ON THE LOCAL RIGIDITY OF EINSTEIN MANIFOLDS WITH CONVEX BOUNDARY

MICHAEL T. ANDERSON

Abstract. Let (M, g) be a compact Einstein manifold with non-empty boundary ∂M. We prove that Killing fields at ∂M extend to Killings fields of (any) (M, g) provided ∂M is (weakly) convex and $\pi_1(M, \partial M) = \{e\}$. This gives a new proof of the classical infinitesimal rigidity of convex surfaces in Euclidean space and generalizes the result to Einstein metrics of any dimension.

1. Introduction

A well-known result in classical differential geometry is the infinitesimal isometric rigidity of convex surfaces Σ in Euclidean space $\mathbb{R}^3$. Thus, if X is an infinitesimal deformation of Σ in $\mathbb{R}^3$, i.e. a vector field along the embedding $F : \Sigma \to \mathbb{R}^3$, and if X preserves the metric $\gamma = F^*(g_{Eucl})$ on Σ to first order,

$$\mathcal{L}_X \gamma = 0,$$

then X is a rigid motion of $\mathbb{R}^3$, and so the restriction of a Killing field on $\mathbb{R}^3$ to Σ. In various regularity classes, this was proved by Liebmann, Blaschke and Weyl, cf. [7], [10] for discussion and further references.

This result is false for general smooth surfaces in $\mathbb{R}^3$ and it a question of basic interest to understand how broad the class of surfaces is for which it remains true.

In this paper, we give a new proof of the infinitesimal or local isometric rigidity of convex surfaces in $\mathbb{R}^3$. This is a special case of a much more general result for convex or weakly convex boundaries ∂M of $(n+1)$-dimensional Einstein metrics, $n \geq 2$. To state the result, let M be any compact $(n+1)$-dimensional manifold with boundary ∂M and suppose g is an Einstein metric on M, so that g satisfies the Einstein equations

$$Ric_g = \lambda g, \tag{1.1}$$

for some constant $\lambda \in \mathbb{R}$. The metric g induces a Riemannian metric γ on the closed n-manifold ∂M. Define ∂M to be $(n-1)$-convex if the symmetric form

$$-\tau = H\gamma - A > 0 \tag{1.2}$$

is positive definite. Here A is the second fundamental form of ∂M in M with respect to the outward normal N and $H = trA$ is the mean curvature. The condition (1.2) is equivalent to the statement that the sum of any $(n-1)$ eigenvalues of A is positive. Thus for $n = 2$, $A > 0$ and (1.2) is equivalent to the usual notion of convexity. The condition (1.2) becomes progressively weaker in higher dimensions. The form τ in (1.2) is the conjugate momentum to the boundary metric γ in the setting of general relativity, cf. Section 2 for further discussion.

In the context of Einstein metrics on manifolds with boundary, one may consider two distinct notions of rigidity or uniqueness of the structure (M, g) with respect to the boundary metric $(\partial M, \gamma)$. First, one may consider Einstein deformations of the metric g fixing the boundary metric. At the infinitesimal level, these are symmetric forms κ on M preserving the Einstein equations (1.1) to first order, so $E'(\kappa) = 0$, which vanish on the boundary: $\kappa^T = 0$, where T denotes the restriction

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of a tensor at ∂M to ∂M. Such deformations may change both the local isometry type of the metric g in the interior as well as deform the location of the boundary. Secondly, one may consider deformations of the boundary ∂M within a thickening of M, leaving the metric g on M essentially fixed (up to isometry). Namely, in many situations, the Einstein metric (M,g) will extend to a larger domain (M',g') with $(M,g) \subset (M',g')$ and so one may consider two-sided deformations of $(\partial M,\gamma)$ within (M',g'). At the infinitesimal level, such deformations of the metric are of the form $k = \delta^* X$, for some vector field X on M, not necessarily tangent to ∂M at ∂M. (The existence of an extension of the Einstein metric to a larger domain is immaterial in the context of infinitesimal deformations).

In the first case, infinitesimal rigidity means that any infinitesimal Einstein deformation κ as above with $\kappa^T = 0$ on ∂M is trivial modulo "gauge", i.e. $\kappa = \delta^* Z$ with $Z = 0$ on ∂M. If κ is then in divergence-free gauge, it follows that $\kappa = 0$ on M, cf. Lemma 2.6. However, infinitesimal Einstein rigidity in this form is false in general; counterexamples occur on the curve of Schwarzschild and similar types of Einstein metrics, cf. [3] and Remark 2.4 for further discussion.

The second case, infinitesimal isometric rigidity (rigidity of the boundary structure $(\partial M,\gamma)$ within a fixed (M,g)), means that any deformation of the form $k = \delta^* X$ with $k^T = 0$ on ∂M is again pure gauge, i.e. of the form $k = \delta^* Z$ with $Z = 0$ on ∂M. Thus the infinitesimal isometry X at ∂M extends to a Killing field of (M,g), in that the vector field $X' = X - Z$ is a Killing field on M extending X on ∂M.

Of course in dimension 3, these two notions of rigidity agree, at least when M is simply connected, since 3-dimensional Einstein metrics are of constant curvature and hence locally isometric.

The main result of the paper is the following:

Theorem 1.1. *Let M be a compact $(n+1)$-dimensional manifold with boundary $\partial M \neq \emptyset$ and g an Einstein metric on M, $C^{m,\alpha}$ smooth up to ∂M, with $m \geq 5$. Suppose $\pi_1(M,\partial M) = \{e\}$ and suppose that ∂M is $(n-1)$-convex in M, so that* (1.2) *holds. Then any infinitesimal isometry X at ∂M extends to a Killing field of (M,g).*

Theorem 1.1 shows in particular that any continuous group of symmetries at $(\partial M,\gamma)$ extends to a group of symmetries of any Einstein metric (M,g) bounding $(\partial M,\gamma)$, provided the boundary is $(n-1)$-convex and $\pi_1(M,\partial M) = \{e\}$. Thus

$$Isom_0(\partial M,\gamma) \subset Isom_0(M,g),$$

where $Isom_0$ is the connected component of the identity of the isometry group. For example (under the hypotheses above) any Einstein metric (M,g) whose boundary metric is homogeneous must be of cohomogeneity one. This gives very strong restrictions on the geometry and topology of such Einstein fillings and leads to global rigidity or uniqueness results in certain situations. For instance, it shows that the only Einstein metric (M,g) with $(\partial M,\gamma)$ equal to a round metric on a sphere is a metric of constant curvature on a ball, (again under the convexity and π_1 conditions above).

Theorem 1.1 generalizes to weakly $(n-1)$-convex boundaries where $-\tau \geq 0$, provided the "singular" set $Z = \{det\, \tau = 0\}$ has empty interior. This is discussed further in Remark 2.12. The regularity condition $m \geq 5$ is likely not optimal and can probably be improved.

The condition $\pi_1(M,\partial M) = \{e\}$ means that ∂M is connected and the inclusion map induces a surjection $\pi_1(\partial M) \to \pi_1(M) \to 0$. This condition is necessary for the validity of Theorem 1.1. For instance, if ∂M is not connected, distinct Killing fields on each component of ∂M may not extend to a common Killing field on M.

To place Theorem 1.1 in a general context, consider the space $\mathbb{E}^{m.\alpha} = \mathbb{E}^{m,\alpha}_\lambda(M)$ of all Einstein metrics (1.1) on M (with λ fixed) which are $C^{m,\alpha}$ up to the boundary ∂M. It is proved in [5] that $\mathbb{E}^{m,\alpha}$ is a smooth Banach manifold, at least when $\pi_1(M,\partial M) = \{e\}$ and $m \geq 5$. The group

$\mathrm{Diff}_1^{m+1,\alpha}(M)$ of $C^{m+1,\alpha}$ diffeomorphisms of M equal to the identity on ∂M acts smoothly on $\mathbb{E}^{m,\alpha}$ and the quotient (the moduli space of Einstein metrics)

$$\mathcal{E}^{m,\alpha} = \mathbb{E}^{m,\alpha}/\mathrm{Diff}_1^{m+1,\alpha}, \tag{1.3}$$

is also a smooth Banach manifold. Moreover, the (Dirichlet) boundary map

$$\Pi : \mathcal{E}^{m,\alpha} \to Met^{m,\alpha}(\partial M), \tag{1.4}$$

$$\Pi[g] = \gamma,$$

is C^{∞} smooth. It is not however Fredholm; the range of $D\Pi$ is always of infinite codimension when $m < \infty$. Infinitesimal Einstein rigidity (rigidity within the class of Einstein metrics) is equivalent to the injectivity of the derivative map $D\Pi$. However, as noted above, rigidity in this sense does not always hold.

Next, given $g \in \mathbb{E}^{m,\alpha}$, let $\mathbb{B}^{m,\alpha} = \mathbb{B}_g^{m,\alpha} \subset \mathcal{E}^{m,\alpha}$ be the subset of $C^{m,\alpha}$ Einstein metrics g' on M such that there exists a smooth open domain $U \subset M$, diffeomorphic to M, on which g' is isometric to g. By analytic continuation, g' is thus everywhere locally isometric to g away from ∂M; only the location of the boundary is being changed. Let $\mathcal{B}^{m,\alpha} \subset \mathcal{E}^{m,\alpha}$ be the associated moduli space as in (1.3). The space $\mathcal{B}^{m,\alpha}$ is again a smooth Banach manifold. Tangent vectors to $\mathcal{B}^{m,\alpha}$ are equivalence classes $[h] = [\delta^* X]$, where X is a $C^{m+1,\alpha}$ vector field on M and $\delta^* X_1 \sim \delta^* X_2$ if $X_2 - X_1 = Z$ vanishes on ∂M. One again has a (Dirichlet) boundary map

$$\Pi_B : \mathcal{B}^{m,\alpha} \to Met^{m,\alpha}(\partial M), \tag{1.5}$$

$$\Pi_B[g] = \gamma.$$

Theorem 1.1 implies that $D\Pi_B$ is injective when ∂M is $(n-1)$-convex. This is also equivalent to the statement that, for forms in zero divergence gauge,

$$Ker D\Pi \cap Im\delta^* = 0.$$

It is interesting to consider the relation of the injectivity and surjectivity of $D\Pi$. When $n = 2$, ∂M is convex and M is simply connected, the method of proof of Theorem 1.1 easily shows that $D\Pi = D\Pi_B$ is surjective, modulo loss of one derivative, cf. Proposition 2.11. One may then use the Nash-Moser inverse function theorem [12], [11], (cf. in particular [16] which deals with the case of finite differentiability), to show that (M, g) is also locally rigid, so that the map Π is also everywhere locally one-to-one, hence locally a diffeomorphism, modulo loss of one derivative. In higher dimensions this is false in general, cf. again Remark 2.4. Also, it is interesting to note that when ∂M is not convex or simply connected, $D\Pi_B$ may be injective without being surjective, cf. Remark 2.3 for further discussion.

The proof of Theorem 1.1 is based on and related to the proof of the isometry extension theorem in [3], where the same result is proved for infinitesimal isometric deformations X which preserve the mean curvature $X(H) = 2H'_{\delta^* X} = 0$, without however any assumptions on the convexity of ∂M in M.

The starting point of the proof is an analysis of the 2^{nd} variation of the Einstein-Hilbert action for Riemannian metrics with Gibbons-Hawking-York boundary term, well-studied in general relativity. This is also the point of view in an interesting recent work of Izmestiev [10], which gives another new proof of infinitesimal isometric rigidity of convex surfaces in dimension 3. (Theorem 1.1 answers a conjecture in [10]). However, the use of the Einstein-Hilbert action and its analysis here (and in [3]) appears to be quite different than that in [10]. We also point out earlier interesting work of Schlenker [13] which proves analogs of Theorem 1.1 in certain special situations.

2. Proofs

The Einstein-Hilbert action with Gibbons-Hawking-York boundary term on M is

$$I(g) = I_{EH}(g) = -\int_M (R_g - 2\Lambda) dV_g - 2\int_{\partial M} H dv_\gamma, \tag{2.1}$$

where R_g is the scalar curvature of g and $\Lambda = \frac{n-1}{2}\lambda$, cf. [9], [15]. The 1^{st} variation of I in the direction h is given by

$$\frac{d}{dr} I(g + rh) = \int_M \langle E_g, h\rangle dV_g + \int_{\partial M} \langle \tau, h^T\rangle dv_\gamma, \tag{2.2}$$

where E is the Einstein tensor,

$$E_g = Ric_g - \frac{R}{2} g + \Lambda g, \tag{2.3}$$

and $\tau = A - H\gamma$ is the conjugate momentum to γ. (The conjugate momentum arises frequently in connection with the constraint and evolution equations in general relativity). Observe that Einstein metrics with $Ric_g - \lambda g = 0$ are exactly the critical points of I, among variations h vanishing on ∂M, i.e. $h^T = 0$ on ∂M.

Remark 2.1. The formulas (2.1)-(2.2) give a very simple proof (and generalization) of the results of Almgren-Rivin [2] and Rivin-Schlenker [14] on the invariance of the total mean curvature of ∂M under metric or boundary deformations. Thus suppose h is any deformation of (M, g) preserving the metric γ on ∂M to first order, so that $h^T = 0$. Then (2.2) gives, since g is Einstein, $I'(h) = 0$, and hence by (2.1)

$$\left(\int_{\partial M} H dv_g\right)' = -\frac{1}{2}\left(\int_M (R - 2\Lambda)\right)'.$$

In particular if $\lambda = 0$, (so $R = 0$) and if the deformation h preserves the scalar curvature, i.e. $R'_h = 0$, then

$$\left(\int_{\partial M} H dv_g\right)' = 0.$$

Now consider a 2-parameter family of metrics $g_{r,s} = g + rh + sk$ where $E_g = 0$. Then

$$\frac{d^2}{dsdr} I(g_{r,s}) = \frac{d^2}{drds} I(g_{r,s}). \tag{2.4}$$

Computing the left side of (2.4) by taking the derivative of (2.2) in the direction k gives

$$\frac{d^2}{dsdr} I(g_{r,s}) = \int_M \langle E'(k), h\rangle dV_g + \int_{\partial M} \langle \tau'_k + a(k^T), h^T\rangle dv_\gamma. \tag{2.5}$$

Since $E_g = 0$, there are no further derivatives of the bulk integral in (2.2). Also, $a(k) = -2\tau \circ k + \frac{1}{2}(tr_\gamma k)\tau$ arises from the variation of the metric and volume form in the direction k; by definition $(\tau \circ k)(V, W) = \frac{1}{2}\{\langle \tau(V), k(W)\rangle + \langle \tau(W), k(V)\rangle\}$.

Similarly, for the right side of (2.4) one has

$$\frac{d^2}{drds} I(g_{r,s}) = \int_M \langle E'(h), k\rangle dV_g + \int_{\partial M} \langle \tau'_h + a(h^T), k^T\rangle dv_\gamma. \tag{2.6}$$

In particular, suppose k is an infinitesimal Einstein deformation $E'(k) = 0$ preserving the boundary metric γ to first order, so that $k|_{\partial M} = k^T = 0$. If $h \in T\mathbb{E}$ is any infinitesimal Einstein deformation, then (2.4)-(2.6) gives,

$$\int_{\partial M} \langle \tau'_k, h^T\rangle dv_\gamma = \int_{\partial M} \langle \tau'_h, k^T\rangle dv_\gamma = 0. \tag{2.7}$$

One thus has

$$I''(k,h) = 0,$$

on-shell, i.e. for all infinitesimal Einstein deformations h. It follows that

$$\tau'_k \perp ImD\Pi. \tag{2.8}$$

Suppose for the moment that the derivative $D\Pi$ of the Dirichlet boundary map Π in (1.4) is surjective, or more precisely has dense range in $S^{m,\alpha}(\partial M) = TMet^{m,\alpha}(\partial M)$. It then follows from (2.8) that

$$(\tau'_k)^T = 0. \tag{2.9}$$

Taking the trace of this equation, using $k^T = 0$, it follows that $(A'_k)^T = 0$, so that, roughly speaking, k vanishes to 2nd order at ∂M, (modulo "gauge" terms involving $k(N,\cdot)$). We then use the following unique continuation result from [4]:

Proposition 2.2. *Suppose $(M,g) \in \mathbb{E}^{m,\alpha}$ with $m \geq 5$ and k is an infinitesimal Einstein deformation satisfying*

$$k^T = (A'_k)^T = 0, \tag{2.10}$$

on some open set $U \subset \partial M$. Then in a neighbhorhood of U in M, there is a vector field Z, with $Z = 0$ on U, such that

$$k = \delta^* Z. \tag{2.11}$$

Furthermore, if (2.10) *holds on all of ∂M and $\pi_1(M, \partial M) = \{e\}$, then Z is globally defined on M and* (2.11) *holds globally on M. In particular, if k is assumed to be in divergence-free gauge,*

$$\delta k = 0,$$

on M (which can always be arranged without loss of generality) then $k = 0$ on M.

∎

It follows from the discussion above that if $D\Pi$ in (1.4) has dense range, (thus is surjective in a weak sense) then (M,g) is infinitesimally rigid among Einstein metrics, so that $D\Pi$ is injective. We note also that the condition $m \geq 5$ in Theorem 1.1 comes mainly from this assumption in Proposition 2.2.

Remark 2.3. It would be interesting to know if the converse of the statement above holds, i.e. whether injectivity of $D\Pi$ implies $D\Pi$ is weakly surjective. This seems to be unknown even in 3 dimensions, i.e. $n = 2$.

For example, consider a standard round torus of revolution $(T^2, \gamma_0) \subset \mathbb{R}^3$. This bounds a C^∞ flat metric g_0 on the solid torus $M = D^2 \times S^1 \subset \mathbb{R}^3$. It is known that (M, g_0) or (T^2, γ_0) is infinitesimally rigid among isometric embeddings of $T^2 \subset \mathbb{R}^3$, so that $D\Pi_B$ is injective. However, Han-Lin [8] have shown that $D\Pi_B$ is not weakly surjective at g_0. The space of metrics on T^2 near γ_0 which isometrically embed in $\mathbb{R}^3$ is of codimension 1 in $Met(T^2)$.

However, it seems plausible that the codimension 1 restriction (which comes from a period condition) is related to the fact that flat metrics on M may have non-trivial holonomy around the generator of $\pi_B(M) \simeq \mathbb{Z}$ and so do not embed or immerse in $\mathbb{R}^3$; (only the universal cover $\widetilde{M}$ immerses isometrically in $\mathbb{R}^3$). Thus $\Pi_B \neq \Pi$ in this situation; infinitesimal deformations are of the form $k = \delta^* X$ locally, but not necessarily globally on M. While $D\Pi_B$ is not weakly surjective, it is possible that $D\Pi$ is.

Remark 2.4. We note there are a number of examples of Einstein metrics (M, g) which are not infinitesimally Einstein rigid, even when ∂M is convex and umbilic. Consider for example the curve of Riemannian Schwarzschild metrics g_m on $\mathbb{R}^2 \times S^2$, given by

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^2(1)},$$

where $V = V(r) = 1 - \frac{2m}{r}$, $r \geq 2m > 0$ with $\theta \in [0, \beta]$, $\beta = 8\pi m$. This is a curve of complete Ricci-flat metrics, but the metrics g_m differ from each other just by rescalings and diffeomorphisms. Taking the derivative with respect to m gives an infinitesimal Einstein deformation κ of g_m:

$$\kappa = \frac{2}{r}[1 - \frac{2m}{r}]^{-2}dr^2 + [1 - \frac{3m}{r}]d\theta^2.$$

On the compact manifold $M = \{2m \leq r \leq 3m\} \simeq S^2 \times D^2$ with boundary $S^2 \times S^1$ at $r = 3m$ one has $\kappa^T = 0$, so that $\kappa \in KerD\Pi$, with $\kappa \neq 0$. Thus (M, g_m) is not infinitesimally Einstein rigid. A simple computation shows that ∂M is both convex and umbilic, cf. [3] for further details. Of course κ is not of the form $\delta^* X$ for some vector field X.

Further examples in both four and higher dimensions can be deduced from the work in [1] and further references therein.

We turn now to infinitesimal isometric deformations of (M, g) in place of the more general Einstein deformations. These are of the form $k = \delta^* X$, for some vector field X on M, and as above we assume k preserves the boundary metric to first order, so that $(\delta^* X)^T = 0$. Using (2.4)-(2.6) but now with the deformation h arbitrary gives

$$\int_{\partial M} \langle \tau'_k, h^T \rangle = \int_{\partial M} \langle \tau'_h, (\delta^* X)^T \rangle + \int_M \langle E'_h, \delta^* X \rangle = \int_M \langle E'_h, \delta^* X \rangle.$$

Applying the divergence theorem and using the fact that $\delta E'_h = 0$ (from the linearization of Bianchi identity) gives

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h^T \rangle = \int_{\partial M} E'_h(N, X). \tag{2.12}$$

Equation (2.12) holds for any h.

Arguing as above, the issue is to understand when the right side of (2.12) can be shown to vanish, for arbitrary h^T on ∂M. When such holds, then Proposition 2.2 implies that $k = \delta^* X = 0$ so that Theorem 1.1 follows; here we implicitly use Lemma 2.6 below to see that k may be brought to divergence-free gauge.

First we note the following general result on the form of τ'_k.

Lemma 2.5. *If k is an infinitesimal Einstein deformation of (M, g) with $k^T = 0$ on ∂M, then*

$$\delta(\tau'_k) = 0, \quad \text{and} \quad \langle \tau'_k, A \rangle = 0, \tag{2.13}$$

pointwise on ∂M.

Proof: The first equation in (2.13) follows by applying (2.12), with $h = \delta^* V$ where V is any smooth vector field on M tangent to ∂M. Such deformations are infinitesimal Einstein deformations so that the right hand side of (2.12) vanishes, and the result follows from the divergence theorem on ∂M.

In a similar way, the second equation in (2.13) follows by applying (2.12) with $h = \delta^* V$ where V is a smooth vector field on M with $V = fN$ at ∂M, where N is the unit outward normal vector field and f is arbitrary.

Alternately, (and equivalently), (2.13) follows by differentiating the Gauss-Codazzi equations for $(\partial M, \gamma) \subset (M, g)$, i.e. from the linearization of the divergence and scalar (or Hamiltonian) constraints

$$\delta\tau = 0, \tag{2.14}$$

$$|\tau|^2 - \frac{1}{n-1}(tr\tau)^2 + R_\gamma = |A|^2 - H^2 + R_\gamma = R_g - 2Ric_g(N, N), \tag{2.15}$$

in the direction of an infinitesimal Einstein deformation vanishing on ∂M.

■

To carry the analysis further, we use elliptic equations to analyse in more detail $ImD\Pi$. To do this, one needs to introduce a gauge since the diffeomorphism invariance of the Einstein equations implies they do not form an elliptic system. The most natural gauge for our purposes is the divergence-free gauge. Thus, given any background Einstein metric $\widetilde{g}$, consider the operator

$$\Phi_{\widetilde{g}} : Met^{m,\alpha}(M) \to S^{m-2,\alpha}(M),$$

$$\Phi_{\widetilde{g}}(g) = Ric_g - \frac{s}{2}g + \Lambda g + \delta_g^*\delta_{\widetilde{g}}(g). \tag{2.16}$$

The linearization of Φ at $g = \widetilde{g}$ is

$$L(h) = \tfrac{1}{2}[D^*Dh - 2R(h) - D^2trh - (\delta\delta h)g + \Delta trh\, g + \lambda trh\, g], \tag{2.17}$$

where D^2 is the Hessian and $\Delta = trD^2$ the Laplacian (with respect to g). It is straightforward to see that L is an elliptic operator (in the interior of M). Moreover, since $\Phi_{\widetilde{g}} = E + \delta^*\delta$, one has

$$E' = L - \delta^*\delta. \tag{2.18}$$

It is easy to see that L is formally self-adjoint; this also follows directly from the symmetry of the 2^{nd} derivatives in (2.4)-(2.6). Of course solutions of $L(h) = 0$ with $\delta h = 0$ on M are infinitesimal Einstein deformations.

The following simple Lemma gives a converse of the statement above and will be used often.

Lemma 2.6. *If $L(h) = 0$ on M and $\delta h = 0$ on ∂M, then*

$$\delta h = 0 \quad \text{on} \quad M,$$

so that h is an infinitesimal Einstein deformation.

For any $h \in S^{m,\alpha}(M)$ there is a $C^{m+1,\alpha}$ vector field Z on M with $Z = 0$ on ∂M such that $\widetilde{h} = h + \delta^*Z$ satisfies

$$\delta\widetilde{h} = 0.$$

Proof: By (2.18), $\delta(L(h)) = \delta\delta^*(\delta h)$, since $\delta E' = 0$, by the linearized Bianchi identity. Thus

$$\delta\delta^*(\delta h) = 0.$$

Pairing this with δh and applying the divergence theorem gives the first result.

For the second result, consider the equation $\delta\delta^*Z = -\delta h$ with Dirichlet boundary condition $Z = 0$. This is an elliptic boundary value problem, with trivial kernel, and so has a unique solution. This gives the result.

■

Since M is a manifold with boundary, one should consider elliptic boundary conditions for

$$L : S^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

In [3], it was shown that boundary data of the form

$$\delta h = 0, \quad [h^T]_B = h_1, \quad tr_\sigma A'_h = \langle A'_h, \sigma\rangle = h_2 \;\; \text{at} \;\; \partial M, \tag{2.19}$$

form a well-posed elliptic boundary value system for L. Here $\sigma > 0$ is any $C^{m,\alpha}$ smooth Riemannian metric and

$$-\tau_B = (tr_\gamma B)\gamma - B > 0 \tag{2.20}$$

is also assumed to be a $C^{m,\alpha}$ smooth Riemannian metric on ∂M; $[h^T]_B$ is the usual equivalence relation mod B, i.e. $h_1 \sim h_2$ if and only if $h_2 = h_1 + fB$, for some smooth function f on ∂M. The Fredholm index of the boundary value problem is zero, cf. again [3].

In the following, we choose $B = A$, so B is the 2nd fundamental form of ∂M in M. This is natural in light of the second equation in (2.13). Moreover, the $(n-1)$-convexity condition (1.2) is then just the statement that (2.20) holds.

However, A involves one derivative of the metric, so that if $g \in C^{m,\alpha}(M)$, $A \in C^{m-1,\alpha}(\partial M)$. Hence $h^T \in C^{m,\alpha}(\partial M)$ but $[h^T]_A$ only makes sense as an equivalence relation on $C^{m-1,\alpha}(\partial M)$ symmetric forms; it does not preserve the regularity of $h^T \in C^{m,\alpha}(\partial M)$. This loss of one derivative, well-known in isometric embedding problems, is closely related to the fact that the boundary map (1.4) is not Fredholm (elliptic) and to the diffeomorphism invariance of the Einstein equations. It is also a consequence of the scalar constraint (2.15), cf. [3] for further discussion.

To deal with this issue, we choose a C^∞ or $C^{m,\alpha}$ smoothing A_δ of A, with $A_\delta \to A$ in $C^{m-1,\alpha}(\partial M)$ as $\delta \to 0$. Throughout the following we fix such a smoothing.

Let $S_0^{m,\alpha}(M) = S_{0,\delta,\sigma}^{m,\alpha}(M)$ be the space of $C^{m,\alpha}$ symmetric bilinear forms on M such that at ∂M,

$$\delta h = 0, \quad [h^T]_{A_\delta} = 0, \quad \langle A'_h, \sigma\rangle = 0.$$

The operator

$$L : S_0^{m,\alpha}(M) \to S^{m-2,\alpha}(M) \tag{2.21}$$

is thus an elliptic operator of Fredholm index 0. In particular, ImL is of finite codimension in $S^{m-2,\alpha}(M)$.

Suppose for the moment L is surjective for some $\delta > 0$ small, and for some choice of σ, so that L is then in fact an isomorphism. Consider the boundary value problem

$$L(h) = 0, \quad \delta h = 0, \ [h^T]_{A_\delta} = h_1, \ \langle A'_h, \sigma\rangle = h_2. \tag{2.22}$$

Since L in (2.21) is a bijection, it follows from a standard subtraction procedure that (2.22) has a unique solution, for arbitrary h_1 and h_2. Namely, take any symmetric form v satisfying the boundary conditions in (2.22) and extend v to a smooth form on M (arbitrarily but smoothly) so that $L(v) = w$, for some w. Let h_0 be the unique solution of $L(h_0) = w$ with zero boundary values, as in (2.21). Then $h = v - h_0$ solves (2.22). By Lemma 2.6, solutions of (2.22) are infinitesimal Einstein deformations.

It follows then that all equivalence classes $[h^T]_{A_\delta}$ are realized as boundary values of $C^{m,\alpha}$ infinitesimal Einstein deformations. Moreover, each point fA in the "fiber" is also the boundary value of an $C^{m-1,\alpha}$ infinitesimal Einstein deformation, namely $h^T = \delta^*(fN)$. At $g \in \mathcal{E}^{m,\alpha}$, consider the extension of $D\Pi = D_g\Pi$ to $C^{m-1,\alpha}$ deformations, so $D_g\Pi : T_g\mathcal{E}^{m-1,\alpha} \to S^{m-1,\alpha}(\partial M)$. Note that $C^{m,\alpha}$ is dense in $C^{m-1,\alpha}$ in the $C^{m-1,\alpha'}$ topology, for any $\alpha' < \alpha$. It follows that if δ is sufficiently small, $ImD_g\Pi$ is dense in $S^{m-1,\alpha}(\partial M)$:

$$\overline{ImD_g\Pi} = S^{m-1,\alpha}(\partial M), \tag{2.23}$$

where the closure is taken in the $C^{m-1,\alpha'}$ topology, $\alpha' < \alpha$. Thus again via Proposition 2.2 as above, Theorem 1.1 is proved in this situation.

However, there is no good reason to believe that L is surjective (for some δ small) and so we will carry the method above further to include cases where L is not surjective. Since the Fredholm index is zero, this occurs when the kernel K of L in (2.21) is non-trivial. The kernel consists forms k satisfying

$$L(k) = 0, \quad \delta k = 0, \quad [k^T]_{A_\delta} = 0, \quad \langle A'_k, \sigma\rangle = 0.$$

To proceed further, we will need to choose σ so that L in (2.21) is self-adjoint. The following proposition shows this occurs for $\sigma = \tau$, when $\delta = 0$.

Proposition 2.7. *For the equation $L(h) = \ell$, the boundary data*

$$\delta h = 0, \quad [h^T]_A = 0, \quad tr_\tau A'_h = \langle A'_h, \tau\rangle = 0 \tag{2.24}$$

forms an elliptic formally self-adjoint boundary value problem.

Proof: The proof is a simple modification of the proof of the same result in the case $B = \sigma = \gamma$ given in [6], [5]. To begin, via integration by parts one has

$$\int_M \langle L(h), k\rangle + \int_{\partial M} \langle \beta(h), k\rangle = \int_M \langle h, L(k)\rangle + \int_{\partial M} \langle \beta(k), h\rangle,$$

where

$$\langle \beta(h), k\rangle = \langle \nabla_N k, h\rangle + h(N, dtrk) - (\delta k)(N)trh - trhN(trk). \tag{2.25}$$

To prove the self-adjoint property, one needs to show that

$$\int_{\partial M} \zeta(h, k) = 0,$$

where $\zeta(h, k) = \beta(h, k) - \beta(k, h)$ is the skew-symmetric part of β. The forms h and k are arbitrary, subject to the conditions that

$$h^T = \varphi_h A,\ k^T = \varphi_k A,\ \delta h = \delta k = 0,\ \text{and } \langle A'_h, \tau\rangle = \langle A'_k, \tau\rangle = 0.$$

A straightforward computation (cf. also [3]) shows that the gauge equations $(\delta k)(T) = 0$ and $(\delta k)(N) = 0$ on ∂M are equivalent to

$$(\nabla_N k)(N)^T = \delta(k^T) - \alpha(k(N)), \tag{2.26}$$

$$N(k_{00}) = \delta(k(N)^T) + \varphi_k|A|^2 - k_{00}H, \tag{2.27}$$

respectively, where $\alpha(k(N)) = A(k(N)) + Hk(N)^T$. (Of course the same equations hold for h). Here N is the unit outward normal at ∂M and $k_{00} = k(N, N)$.

To start with (2.25), one has

$$\langle \nabla_N k, h\rangle = N(k_{00})h_{00} + 2\langle \nabla_N k(N), h(N)^T\rangle + \langle (\nabla_N k)^T, h^T\rangle.$$

Using the standard formula

$$2A'_k = \nabla_N k + 2A \circ k - 2\delta^*(k(N)^T) - \delta^*(k_{00}N), \tag{2.28}$$

the last term above may be rewritten as

$$\langle (\nabla_N k)^T, h^T\rangle = \langle 2A'_k - 2A \circ k + 2\delta^*(k(N)^T), \varphi_h A\rangle + k_{00}\varphi_h|A|^2.$$

Hence

$$\langle \nabla_N k, h\rangle = N(k_{00})h_{00} + 2\langle \delta(k^T), h(N)^T\rangle \tag{2.29}$$

$$-\langle \alpha(k(N)), h(N)^T\rangle + \langle 2A'_k - 2A \circ k + 2\delta^*(k(N)^T), \varphi_h A\rangle + k_{00}\varphi_h|A|^2.$$

In the following, we drop terms which are symmetric in h and k, since they disappear when passing to ζ. Thus the term $\langle A \circ k, \varphi_h A\rangle$ is symmetric, since $k = \varphi_k A$, as is the α term. Similarly, since

$2\delta^*(k(N)^T), \varphi_h A\rangle = 2\langle k(N)^T, \delta(h^T)\rangle$ modulo divergence terms, the second and second-to-last terms on the right in (2.29) drop away. Hence the remaining part of (2.29) is:

$$N(k_{00})h_{00} + 2\langle A'_k, \varphi_h A\rangle + k_{00}\varphi_h|A|^2.$$

Next, again mod divergence terms,

$$h(N, dtrk) - trhN(trk) = \langle h(N)^T, dtrk\rangle - N(trk)(trh - h_{00}) = trk\delta(h(N)^T) - \varphi_h HN(trk).$$

Taking the trace of (2.28) gives

$$2H'_k = N(trk) + 2\delta(k(N)^T) - k_{00}H - N(k_{00}), \tag{2.30}$$

so that

$$h(N, dtrk) - trhN(trk) = (k_{00} + \varphi_k H)\delta(h(N)^T) - \varphi_h H(2H'_k - 2\delta(k(N)^T) + k_{00}H + N(k_{00})).$$

Thus one is left to skew-symmetrize:

$$N(k_{00})h_{00} + 2\langle A'_k, \varphi_h A\rangle + k_{00}\varphi_h|A|^2 + (k_{00} + \varphi_k H)\delta(h(N)^T)$$

$$-\varphi_h H[2H'_k - 2\delta(k(N)^T) + k_{00}H + N(k_{00})].$$

Using the gauge condition (2.27), one obtains after some cancelations,

$$\zeta(h,k) = 2\varphi_h[\langle A'_k, A\rangle - HH'_k] - 2\varphi_k[\langle A'_h, A\rangle - HH'_h],$$

again modulo divergence terms. Next, note that $\langle A'_h, \tau\rangle = \langle A'_h, A\rangle - Htr_\gamma A'_h = \langle A'_h, A\rangle - HH'_h - H\langle A, h\rangle = \langle A'_h, A\rangle - HH'_h - \varphi_h H|A|^2$. Thus modulo divergence terms

$$\zeta(h,k) = 2\varphi_h\langle A'_k, \tau\rangle - 2\varphi_k\langle A'_h, \tau\rangle.$$

This proves that the boundary value problem (2.24) is self-adjoint.

The fact that (2.24) forms a elliptic boundary value problem (in a formal sense) follows directly from [3], [6].

■

Remark 2.8. It would be interesting to know if the self-adjoint property in Proposition 2.7 arises from a variational problem. For example, Dirichlet boundary data $h^T = 0$ are self-adjoint for the operator L; this follows directly from the equality of mixed derivatives in (2.4) and the formula (2.5). The question is whether there is an analog of the Einstein-Hilbert action with Gibbons-Hawking-York boundary term whose critical points correspond to the vanishing of the last two terms in (2.24).

Since the boundary value problem is formally self-adjoint, it follows that on the space

$$S_0^{m,\alpha}(M) = \{h \in S^{m,\alpha}(M) : \delta h = 0,\ [h^T]_A = 0,\ \langle A'_h, \tau\rangle = 0 \ \text{ on } \ \partial M\},$$

the operator

$$L : S_0^{m,\alpha}(M) \to S^{m-2,\alpha}(M)$$

satisfies

$$K \subset (ImL)^\perp, \tag{2.31}$$

i.e. the kernel K is contained in the annihilator of ImL on L^2. As above, let

$$S_{0,\delta}^{m,\alpha}(M) = \{h \in S^{m,\alpha}(M) : \delta h = 0,\ [h^T]_{A_\delta} = 0,\ \langle A'_h, \tau\rangle = 0 \ \text{ on } \ \partial M\},$$

so that

$$L : S_{0,\delta}^{m,\alpha}(M) \to S^{m-2,\alpha}(M).$$

Then L is Fredholm, of Fredholm index 0, and it follows from (2.31) and the regularity properties of elliptic operators that for $\delta > 0$ sufficiently small one has a direct sum decomposition

$$ImL \oplus K = S_2^{m-2,\alpha}(M), \tag{2.32}$$

nearly orthogonal in L^2; here $K = K_\delta$.

We will need to go beyond this setting and consider other, related boundary conditions. Thus we also consider boundary value problems

$$L(h) = \ell, \quad \delta h = h_0, \quad [h^T]_{A_\delta} = h_1, \quad \langle A'_h, \sigma\rangle + \varepsilon H'_h = h_2, \tag{2.33}$$

where $\sigma > 0$ is any smooth symmetric form on ∂M close to τ and ε is sufficiently small, (depending only on (M, g)). The associated kernel $K_\sigma = K_{\sigma,\varepsilon,\delta}$ consists of forms k_σ satisfying

$$L(k_\sigma) = 0, \quad \delta k_\sigma = 0, \quad [k_\sigma^T]_{A_\delta} = 0, \quad \langle A'_{k_\sigma}, \sigma\rangle + \varepsilon H'_{k_\sigma} = 0.$$

Note that $dimK_\sigma$ may depend on σ, (and ε, δ), but for σ close to A, $dimK_\sigma \leq dimK$. Although the boundary value problem (2.33) is no longer formally self-adjoint, for σ sufficiently close to τ and ε, δ sufficiently small, one still has the analog of (2.32),

$$ImL \oplus K_\sigma = S^{m-2,\alpha}(M). \tag{2.34}$$

In (2.34), the operator L acts as

$$L : S_0^{m,\alpha}(M) \to S^{m-2,\alpha}(M),$$

where, abusing notation, $S_0^{m,\alpha}(M) = S_{0,\delta,\varepsilon,\sigma}^{m,\alpha}(M)$ is given by $C^{m,\alpha}$ forms h on M satisfying

$$\delta h = 0, \quad [h^T]_{A_\delta} = 0, \quad \langle A'_h, \sigma\rangle + \varepsilon H'_h = 0, \tag{2.35}$$

on ∂M.

Again if $K_\sigma = 0$ for some σ (and δ, ε small) then Theorem 1.1 is proved in the same way as above: since $D\Pi$ has dense range in the space of equivalence classes $S^{m-1,\alpha}(\partial M)/[A]$ and also maps onto the fibers fA, $ImD\Pi$ is dense in $S^{m-1,\alpha}(\partial M)$ and the result follows as following (2.23).

If $K_\sigma \neq 0$ for all σ, one needs to understand in more detail the form of elements $k \in K_\sigma$. This appears however rather difficult. Instead we seek a substitute space Q_σ for K_σ which has the same basic properties as K_σ whose elements $q \in Q_\sigma$ are effectively computable and have desired properties. This is accomplished in the following Proposition.

Proposition 2.9. *There exist smooth forms σ (arbitrarily) near τ and a finite dimensional space Q_σ with $dimQ_\sigma = dimK_\sigma$, consisting of smooth forms of the type*

$$q = \psi g + D^2 f, \tag{2.36}$$

satisfying the conditions:

$$\delta q = 0, \tag{2.37}$$

$$ImL \oplus Q_\sigma = S^{m-2,\alpha}(M), \tag{2.38}$$

as well as

$$\int_{\partial M} fX(H) = 0, \tag{2.39}$$

where X is the given infinitesimal isometry at ∂M.

(The reason for the requirement (2.39) will be clear below, cf. (2.58)).

Proof: For forms q of the type (2.36), one has

$$\delta q = -d\psi - d\Delta f - \lambda df,$$

since $\delta D^2 f = -d\Delta f - Ric(df)$. Define ψ by $\psi = -\Delta f - \lambda f$ and, for the moment, let f be arbitrary in $C^{m,\alpha}(M)$. Thus (2.37) holds.

To establish the slice property (2.38), by the direct sum decomposition (2.34) it suffices to show that for each $q \in Q_\sigma$ there exists $k \in K_\sigma$ such that

$$\int_M \langle q, k\rangle \neq 0, \tag{2.40}$$

so that Q_σ has no elements orthogonal to K_σ. In addition, we require that the forms q satisfy

$$\int_{\partial M} fX(H) = 0.$$

Now computing (2.40) gives, since $\delta k = 0$,

$$\int_M \langle q, k\rangle = \int_M \psi trk + \int_{\partial M} \langle k(N), df\rangle = \int_M \psi trk + \int_{\partial M} \delta(k(N)^T)f + k_{00}N(f).$$

Set

$$\alpha = \int_{\partial M} \delta(k(N)^T)f + k_{00}N(f),$$

so

$$\int_M \langle q, k\rangle = \int_M \psi trk + \alpha = -\int_M (\Delta f + \lambda f)trk + \alpha. \tag{2.41}$$

On the other hand, since $L(k) = 0$ and $\delta k = 0$ (by Lemma 2.6), taking the trace of (2.17) gives

$$\Delta trk + \lambda trk = 0.$$

Using this and integration by parts gives

$$-\int_M \psi trk = \int_M (\Delta f + \lambda f)trk = \int_{\partial M} N(f)trk - N(trk)f$$

$$= \int_{\partial M} N(f)k_{00} + N(f)H\varphi - N(trk)f = \alpha + \int_{\partial M} -\delta(k(N)^T)f + N(f)H\varphi - N(trk)f,$$

$$= \alpha + \int_{\partial M} N(f)H\varphi - 2H'_k f - H\varphi f,$$

where for the last equality we have used (2.30) and (2.27). Also $k^T = \varphi A$ on ∂M. Substituting this in (2.41) gives then the basic formula

$$\int_M \langle q, k\rangle = -\int_{\partial M} \varphi[HN(f) - |A|^2 f] - 2H'_k f. \tag{2.42}$$

This holds for all $k = k_\sigma \in K_\sigma$, for any σ, with $\delta = 0$ and holds approximately for $\delta > 0$ sufficiently small.

The basic issue is to show that for each non-zero $k \in K_\sigma$ there is q such that (2.42) is non-zero. This, together with showing that such forms are linearly independent over a basis of K_σ gives the slice property (2.40).

Observe that if (2.42) vanishes for all choices of f, then necessarily

$$\varphi = 0 \ \text{ and } \ H'_k = 0.$$

Namely one can set $f = 0$ and $N(f)$ arbitrary on ∂M to obtain $\varphi = 0$; given this one can then choose f arbitrary to obtain $H'_k = 0$. Regarding the condition (2.39), if $\varphi \neq 0$, one can choose $f = 0$ in (2.42), so that (2.39) holds trivially. If $\varphi = 0$, this requires further work.

The discussion above holds for each choice of smooth symmetric form σ and each $k_\sigma \in K_\sigma$. In particular, it applies to the "original" case $\sigma = \tau$. For any σ as above, consider the "reduced kernel" $\widetilde{K}_\sigma \subset K_\sigma$ consisting of those $k_\sigma \in K_\sigma$ with $\varphi_{k_\sigma} = 0$. Let L_σ be a complement for $\widetilde{K}_\sigma$ so that $K_\sigma = \widetilde{K}_\sigma \oplus L_\sigma$. If ℓ_j is a basis for L_σ, then the boundary values φ_j ($\ell_j^T = \varphi_j \gamma$ on ∂M) are linearly independent. Hence, by choosing $f_j \in C^{m,\alpha}(M)$ such that $f_j = 0$ on ∂M and $N(f_j)$ suitably on ∂M, one obtains the slice property

$$\int_M \langle q, \ell \rangle \neq 0 \tag{2.43}$$

for L_σ, i.e. for all $\ell \in L_\sigma$ there exists q such that (2.43) holds. This gives a space Q_{L_σ} with $dim Q_{L_\sigma} = dim L_\sigma$. Also (2.39) holds on Q_{L_σ}.

We now choose σ as follows. For the original choice $\sigma = \tau$, consider first an "enlarged" kernel $\mathcal{K}_\tau$ consisting of forms k such that

$$L(k) = 0, \ \delta k = 0, \ [k^T]_A = 0 \text{ and } \langle A'_k, \tau \rangle + \varepsilon H'_k \in \langle X(H) \rangle.$$

Similarly, consider the reduced part $\widetilde{\mathcal{K}}_\tau \subset \mathcal{K}_\tau$ where $\varphi = 0$. Now choose σ such that (as functions on ∂M)

$$\langle A'_k, \sigma \rangle + \varepsilon H'_k \notin \langle X(H) \rangle, \tag{2.44}$$

for all non-zero $k \in \widetilde{\mathcal{K}}_\tau$. (If $\widetilde{\mathcal{K}}_\tau = 0$, then $K_\tau = L_\tau$, so that (2.43) gives the required slice property for K_σ, $\sigma = \tau$). If some $k_\sigma \in K_\sigma$ satisfies $k_\sigma = k \in \widetilde{\mathcal{K}}_\tau$, then one has of course $\langle A'_k, \sigma \rangle + \varepsilon H'_k \notin \langle X(H) \rangle$ by (2.44) but by definition of K_σ, $\langle A'_{k_\sigma}, \sigma \rangle + \varepsilon H'_{k_\sigma} = 0$, a contradiction. Thus $k_\sigma \notin \widetilde{\mathcal{K}}_\tau$ for all k_σ, i.e.

$$K_\sigma \cap \widetilde{\mathcal{K}}_\tau = 0,$$

for all σ satisfying (2.44).

Now the defining property of $\widetilde{\mathcal{K}}_\tau$ is that $k \in \widetilde{\mathcal{K}}_\tau$ if and only if $\langle A'_k, \tau \rangle + \varepsilon H'_k \in \langle X(H) \rangle$ and $\varphi = \varphi_k = 0$. Hence $k_\sigma \notin \widetilde{\mathcal{K}}_\gamma$ if and only if either $\langle A'_k, \tau \rangle + \varepsilon H'_k \notin \langle X(H) \rangle$ or $\varphi_{k_\sigma} \neq 0$. If the latter holds, then $k_\sigma \in L_\sigma$ and so (2.43) gives the slice property. If $\varphi_{k_\sigma} = 0$, $k_\sigma \in \widetilde{K}_\sigma$ but $\langle A'_{k_\sigma}, \tau \rangle + \varepsilon H'_{k_\sigma} \notin \langle X(H) \rangle$. However, $\langle A'_k, \tau \rangle = 0$ on any infinitesimal Einstein deformation k with $\varphi_k = 0$; this follows by differentiating the scalar constraint (2.15). Hence $\langle A'_k, \tau \rangle + \varepsilon H'_{k_\sigma} \notin \langle X(H) \rangle$ if and only if $\varepsilon H'_{k_\sigma} \notin \langle X(H) \rangle$, (and so in particular $H'_{k_\sigma} \neq 0$). If k_j is a basis of $\widetilde{K}_\sigma$, then the functions H'_{k_j} are linearly independent. Thus again via (2.42) a suitable choice of basis functions $\{f_j\}$ gives the slice property as in (2.43) on $\widetilde{K}_\sigma$; together with (2.43), this gives the slice property for all of K_σ. Moreover, the boundary functions f_j can be chosen so that

$$\int_{\partial M} f_j X(H) = 0,$$

so that (2.39) holds on Q_σ.

To complete the proof, it thus suffices to prove there exists $\sigma > 0$ near τ such that (2.44) holds. To do this, note first that for $k \in \widetilde{\mathcal{K}}_\tau$, (so $\varphi_k = 0$), $A'_k \neq 0$ on ∂M. Namely, if $k^T = (A'_k)^T = 0$ on ∂M, it follows by Proposition 2.2 that $k = 0$ on M. Hence if k_j is a basis for $\widetilde{\mathcal{K}}_\tau$ then the symmetric forms A'_{k_j} are linearly independent on ∂M.

There are certainly many ways to prove the existence of $\sigma > 0$ for which (2.44) holds. One method is as follows. Note that (2.44) may be reformulated as: find a positive definite linear map

B, close to the identity, such that

$$tr_\tau(BA'_k) + \varepsilon H'_k \notin \langle X(H) \rangle, \tag{2.45}$$

for all $0 \neq k \in \widetilde{\mathcal{K}}_\tau$. Choosing ε sufficiently small, it suffices to find B such that

$$tr_\tau(BA'_k) \notin \langle X(H) \rangle. \tag{2.46}$$

Since each A'_k is trace-free with respect to τ (since $tr_\tau A'_k = 0$ by the linearized scalar constraint (2.15)) each has a non-trivial positive part $(A'_k)^+$ given by composing A'_k with the projection onto the positive eigenspaces of A'_k. In particular, on any basis k_j of $\widetilde{\mathcal{K}}_\tau$, the forms $(A'_{k_j})^+$ are linearly independent on ∂M. Hence they are linearly independent pointwise on some open set $\Omega \subset \partial M$. To simplify the notation, set $A_j^+ = (A'_{k_j})^+$ and $A_j = A'_{k_j}$.

Choose points $p_i \in \Omega$, $1 \leq i \leq dim\widetilde{\mathcal{K}}_\tau$ with disjoint neighborhoods $U_i \subset \Omega$ and positive bump functions η_i supported in U_i, with $\eta_i(p_i) = 1$. For the moment, set $B = \sum_j \eta_j A_j^+$, where for each i, the basis forms $\{A_i^+\}$ satisfy

$$\langle A_i^+, A_j \rangle_\tau(p_i) = 0, \quad \text{for all} \quad j > i. \tag{2.47}$$

One constructs such a basis inductively as follows. At p_1 choose any basis k_i of $\widetilde{K}_\tau$. Fix k_1 and $A_1 = A'_{k_1}$ and then via the standard Gram-Schmidt process, construct the basis forms k_j, $j \geq 2$ satisfying (2.47) at p_1. Next in the space spanned by $\{k_j\}$, $j \geq 2$, repeat the process at p_2, starting with A_2 and constructing forms k_j, $j \geq 3$ satisfying (2.47) at p_2. One continues inductively in this way through to the last point. Note that a different basis of $\widetilde{\mathcal{K}}_\tau$ is thus used at each point p_i. At any given p_r one has

$$tr_\tau(BA_k)(p_r) = \langle B, A_k \rangle_\tau(p_r) = \sum_{i,j} \eta_i c_j \langle A_i^+, A_j \rangle_\tau(p_r), \tag{2.48}$$

where $k = \sum c_j k_j$ in the basis associated to p_r.

Now suppose first that (2.46) fails for some $k \in \widetilde{\mathcal{K}}_\tau$, in the stronger sense that $tr_\tau(BA'_k) = 0$. Evaluating (2.48) at p_1 gives, by (2.47),

$$tr_\tau(BA'_k)(p_1) = c_1|A_1^+|^2(p_1) = 0,$$

so that $c_1 = 0$. Using this, and by the construction of the basis at p_2, one has similarly

$$tr_\tau(BA'_k)(p_2) = c_2|A_2^+|^2(p_2) = 0,$$

so that $c_2 = 0$. Continuing in this way, it follows that $c_r = 0$ for all r, and hence by the construction of the bases at $\{p_r\}$, $k = 0$. This implies $tr_\tau(BA'_k) \neq 0$ for all $k \in \widetilde{\mathcal{K}}_\tau$, so that any possible solution k of (2.46) is unique up to scaling.

Next, choose a further point $q \in \Omega$ disjoint from $\cup U_i$ with neighborhood $q \in V \subset \Omega$, $V \cap (\cup U_i) = \emptyset$. One may then choose a bump function η_q as above suitably so that in V, the right side of (2.48) is linearly independent from $X(H)$. This establishes (2.46) for this choice of B.

Finally, note that for $B' = Id$, $tr_\tau(B'A'_k) = 0$, for all $k \in \widetilde{K}_\tau$. Also, on the unit sphere in $\widetilde{K}_\tau$ the space of functions $tr_\tau(BA'_k)$ is compact, and so bounded away from the zero function. Hence, choosing ε sufficiently small and replacing B by $Id + \varepsilon B$ gives a smooth metric $\sigma > 0$, close to τ on ∂M, satisfying (2.44).

■

Proposition 2.9 gives the existence of a "good" slice Q_σ as in (2.38) to ImL consisting of forms q of the form (2.36) and satisfying (2.39). Given this, now form the operator

$$\widetilde{L}(h) = L(h) + \pi_{Q_\sigma}(h), \tag{2.49}$$

where π_{Q_σ} is the L^2 orthogonal projection onto Q_σ. Proposition 2.9 implies that $\widetilde{L}$ is an isomorphism

$$\widetilde{L}: S_0^{m,\alpha}(M) \to S^{m-2,\alpha}(M), \tag{2.50}$$

where $S_0^{m,\alpha}(M)$ is defined as in (2.35).

Consider now the boundary value problem

$$\widetilde{L}(h) = 0, \quad \delta h = 0, \ [h^T]_{A_\delta} = h_1, \ \langle A'_h, \sigma\rangle + \varepsilon H'_h = h_2. \tag{2.51}$$

Since $\widetilde{L}$ in (2.50) is a bijection, it follows exactly as following (2.22) that (2.51) has a unique solution, for arbitrary h_1 and h_2. Of course solutions of (2.51) are now no longer infinitesimal Einstein deformations in general.

Lemma 2.10. *For any h_1 and h_2 in* (2.51)*, the solution h of* (2.51) *satisfies*

$$\delta h = 0, \tag{2.52}$$

on M.

Proof: To prove this, one has $L(h) = \widetilde{L}(h) - \pi_Q(h) = -\pi_Q(h)$. By (2.18), $\delta L(h) = \delta\delta^*(\delta(h))$ (since $\delta E' = 0$ by the Bianchi identity) which gives

$$\delta\delta^*(\delta(h)) = -\delta(\pi_Q(h)).$$

But $\pi_Q(h) = q$ for some q and $\delta q = 0$, by (2.37). So

$$\delta\delta^*(\delta(h)) = 0, \tag{2.53}$$

on M. By assumption (in (2.51)) $\delta(h) = 0$ on ∂M and so (2.52) follows as in the proof of Lemma 2.6.

We are now in position to assemble the results above to prove Theorem 1.1.

Proof of Theorem 1.1.

The main point is to return to (2.12). Consider the variation h of the form (2.51). For such h, $\delta h = 0$ on M and $L(h) = \widetilde{L}(h) - \pi_Q(h) = -\pi_Q(h) = -q$, for some $q \in Q_\sigma$. Also by (2.18) and (2.52), $L(h) = E'(h) + \delta^*\delta(h) = E'(h)$. Thus

$$E'_h + q = 0, \tag{2.54}$$

so that, for such h,

$$\int_{\partial M} E'_h(N, X) = -\int_{\partial M} q(N, X) = \int_{\partial M} \langle N, X\rangle(\Delta f + \lambda f) - D^2 f(N, X). \tag{2.55}$$

We now claim that for such h, i.e. for any q as above,

$$\int_{\partial M} q(N, X) = 0.$$

To prove this, computing first the second term in (2.55) gives

$$\int_{\partial M} \langle \nabla_X \nabla f, N\rangle = \int_{\partial M} NN(f)\langle X, N\rangle + X^T\langle \nabla f, N\rangle - \langle \nabla f, \nabla_{X^T} N\rangle$$

$$= \int_{\partial M} NN(f)\langle X, N\rangle - div(X^T)N(f) - A(X^T, \nabla f) = \int_{\partial M} NN(f)\langle X, N\rangle - div(X^T)N(f) - f\delta(A(X^T))$$

$$= \int_{\partial M} NN(f)\langle X, N\rangle - div(X^T)N(f) + f\langle A, \delta^* X^T\rangle + f dH(X^T),$$

where we have used the fact that $\delta A(X^T) = -dH(X^T)$. Since $(\delta^* X)^T = 0$, one has $\delta^* X^T + \langle X, N\rangle A = 0$, so that $div(X^T) = -\langle X, N\rangle H$. It follows that

$$\int_{\partial M} D^2 f(N, X) = \int_{\partial M} \langle X, N\rangle [NN(f) + HN(f) - f|A|^2] + fX^T(H). \tag{2.56}$$

On the other hand, for the first term in (2.55) one has $\Delta f = \Delta_{\partial M} f + HN(f) + NN(f)$ so that setting $\nu = \langle X, N\rangle$,

$$\int_{\partial M} \langle N, X\rangle(\Delta f + \lambda f) = \int_{\partial M} f\Delta_{\partial M}\nu + HN(f)\nu + NN(f)\nu + \lambda f\nu. \tag{2.57}$$

Subtracting (2.56) from (2.57) gives

$$\int_{\partial M} E'_h(N, X) = \int_{\partial M} f[\Delta\nu + (|A|^2 + \lambda)\nu - X^T(H)] = -\int_{\partial M} fX(h); \tag{2.58}$$

the second equality here is exactly the formula for the variation of the mean curvature in the direction X, $X(H) = 2H'_{\delta^* X}$. The claim thus follows from (2.39).

It follows from (2.12) that

$$\int_{\partial M} \langle \tau'_{\delta^* X}, h^T\rangle = 0, \tag{2.59}$$

with $[h^T]_{A_\delta}$ arbitrary. Letting $\delta \to 0$, (2.59) holds for arbitrary classes $[h^T]_A$. Via Lemma 2.5, this shows that $\tau'_k = 0$ on ∂M, for $k = \delta^* X$. Proposition 2.2 then implies that $k = 0$ on M. This completes the proof of Theorem 1.1.

■

To conclude, we study the relation between injectivity and surjectivity of $D\Pi$. Note first that if $n = 2$ and M is simply connected then

$$D\Pi = D\Pi_B, \tag{2.60}$$

i.e. all infinitesimal Einstein deformations k are of the form $k = \delta^* X$. If M is not simply connected, then $k = \delta^* X$ locally but not necessarily globally; the defect is measured by the holonomy representation of M.

Proposition 2.11. *Suppose $n = 2$, $(M, g) \in \mathcal{E}^{m,\alpha}$ is simply connected and suppose that ∂M is convex. Then*

$$D\Pi : T\mathcal{E}^{m,\alpha} \to S^{m,\alpha}(\partial M),$$

has dense range and (2.23) *holds.*

Proof: The formula (2.4)-(2.6) with $h^T = k^T = 0$ on ∂M gives

$$\int_M \langle E'_h, k\rangle = \int_M \langle h, E'_k\rangle. \tag{2.61}$$

On the closed subspace of divergence-free forms (h such that $\delta h = 0$ on M) one has $ImL = ImE'$ by (2.18). Hence if there exists $k \perp ImL$, then (2.61) gives $E'_k = 0$, since $\int_M \langle \delta^* Z, E'_k\rangle = 0$ for all vector fields Z with $Z = 0$ on ∂M, (cf. Lemma 2.6). But by Theorem 1.1. $E'_k = 0$ with $k^T = 0$ on ∂M means $k \in KerD\Pi = KerD\Pi_B$. Hence $k = 0$ (since k is divergence-free). Thus ImE' is dense. By the usual subtraction procedure as following (2.22), this implies $D\Pi = D\Pi_B$ also has dense range.

■

Remark 2.12. We point out here that the proof of Theorem 1.1 generalizes easily to situations where ∂M is weakly convex in the sense that $\tau \geq 0$, provided the set $Z = \{det\, \tau = 0\}$ has empty interior in ∂M. To see this, let $U = \partial M \setminus Z$ and let τ_s be a (smooth) approximation to τ with $\tau_s > 0$ on ∂M and $\tau_s = \tau$ on a large domain $V \subset U$. One may then carry out all the arguments above with respect to τ_s in place of τ. One derives then in the same way that $\tau'_k = 0$ on any compact subset of $V \subset \partial M$. Since V may be chosen arbitararily large in U, and U is dense in ∂M, this gives $\tau'_k = 0$ on ∂M and the result follows as before.

When $n = 2$, this recaptures the classical result of Blaschke, (cf. [7]) that (weakly) convex surfaces Σ in $\mathbb{R}^3$ are infinitesimally rigid provided the set Z where Σ is flat has empty interior. Of course it is well known that this result is false if the set $\{A = 0\}$ contains a non-empty open set $O \subset \partial M$; the deformation $k = \delta^*(fN)$ where f is any function with compact support in O is an infinitesimal isometric deformation which is not the restriction of a rigid motion in general.

Remark 2.13. Theorem 1.1 does not hold for asymptotic symmetries of complete Einstein metrics in general (so ∂M is at "infinity"). For instance, the asympotically flat Kerr metric is Ricci flat and has asymptotic symmetry group $SO(3) \times S^1$; the Kerr metric itself does not have such a large isometry group. Similarly, the Ricci-flat Eguchi-Hanson metric is asymptotically Euclidean with asymptotic symmetry group $SO(4)$ but again has itself a smaller isometry group.

Remark 2.14. It may be possible to generalize the proof of Theorem 1.1 to situations where ∂M is not necessarily $(n-1)$-convex. The convexity condition is used to obtain elliptic boundary conditions for the operator L, leading to the splitting (2.32). The proof of Theorem 1.1 remains valid for boundary conditions for which (2.32) holds, possibly even with K infinite dimensional. It would be interesting to know if (2.32), or a suitable analog of it, can be established for other natural geometric conditions on ∂M.

References

[1] M. Akbar and G. Gibbons, Ricci-flat metrics with $U(1)$ action and the Dirichlet boundary-value problem in Riemannian quantum gravity and isoperimetric inequalities, Class. Quantum Grav., **20**, (2003), 1787-1822.

[2] F. Almgren, Jr and I. Rivin, The mean curvature integral is invariant under bending, Geom. & Topology Monographs, Vol 1: Epstein Birthday Schrift, (1998), 1-21.

[3] M. Anderson, Extension of isometries on Einstein manifolds with boundary, Selecta Mathematica, **16**, (2010), 343-375.

[4] M. Anderson and M. Herzlich, Unique continuation results for Ricci curvature and applications, Jour. Geom. & Physics, **58**, (2008), 179-207, and Erratum, ibid, **60**, (2010), 1062-1067.

[5] M. Anderson, On boundary value problems for Einstein metrics, Geometry & Topology, **12**, (2008), 2009-2045.

[6] M. Anderson and M. Khuri, The static extension problem in general relativity, arXiv:0909.4550v2 [math.DG]

[7] Q. Han and J-X. Hong, Isometric Embeddings of Riemannian Manifolds in Euclidean Space, Math. Surveys and Monographs, vol. 130, Amer. Math. Soc., Providence, R.I. (2006).

[8] Q. Han and F. Lin, On the isometric embedding of torus in $\mathbb{R}^3$, Methods and Appl. of Analysis, **15**, (2008), 197-204.

[9] S. Hawking, Euclidean quantum gravity, in: Recent Developments in Gravitation, Cargese Lectures, Ed: M. Levy and S. Deser, Plenum Press, New York, (1978), 145-173.

[10] I. Izmestiev, Infinitesimal rigidity of convex surfaces through the second derivative of the Hilbert-Einstein functional II: smooth case, arXiv:1105.5067, [math.DG].

[11] J. Moser, A rapidly convergent iteration method and non-linear differential equations, Ann. Scuola Norm. Sup. Pisa, **20**, (1966), 499-535.

[12] J. Nash, The embedding problem for Riemannian manifolds, Annals of Math., **63**, (1956), 20-63.

[13] J.M. Schlenker, Einstein manifolds with convex boundaries, Comm. Math. Helv., **76**, (2001), 1-28.

[14] I. Rivin and J.M. Schlenker, The Schläfli formula in Einstein manifolds with boundary, Electron. Res. Announc. Amer. Math. Soc., **5**, (1999), 18-23.

[15] J.W. York Jr., Black hole thermodynamics and the Euclidean Einstein action, Phys. Rev. D, **33**, (1986), 2092-2099.

[16] E. Zehnder, Generalized implicit function theorems, in: Topics in Nonlinear Functional Analysis, L. Nirenberg, Courant Inst. Math. Sci., NYU, (1974).

Department of Mathematics, Stony Brook University, Stony Brook, N.Y. 11794-3651, USA
E-mail address: anderson@math.sunysb.edu
URL: http://www.math.sunysb.edu/∼anderson