial_\alpha \tilde{g}^{\alpha\beta} + \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} \partial_\alpha \tilde{g}_{\mu\nu} = 0.$$

In the coordinates x_α , the metric \bar{g} in (2.15) becomes

$$(2.21) \quad \bar{g} = g_{00}(dt^2) + 2g_{0i}dt dx_i + g_{ij}dx_i dx_j,$$

where

$$(2.22) \quad g_{00} = -(\partial_t \rho)^2, \quad g_{0i} = \partial_t \rho \partial_i \rho, \quad g_{ij} = \partial_i \rho \partial_j \rho + (g_\rho)_{ij}.$$

Similarly, for \tilde{g} , one has

$$(2.23) \quad \tilde{g} = \tilde{g}_{00}(dt^2) + 2\tilde{g}_{0i}dt dx_i + \tilde{g}_{ij}dx_i dx_j,$$

with $\tilde{g}_{\alpha\beta} = \sigma^2 g_{\alpha\beta}$.

As long as the coordinates are \tilde{g} -harmonic, the Ricci curvature has the form

$$\tilde{R}ic_{\alpha\beta} = -\frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \partial_\nu \tilde{g}_{\alpha\beta} + Q_{\alpha\beta}(\tilde{g}, \partial\tilde{g}),$$

Similarly at leading order, the Laplacian $\tilde{D}^* \tilde{D}$ has the form $\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu$ in harmonic coordinates. Thus, with these choices of gauge for the conformal and diffeomorphism invariance, the equation $\tilde{\mathcal{H}} = 0$ has the rather simple form

$$(2.24) \quad (\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu)^{\frac{n+1}{2}} \tilde{g}_{\alpha\beta} + L(D^n \tilde{g}) = 0.$$

This is an $N \times N$ system of PDE's for $\tilde{g}_{\alpha\beta}$ which is diagonal, i.e. uncoupled, at leading order, $N = (n+1)(n+2)/2$. These choices for the conformal and diffeomorphism gauges are the simplest; however, they are not necessary and other choices, for instance gauges determined by fixed gauge source functions, cf. [12], could also be used.

Having discussed the equations for the metric, we have left to determine the equations for ρ in (2.15) and σ in (2.17). The fact that ρ is a geodesic defining function for \bar{g} , i.e. $|\bar{\nabla}\rho|_{\bar{g}}^2 = -1$, implies that

$$\partial_t (g^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho) = 0,$$

or equivalently,

$$(2.25) \quad \partial_t (\sigma^2 \tilde{g}^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho) = 0.$$

To derive the equation for σ , the equation for the Ricci curvature relating \tilde{g} and \bar{g} is

$$(2.26) \quad \bar{R}ic = \tilde{R}ic + (n-1) \frac{\tilde{D}^2 \sigma}{\sigma} + \left\{ \frac{\tilde{\square} \sigma}{\sigma} - n |d \log \sigma|^2 \right\} \tilde{g}.$$

Taking the trace gives the equation relating the scalar curvatures as

$$(2.27) \quad \sigma^{-2} \bar{R} = \tilde{R} + 2n \frac{\tilde{\square} \sigma}{\sigma} - n(n+1) |d \log \sigma|_{\tilde{g}}^2.$$

Using the formula analogous to (2.26) relating the Ricci curvature of g and \bar{g} , together with the fact that g satisfies (2.1) and ρ is a geodesic defining function gives

$$(2.28) \quad \bar{R} = -2n \frac{\bar{\square}\rho}{\rho} = -2n \bar{Ric}(T, T),$$

where $T = \partial_\rho = -\nabla_{\bar{g}}\rho$. (Observe that the middle term in (2.28) is degenerate at \mathcal{I}^+ , since $\rho = 0$ there; however, the last term in (2.28) is non-degenerate at $\rho = 0$). Substituting (2.28) in (2.27) and using (2.26) gives then the equation

$$\tilde{R} + n(n-1)|d \log \sigma|_{\tilde{g}}^2 = -2n\sigma^{-2}[\tilde{Ric}(T, T) + (n-1)\frac{\tilde{D}^2\sigma}{\sigma}(T, T)],$$

or equivalently,

$$(2.29) \quad TT(\sigma) - \langle \tilde{\nabla}\sigma, \tilde{\nabla}_T T \rangle = -\frac{1}{n-1}\sigma \tilde{Ric}(T, T) - \frac{\sigma^3}{2n(n-1)}c_0 - \frac{1}{2}\sigma|d \log \sigma|_{\tilde{g}}^2,$$

where we have also used (2.19).

The equations (2.24), (2.25), and (2.29) represent a coupled system of evolution equations for the variables $(\tilde{g}_{\alpha\beta}, \rho, \sigma)$ on a domain U in $(\mathbb{R}^{n+1})^+$ with coordinates (t, x_i) ; the boundary $\partial_0 U = U \cap \{t = 0\}$ corresponds to a portion of \mathcal{I}^+ . Written out in more detail, these are:

$$(2.30) \quad (\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu)^{\frac{n+1}{2}} \tilde{g}_{\alpha\beta} = L_1(D^n \tilde{g})_{\alpha\beta},$$

$$(2.31) \quad \tilde{g}^{00} \partial_t \partial_t \rho + 2(\tilde{\nabla}^i \rho)(\partial_i \partial_t \rho) = L_2(D\rho, D\sigma, D\tilde{g}),$$

$$(2.32) \quad (\tilde{\nabla}^0 \rho)^2 \partial_t \partial_t \sigma + 2(\tilde{\nabla}^0 \rho)(\tilde{\nabla}^i \rho) \partial_i \partial_t \sigma + (\tilde{\nabla}^i \rho)(\tilde{\nabla}^j \rho) \partial_i \partial_j \sigma = L_3(D\sigma, D^2 \tilde{g}).$$

Here $D^k w$ denotes derivatives up to order k in the variable w and $\tilde{\nabla}^\alpha \rho$ denotes the α -component of $\tilde{\nabla} \rho$, $\tilde{\nabla} \rho = \tilde{g}^{\alpha\beta} \partial_\beta \rho \partial_\alpha$. The terms L_i are lower order terms. Observe that the system (2.30) for the metric $\tilde{g}_{\alpha\beta}$ is a closed sub-system, i.e. it does not involve ρ or σ . Moreover, although the equations (2.31) and (2.32) for ρ and σ are coupled to each other and to (2.30), the system (2.30)-(2.32) is uncoupled at leading order.

Following common practice, we now reduce the system (2.30)-(2.32) to a system of 1st order equations. There is not a unique way to do this, but we will discuss perhaps the simplest method, which uses pseudodifferential operators. As usual, the domain $\partial_0 U \subset \mathbb{R}^n$ is viewed as a domain in the n -torus T^n and the variables $(\tilde{g}_{\alpha\beta}, \sigma, \rho)$ are extended to functions on $I \times T^n$.

Recall that a system of 1st order evolution equations

$$(2.33) \quad \partial_t u = \sum_{j=1}^m B_j(t, x, u) \partial_j u + c(t, x, u)$$

is symmetrizable in the sense of Lax, cf. [20], [21], if there is a smooth matrix valued function $R(t, u, x, \xi)$ on $\mathbb{R} \times \mathbb{R}^p \times T^*(T^n) \setminus 0$, homogeneous of degree 0 in ξ , such that R is a positive definite $p \times p$ matrix with $R(t, u, x, \xi) \sum B_j(t, u, x) \xi_j$ self-adjoint, for each (t, u, x, ξ) . It is well-known [20], [21] that strictly hyperbolic systems of PDE, diagonal at leading order, are symmetrizable. A symmetrizer R is given by $R = \sum P_k P_k^*$, where P_k is the projection onto the k^{th} eigenspace of the symbol $\sum B_j(t, u, x) \xi_j$, $1 \leq k \leq p$.

Proposition 2.1. *There is a reduction of the system (2.30)-(2.32) to a symmetrizable system of 1st order evolution equations on $I \times T^n$.*

Proof: Consider first the closed system (2.30) for \tilde{g} . This system is not strictly hyperbolic; the leading order symbol is diagonal and has two distinct real eigenvalues, each of multiplicity $(n+1)/2$. However, the eigenspaces of the symbol of $\tilde{\square}^{\frac{n+1}{2}}$ vary smoothly and do not coalesce. Thus the operator $\tilde{\square}^{\frac{n+1}{2}}$ is strongly hyperbolic, cf. [12] and references therein. In these circumstances, it is essentially standard that the operator $\tilde{\square}^k$ is symmetrizable, for any k ; for completeness we sketch the proof following [20, §5.3].

Let $\tilde{u} = \tilde{g}_{\alpha\beta}$ be the variable in \mathbb{R}^N , $N = (n+1)(n+2)/2$. Write \square , (we drop the tilde here and below), in the form

$$(g^{00})^{-1}\square = \partial_t^2 - \sum_{j=0}^1 A_j(\tilde{u}, D_x)\partial_t^j,$$

where A_j is a differential operator in x , homogeneous of order $2-j$, depending smoothly \tilde{u} . Then

$$(2.34) \quad [(g^{00})^{-1}\square]^{(n+1)/2} = \partial_t^{n+1} - \sum_{j=0}^n B_j(\tilde{u}, D_x)\partial_t^j,$$

where B_j are differential operators in x , homogeneous of order $n+1-j$. Set $u_j = \partial_t^j \Lambda^{n-j} \tilde{u}$, for $j = 0, \dots, n$, where $\Lambda = (1 - \Delta)^{1/2}$ and Δ is the standard Laplacian on T^n . Then (2.30) becomes

$$\partial_t u_j = \Lambda u_{j+1}, \quad 0 \leq j < n, \quad \partial_t u_n = \sum_{j=0}^n B_j(Pu, D_x)\Lambda^{j-n}u_j + C(Pu),$$

where $Pu = D^n \tilde{u}$ involves \tilde{u} up to n derivatives. More precisely, for $\beta + j \leq n$, $\partial_x^\beta \partial_t^j \tilde{u} = \partial_x^\beta \Lambda^{j-n} u_j$; for example $\Lambda^{-n} u_0 = \tilde{u}$. This is a system of 1st order pseudodifferential equations in the variables $u = \{u_j\}$, $j = 0, \dots, n$ of the form

$$(2.35) \quad \partial_t u = L(Pu, D_x)u + C(Pu).$$

The eigenvalues $\lambda_\nu(w, \xi)$ of the matrix $L(w, \xi)$ are the roots of the characteristic equation $\tau^{n+1} - \sum B_j(w, \xi)\tau^j$, (up to an overall factor of i). Hence, from (2.34), one sees that for each (w, ξ) , $\xi \neq 0$, there are two distinct roots, each of multiplicity $(n+1)/2$. The eigenvalues vary smoothly with (w, ξ) and remain a bounded distance apart on the sphere $|\xi| = 1$. The same is true of the corresponding eigenspaces. Hence, the system (2.35) has a symmetrizer R constructed in the same way as following (2.33), cf. [20, Prop.5.2.C] or [21, Prop.16.2.2].

Next we show that the equation (2.31) is also symmetrizable. Let $\phi_i = -2\tilde{\nabla}^i \rho / \tilde{g}^{00}$. Introducing the vector variable $v = (\rho, \rho_0, \dots, \rho_n)$ with $\rho_0 = \partial_t \rho$, $\rho_j = \partial_j \rho$, the equation is equivalent to the system

$$\partial_t \rho = \rho_0, \quad \partial_t \rho_0 = \sum \phi_j \partial_j \rho_0 + c(t, x, v), \quad \partial_t \rho_j = \partial_j \rho_0.$$

This has the form

$$(2.36) \quad \partial_t v = \sum_{j=1}^n B_j(x, t, v)\partial_j v + c(x, t, v),$$

where B_j is an $(n+2) \times (n+2)$ matrix with ϕ_j in the $(2, 2)$ slot, 1 in the $(j+2, 2)$ slot, and 0 elsewhere. The system (2.36) is coupled at lower order to the equations (2.30) and (2.32) for \tilde{g} and σ respectively, in that B_j depends on \tilde{g} to order 0, while c depends on \tilde{g} and σ to order 1; for the moment, these dependencies are placed in the (x, t) dependence of B_j and c .

The matrix $\sum B_j \xi_j$ has the entry $\sum \phi_j \xi_j$ in the $(2, 2)$ slot, ξ_j in the $(j+2, 2)$ slot for $3 \leq j \leq n$, and 0 elsewhere. By a direct but uninteresting computation, it is straightforward to see that this matrix is symmetrizable in the sense following (2.33).

Essentially the same argument shows that the equation (2.32) for σ is again symmetrizable. Thus let $w = (\sigma, \sigma_0, \dots, \sigma_n)$ with $\sigma_0 = \partial_t \sigma$, $\sigma_j = \partial_j \sigma$. The equation (2.32) is equivalent to the system

$$\partial_t \sigma = \sigma_0, \quad \partial_t \sigma_0 = \sum \phi_j \partial_j \sigma_0 + \sum \psi_{ij} \partial_j \sigma_i + c(t, x, v), \quad \partial_t \rho_j = \partial_j \rho_0,$$

where $\phi_j = -2(\tilde{\nabla}^i \rho)/|\tilde{\nabla}^0 \rho|$, $\psi_{ij} = (\tilde{\nabla}^i \rho)(\tilde{\nabla}^j \rho)/|\tilde{\nabla}^0 \rho|^2$. This system has the form

$$(2.37) \quad \partial_t w = \sum_{j=1}^n B_j(x, t, w) \partial_j w + c(x, t, w),$$

where B_j is the $(n+2) \times (n+2)$ matrix with ϕ_j in the $(2, 2)$ slot, 1 in the $(j+2, 2)$ slot, and ψ_{ij} in the $(i+2, j+2)$ slot. The system (2.37) is again coupled at lower order to the equations (2.30) and (2.31) for \tilde{g} and ρ respectively, in that c depends on \tilde{g} to order 2, while B_j depends on \tilde{g} to order 0 and ρ to order 1. Again a straightforward but longer (uninteresting) computation shows that the matrix $\sum B_j \xi_j$ is symmetrizable in the sense of (2.33).

One may then combine the three systems (2.35), (2.36), and (2.37) to a single large system in the variable $U = (u, v, w)$. The resulting system is then a symmetrizable system of 1st order pseudodifferential equations, cf. [20], [22]. \blacksquare

Next consider the Cauchy data for the system (2.30)-(2.32). If one is interested in general solutions of this system, then the Cauchy data are essentially arbitrary, subject only to the constraint equation $\mathcal{H}(\tilde{\nabla} \rho, \cdot) = 0$ on \mathcal{I}^+ . However, as will be seen in Proposition 2.3 below, it is the specification of the Cauchy data which determines the class of conformally Einstein metrics among all solutions of (2.30)-(2.32). This is of course closely related to the FG expansion (2.8) of the dS metric g .

The Cauchy data for σ are

$$(2.38) \quad \sigma = 1, \text{ and } \partial_t \sigma = 0 \text{ at } \mathcal{I}^+,$$

while the Cauchy data for ρ are

$$(2.39) \quad \rho = 0, \text{ and } \partial_t \rho = 1 \text{ at } \mathcal{I}^+.$$

For the metric \tilde{g} , the closed subsystem (2.30) is of order $n+1$, so Cauchy data are specified by prescribing $(\partial_t)^k \tilde{g}_{\alpha\beta}$, $k = 0, 1, \dots, n$ at \mathcal{I}^+ . We compute the data inductively. First, the condition (2.38) implies that $\tilde{g}_{ij} = \bar{g}_{ij}$ at \mathcal{I}^+ . Thus at order 0, set

$$(2.40) \quad \tilde{g}_{00} = -1, \quad \tilde{g}_{0i} = 0, \quad \tilde{g}_{ij} = \gamma_{ij} \text{ at } \mathcal{I}^+,$$

since $\rho_i = 0$ at \mathcal{I}^+ .

At 1st order, (2.38) and (2.39) together with (2.23) show that $\partial_t \tilde{g}_{ij} = \partial_t \bar{g}_{ij}$ at \mathcal{I}^+ , and the FG expansion (2.8) gives $\partial_t \bar{g}_{ij} = 0$ at \mathcal{I}^+ . Thus, set

$$(2.41) \quad \partial_t \tilde{g}_{ij} = 0 \text{ at } \mathcal{I}^+.$$

(This condition, and related ones below, are necessary to obtain Einstein metrics). The first derivatives of the mixed components $\tilde{g}_{0\alpha}$ of \tilde{g} are determined by the requirement that the coordinates $x_\alpha = (t, x_i)$ are harmonic at \mathcal{I}^+ with respect to \tilde{g} , i.e. for each β ,

$$(2.42) \quad \partial_\alpha \tilde{g}^{\alpha\beta} + \frac{1}{2} \tilde{g}^{\alpha\beta} \tilde{g}^{\mu\nu} \partial_\alpha \tilde{g}_{\mu\nu} = 0.$$

Via (2.40)-(2.41), this determines $\partial_t \tilde{g}_{0\alpha}$ at \mathcal{I}^+ .

At 2nd order, the equation (2.27) implies, using the normalization (2.19), that $\partial_t^2 \sigma = 0$ at \mathcal{I}^+ . Also, $\partial_t^2(\rho_i \rho_j) = 0$, and hence, from the FG expansion (2.8), we set

$$(2.43) \quad \partial_t^2 \tilde{g}_{ij} = 2g_{(2)} \text{ at } \mathcal{I}^+.$$

The 2nd derivatives $\partial_t \tilde{g}_{0\alpha}$ are then determined by (2.43), the lower order Cauchy data, (2.38)-(2.41), and the t -derivative of (2.42) at $t = 0$.

At 3rd order, suppose first $n = 3$, so that $\dim M = 4$. Then a straightforward computation, using (2.8), the Raychaudhuri equation on \bar{g} and (2.28), shows that $\partial_t \bar{R} = 6n \text{tr} g_{(3)} = 0$ at \mathcal{I}^+ , where the last equality follows from (2.7). Hence, (2.27) gives $\partial_t^3 \sigma = 0$. Similarly, (2.39) gives $\partial_t^3 (\rho_i \rho_j) = 0$. Thus, set

$$(2.44) \quad \partial_t^3 \tilde{g}_{ij} = 6g_{(3)} \quad \text{at } \mathcal{I}^+.$$

This term is free or unconstrained, subject to the transverse-traceless constraint (2.7). As before the mixed term at order 3, $\partial_t^3 \tilde{g}_{0\alpha}$ is determined by taking two t -derivatives of (2.42) at $t = 0$, and using (2.44) together with the determination of the lower order Cauchy data.

Suppose instead $n > 3$ and hence $n \geq 5$. Then $g_{(3)} = 0$ and same arguments as above give

$$(2.45) \quad \partial_t^3 \tilde{g}_{ij} = 0 \quad \text{at } \mathcal{I}^+,$$

with $\partial_t^3 \tilde{g}_{0\alpha}$ again determined from two t -derivatives of (2.42), (2.45) and lower order Cauchy data.

At 4th order on \mathcal{I}^+ , (assuming $n \geq 5$), by (2.27) and the fact that $\partial_t^k \sigma = 0$, $k \leq 3$ on \mathcal{I}^+ , one has $\partial_t^4 \sigma = -2n \partial_t^2 \bar{R}$, and again computations as above then give $\partial_t^2 \bar{R} = 24n \text{tr} g_{(4)}$. Also, taking i -derivatives of (2.31) or (2.25) and using (2.38)-(2.39) shows that $\partial_t^4 (\rho_i \rho_j) = 0$ at \mathcal{I}^+ . It follows that, at $t = 0$,

$$(2.46) \quad \partial_t^4 \tilde{g}_{ij} = 24(g_{(4)} - 2n^2 \text{tr} g_{(4)}) \gamma.$$

Again, $\partial_t^4 \tilde{g}_{0\alpha}$ is determined by taking three t -derivatives of (2.42) at $t = 0$, and using (2.46) with the determination of the lower order Cauchy data.

At 5th order, suppose $n = 5$. As in the case $n = 3$, $\partial_t^3 \bar{R} = \text{ctr} g_{(5)} = 0$ at \mathcal{I}^+ while $\partial_t^5 (\rho_i \rho_j) = 0$. Hence, as in the case $n = 3$,

$$(2.47) \quad \partial_t^5 \tilde{g}_{ij} = (5!) g_{(5)},$$

which is freely specifiable, subject to the transverse-traceless constraint. As before the mixed term at order 5, $\partial_t^5 \tilde{g}_{0\alpha}$ is determined by taking four t -derivatives of (2.42) at $t = 0$, and using (2.47) with the determination of the lower order Cauchy data.

If $n > 5$, then as before,

$$(2.48) \quad \partial_t^5 \tilde{g}_{ij} = 0,$$

with $\partial_t^5 \tilde{g}_{0\alpha}$ again determined from (2.42). It is clear that one can continue inductively in this way to determine the Cauchy data $\partial_t^k \tilde{g}_{\alpha\beta}$ up to order n . Since $\partial_t^6 (\rho_i \rho_j) \neq 0$ at $t = 0$, these and higher derivative terms contribute to the Cauchy data at order 6 and above. However, one sees by differentiations of (2.31) and (2.42) that these terms are all determined by lower order Cauchy data for \tilde{g} .

In sum, Cauchy data for $\tilde{g}_{\alpha\beta}$, i.e. the data $\partial_t^k \tilde{g}_{\alpha\beta}$, $0 \leq k \leq n$, are determined by the Cauchy data (2.38), (2.39) for ρ and σ , the equations (2.31)-(2.32), (or (2.25), (2.27)) for ρ and σ , the harmonic equation (2.42), and the coefficients $g_{(k)}$ in the FG expansion (2.8). Thus, the Cauchy data are uniquely determined in terms of the free data

$$(2.49) \quad \gamma = g_{(0)}, \quad \text{and} \quad \tau = g_{(n)},$$

which are arbitrary, subject to the transverse-traceless constraint (2.7) on $g_{(n)}$ and the constant scalar curvature constraint (2.16) or (2.19) on the representative $\gamma \in [\gamma]$. Abusing notation slightly, we will call (γ, τ) Cauchy data for $\tilde{g}_{\alpha\beta}$, since this data determines the rest of the Cauchy data $\partial_t^k \tilde{g}_{\alpha\beta}$, $0 < k < n$ uniquely.

The analysis above then gives:

Proposition 2.2. *The system (2.30)-(2.32) for $(\tilde{g}_{\alpha\beta}, \rho, \sigma)$ is well-posed in $H^{s+n}(\mathcal{I}^+) \times H^s(\mathcal{I}^+)$, $s > \frac{n}{2} + 2$. Thus, given Cauchy data $(\gamma, \tau) \in (H^{s+n}(\mathcal{I}^+), H^s(\mathcal{I}^+))$ satisfying (2.38), (2.39) and (2.49), and satisfying the constraints (2.7), (2.19), there is a unique solution $(\tilde{g}_{\alpha\beta}, \rho, \sigma)$ of (2.30)-(2.32) with*

$$(2.50) \quad (\tilde{g}_{\alpha\beta}, \rho, \sigma) \in C(I, H^{s+n}(\mathcal{I}^+)) \cap C^n(I, H^s(\mathcal{I}^+)).$$

Further, if (γ_s, τ_s) is a continuous curve in $H^{s+n}(\mathcal{I}^+) \times H^s(\mathcal{I}^+)$, then the solutions $(\tilde{g}_s, \rho_s, \sigma_s)$ vary continuously with s .

Proof: In the local coordinates (t, x_i) , the system (2.30)-(2.32) is symmetrizable and so for given Cauchy data on T^n , it has a unique solution on $I \times T^n$ satisfying (2.50), with T^n in place of \mathcal{I}^+ , cf. [20, §5.2-5.3] or also [21], [22]. The existence of such local solutions holds for $s > \frac{n}{2} + 1$. By restriction, one thus obtains local solutions on the domain $I \times U \subset I \times T^n$, for $U \subset \mathcal{I}^+$ as preceding (2.30).

To prove these local solutions obtained from domains on $I \times \mathcal{I}^+$ patch together to give a unique solution on $I \times \mathcal{I}^+$, it is necessary and sufficient to prove that the system (2.30)-(2.32) has finite domains of dependence (or equivalently uniqueness in the local Cauchy problem). This is well-known to be true for symmetric hyperbolic systems of PDE's, (cf. [22, §IV.4] for example); however, the reduction of (2.30) is to a symmetric system of first order pseudodifferential equations, for which finite propagation speed is not true in general. Nevertheless, standard methods show that solutions of (2.30)-(2.32) do have local uniqueness in the Cauchy problem.

Thus, consider the closed subsystem (2.30) for \tilde{g} . A standard argument using the mean-value theorem shows that it suffices to prove local uniqueness in the Cauchy problem for the associated linear equation $L(\tilde{g}) = 0$, where the coefficients of (2.30) are frozen at a given metric \tilde{g}_0 , cf. [12] for instance. The linear operator L is self-adjoint at leading order, and hence by Proposition 2.1, one has existence and uniqueness of solutions to the Cauchy problem on $I \times T^n$ for the adjoint equation $L^*u = \phi$. It is then well-known, cf. [22, Thm.IV.4.3], that this suffices to prove local uniqueness in the Cauchy problem for L , and hence for the system (2.30). Given the local uniqueness for \tilde{g} , exactly the same method can be applied to (2.31)-(2.32) to give local uniqueness for ρ and σ .

It follows that the system (2.30)-(2.32) does have finite propagation speed, and hence the local solutions patch together uniquely to give a unique global solution on $I \times \mathcal{I}^+$ satisfying (2.50). As is well-known in the case of Einstein metrics, (cf. [12] for example), the patching of local coordinate charts requires an extra derivative; the same analysis holds here, so that we assume $s > \frac{n}{2} + 2$.

A lower bound for the time of existence $I = [0, t_0)$ of the solution depends only on an upper bound on the norm of the initial data in $H^{s+n} \times H^s$. This implies that the last statement is an immediate consequence of the existence and uniqueness theorem. ■

Given a solution $(\tilde{g}_{\alpha\beta}, \rho, \sigma)$ of the Cauchy problem, one may then construct the “physical” metric g by setting

$$(2.51) \quad g = \rho^{-2} \sigma^{-2} \tilde{g}.$$

Since σ is bounded away from 0 and ∞ near \mathcal{I}^+ and ρ is a geodesic defining function, it is easy to see that the metric g is future geodesically complete, i.e. geodesically complete to the future of some Cauchy surface $\Sigma \subset (M^{n+1}, g)$. However, it is not so clear that the metric g is Einstein, or equivalently that the metric \tilde{g} is conformally Einstein. For this, one needs to verify first that the gauge condition (2.42) that the coordinates remain harmonic, is preserved for the solution \tilde{g} . If this is so, it then follows that \tilde{g} is a solution of the equation (2.11). Secondly, the equation (2.11) admits solutions which are not conformally Einstein, and so one needs to verify that the constructed solution \tilde{g} actually is conformally Einstein.

Both of these conditions on \tilde{g} can be verified by computation; however, the computations will be somewhat long and involved. Instead, we verify both conditions together by using the following simple conceptual technique, based on analyticity.

Proposition 2.3. *Any solution $(\tilde{g}_{\alpha\beta}, \rho, \sigma)$ of the Cauchy problem in Proposition 2.2 defines an Einstein metric in dS^+ via (2.51).*

Proof: Suppose first that the free data $(\gamma, g_{(n)})$ are analytic on \mathcal{I}^+ (or on a domain in \mathcal{I}^+). As noted above, the expansion (2.5) then converges to g_ρ and gives a solution, denoted g_E , to the Einstein equations (1.1); this metric has the form (2.4), with compactification \bar{g}_E , in a neighborhood of \mathcal{I}^+ . In particular, both \bar{g}_E and ρ are analytic near \mathcal{I}^+ . Moreover, since the coefficients of the equation (2.27) defining σ are then also analytic, and since the Cauchy data (2.38) for σ are analytic, the Cauchy-Kowalewsky theorem ([21, §16.4]) shows that σ and hence $\tilde{g}_E = \sigma^2 \bar{g}_E$ are analytic near \mathcal{I}^+ . Of course the metric \tilde{g}_E is a solution of (2.11), and hence a solution of (2.30)-(2.32) with Cauchy data determined by (2.38)-(2.39) and $(\gamma, g_{(n)})$.

On the other hand, let $(\tilde{g}, \rho, \sigma)$ be the solution to the Cauchy problem (2.30)-(2.32) with Cauchy data determined by (2.38)-(2.39) and $(\gamma, g_{(n)})$ given by Proposition 2.2. Since $(\tilde{g}_E, \rho, \sigma)$ is also a solution of this Cauchy problem, with the same initial data, it follows from the uniqueness part of Proposition 2.2 that $\tilde{g}_E = \tilde{g}$ in a neighborhood of \mathcal{I}^+ . Moreover, the results in [1] show that \tilde{g}_E remains analytic within its globally hyperbolic development, so that $\tilde{g}_E = \tilde{g}$ everywhere in the domain of \tilde{g} . Hence $g = \rho^{-2} \sigma^{-2} \tilde{g}$ is an Einstein metric in dS^+ , realizing the given data on \mathcal{I}^+ . (In particular, this shows that the harmonic gauge (2.42) must be preserved for analytic initial data).

Now analytic data $(\gamma, g_{(n)})$ are dense in $H^{s+n} \times H^s$ data with respect to the $H^{s+n} \times H^s$ topology on \mathcal{I}^+ . The Cauchy stability given by Proposition 2.2 implies that if $(\gamma_i, g_{(n),i})$ is a sequence of analytic Cauchy data converging in $H^{s+n} \times H^s$ to $H^{s+n} \times H^s$ data $(\gamma, g_{(n)})$, then the corresponding solutions $(\tilde{g}_i, \rho_i, \sigma_i)$ also converge to the (unique) solution $(\tilde{g}, \rho, \sigma)$ with Cauchy data $(\gamma, g_{(n)})$. Hence the metric g in (2.51) is Einstein. ■

Propositions 2.2-2.3 give the existence of an Einstein metric $(M^{n+1}, g) \in dS^+$, with arbitrarily prescribed asymptotic behavior (γ, τ) on \mathcal{I}^+ , subject to the constraints (2.7) and (2.19). Suppose $[\gamma'] = [\gamma]$ and $[\tau'] = [\tau]$ as following (1.5), so that $\gamma' = \lambda^2 \gamma$ and $\tau' = f(\lambda) \tau$. Given λ , there exists a diffeomorphism $\phi : \bar{M} \rightarrow \bar{M}$, with $\phi|_{\mathcal{I}^+} = id$, such that $\lim_{\rho \rightarrow 0} \phi^*(\rho) / \rho = \lambda^{-1}$, where ρ is the geodesic defining function determined by γ . Setting $g' = \phi^* g$, one has $\bar{g}' = \rho^2 \phi^*(g) = \phi^*(\bar{g}) (\frac{\rho}{\phi^*(\rho)})^2$. Hence, the boundary metric of \bar{g}' equals γ' . Similarly, by the uniqueness, any Einstein metric (M^{n+1}, g') with Cauchy data (γ', τ') satisfying (1.5) differs from (M^{n+1}, g) by a diffeomorphism ϕ equal to the identity on \mathcal{I}^+ . This completes the proof of Theorem 1.1. ■

Remark 2.4. Theorem 1.1 is formulated as a global result, in the sense that future conformal infinity \mathcal{I}^+ is a compact smooth manifold. However, Propositions 2.1-2.3 all hold locally, by the finite propagation speed of the system (2.30)-(2.32). Hence, Theorem 1.1 also holds locally, where \mathcal{I}^+ is an open manifold with a finite number of local charts. Of course the uniqueness statement then holds only within the domain of dependence of the initial data.

Many of the standard solutions of the Einstein equations (2.1) have \mathcal{I} non-compact; this is the case for instance for the dS Schwarzschild metrics.

Proof of Theorem 1.2.

Let (M, g_0) be a dS Einstein metric in dS^\pm with Cauchy data $([\gamma^+], [g_{(n)}^+])$ and $([\gamma^-], [g_{(n)}^-])$ induced on \mathcal{I}^+ and \mathcal{I}^- respectively. Thus, there exists a smooth defining function $\Omega : \bar{M} \rightarrow \mathbb{R}$ such that $\bar{g}_0 = \Omega^2 g_0$ extends to a metric on $\bar{M} = M \cup \mathcal{I}^+ \cup \mathcal{I}^- \simeq I \times \Sigma$; here Σ is a Cauchy surface for

(M, g_0) and I is a *compact* time interval. We will choose Ω to be a geodesic defining function ρ in a neighborhood of \mathcal{I}^+ and \mathcal{I}^- so that \bar{g}_0 is $C(I \times H^{s+n}(\Sigma)) \cap C^n(I \times H^s(\Sigma))$ up to \bar{M} . The choice of Ω defines representatives $(\gamma^+, g_{(n)}^+)$, $(\gamma^-, g_{(n)}^-)$ in the conformal classes $([\gamma^+], [g_{(n)}^+])$, $([\gamma^-], [g_{(n)}^-])$. In the following, we work in the $H^{s+n} \times H^s$ topology.

Let \mathcal{U}^+ be an open neighborhood of Cauchy data on \mathcal{I}^+ containing the given data $(\gamma^+, g_{(n)}^+)$. Then for all data $(\hat{\gamma}^+, \hat{g}_{(n)}^+) \in \mathcal{U}^+$, there exists $T < \infty$, depending on \mathcal{U}^+ , such that the maximal globally hyperbolic dS Einstein metric (M^{n+1}, \hat{g}) having Cauchy data $(\hat{\gamma}^+, \hat{g}_{(n)}^+)$ given by Theorem 1.1 is defined on $[T, \infty) \times \Sigma$; here the time factor is proper time $t = -\log \frac{\rho}{2}$, where ρ is the geodesic defining function. The Cauchy data of such solutions \hat{g} at $\Sigma = \{T\} \times \Sigma$ then forms an open set \mathcal{U}^T in the space of Cauchy data for the Einstein equations (2.1) on Σ . By passing to an open subset $\mathcal{V}^T \subset \mathcal{U}^T$ if necessary, the Cauchy stability theorem for the (standard) Einstein equations implies that the maximal globally hyperbolic development of any \hat{g} with data in \mathcal{V}^T contains the region $[-T, T] \times \Sigma$, and induces again an open set of Cauchy data \mathcal{V}^{-T} at $\{-T\} \times \Sigma$. Then, as above with \mathcal{I}^+ , there is an open set \mathcal{U}^- of Cauchy data on \mathcal{I}^- whose future development gives a non-empty open subset of \mathcal{V}^{-T} . Combining these three unique developments gives a global solution $(M^{n+1}, \hat{g}) \in dS^\pm$, which completes the proof of Theorem 1.2. ■

Remark 2.5. The proof above shows that the space dS^\pm is also stable with respect to perturbations of the Cauchy data (Σ, γ, K) , (satisfying the constraint equations), on a compact Cauchy surface $\Sigma \subset (M^{n+1}, g)$ in the $H^{s+n} \times H^{s+n-1}$ topology on Σ . It is well-known, cf. [4] that this is not the case for perturbations of asymptotically flat Cauchy data when $\Lambda = 0$, in that smoothness of the resulting space-time at conformal infinity is lost for generic perturbations.

Proof of Theorem 1.3.

This result is proved for $n = 3$ in [2], and it is pointed out there that the same proof holds provided one has the Cauchy stability result of Theorem 1.1, (i.e. Friedrich's result [8] when $n = 3$). Given then Theorem 1.1, the proof of Theorem 1.3 is exactly the same as that given in [2], to which we refer for details. ■

Remark 2.6. This paper has focussed on the de Sitter case $\Lambda > 0$ mainly for simplicity, but also because there are no direct analogues of Theorems 1.2 or 1.3 when $\Lambda = 0$ or $\Lambda < 0$, due to the more complicated nature of conformal infinity. Nevertheless, one expects that the analogues of Theorem 1.1 for $\Lambda = 0$ and $\Lambda < 0$, as formulated and proved by Friedrich [8] in the case $n = 3$, hold for all even dimensions. When $\Lambda < 0$, future space-like infinity \mathcal{I}^+ is replaced by time-like infinity \mathcal{I} , while when $\Lambda = 0$, \mathcal{I}^+ is replaced by future null infinity.

We hope to discuss these situations elsewhere.

Remark 2.7. We close with a brief remark on the applicability of the methods used above to the Einstein equations coupled to other matter fields. As noted above, up to multiplicative constants, the tensor \mathcal{H} is the metric variation, or stress-energy tensor, of the conformal anomaly or of the Q -curvature. The conformal invariance of these functionals corresponds to the conformal invariance of \mathcal{H} . For functionals containing the metric coupled to other fields which are conformally invariant, and whose field equations are symmetric hyperbolic, it seems very likely that the methods used above will again lead to a well-posed Cauchy problem at \mathcal{I}^+ , as in Theorem 1.1. In dimensions $3 + 1$, this is the case for the Einstein equations coupled to gauge fields, i.e. Einstein-Maxwell or Einstein-Yang-Mills fields. Theorem 1.1 has already been proved in this situation by Friedrich [11], and so at best one would have a different method of proof of this result. In higher dimensions, the

EM or YM action is not conformally invariant, and it is less clear if the method can be adapted to this situation.

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