THE CONFORMAL METHOD FOR THE EINSTEIN CONSTRAINT EQUATIONS

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Abstract. This work presents a simple criterion for the existence on compact manifolds of solutions to the vacuum Einstein constraint equations via the well-known conformal method of Lichnerowicz-Choquet-Bruhat-York in the far from CMC regime, extending previous work in the CMC and near-CMC regime. The main perspective is based on Smale's version of the Sard theorem for Fredholm maps between Banach manifolds and related transversality properties. To implement this, we prove the existence of apriori estimates along curves of solutions.

1. Introduction

Let (M, g, K) be a triple consisting of a closed 3-manifold M, a Riemannian metric g and a symmetric bilinear form K on M. The constraint equations for the vacuum Einstein equations are given by

$$\delta(K - Hg) = 0, \tag{1.1}$$

$$|K|^2 - H^2 - R_g = 0, \tag{1.2}$$

where δ is the divergence with respect to g, $H = tr_g K$ and R_g is the scalar curvature of (M, g). The equation (1.1) is called the divergence or momentum constraint while (1.2) is the Hamiltonian or scalar constraint. They are the Gauss-Codazzi and Gauss equations respectively of a hypersurface embedded in a 4-dimensional Ricci-flat Lorentzian space-time $(\mathcal{M}, g^{(4)})$.

The fundamental theorem of Choquet-Bruhat [7] guarantees that a smooth triple (M, g, K) satisfying the constraints (1.1)-(1.2) form an initial data or Cauchy hypersurface of a space-time solution $(\mathcal{M}, g^{(4)})$ of the vacuum Einstein equations $Ric_{g^{(4)}} = 0$. The metric and second fundamental form of $g^{(4)}$ induced on M are given by (g, K).

The equations (1.1)-(1.2) are highly underdetermined; there are 4 equations for the 12 unknown components of (g, K). A basic issue of interest has been to determine whether there is a natural space of "free" or "unconstrained" data $\mathcal{D}$, formally with 8 degrees of freedom, which upon specifying an element in $\mathcal{D}$, reduce the equations (1.1)-(1.2) to a determined set of equations. Ideally, one would then be able to uniquely solve these equations, giving then an effective parametrization of the dynamical gravitational degrees of freedom from the data in $\mathcal{D}$.

A priori there are of course many possible choices for the free data space $\mathcal{D}$. One would like $\mathcal{D}$ to be as simple as possible topologically. On the other hand, very little seems to be known about the topology of the space $\mathcal{C}$ of solutions of the constraint equations.

By far the best understood and most well-studied choice, especially for the case of closed manifolds considered here, is that given by the conformal method of Lichnerowicz-Choquet-Bruhat-York, cf. for instance [3], [8], [22], or one of its variants [3], [26]. For the conformal method, $\mathcal{D}$ has the following product structure. Let $\mathcal{C}_0$ be the space of (pointwise) conformal equivalence classes $[g]$ of C^∞ smooth metrics g on M and let $\mathcal{T}$ be the fibration over $\mathcal{C}_0$ with fiber over $[g]$ given by the space of C^∞ smooth symmetric 2-tensors σ which are transverse-traceless with respect to $[g]$, i.e. $\delta_{g_0}\sigma = tr_{g_0}\sigma = 0$, where g_0 is any representative of $[g]$. Next, let $C^\infty(M)$ denote the space of smooth scalar functions H on M. Then $\mathcal{D}$ (i.e. $\mathcal{D}_{C^\infty}$) is given by

$$\mathcal{D} = \mathcal{T} \times C^\infty(M).$$

The class $[g]$ represents the conformal class of the metric g, while σ and H represent the transverse-traceless part and pure trace part of K; H thus represents the mean curvature. It is easily verified that $\mathcal{D}$ has formally 8 degrees of freedom and is contractible.

Given then a point in $\mathcal{D}$, i.e. a triple (g_0, σ, H) where g_0 is a fixed representative in the conformal class $[g]$, the remaining data determining (g, K) are a conformal factor φ for the metric and a vector field X for the action of diffeomorphisms on symmetric bilinear forms. More precisely, using the York decompostion of symmetric bilinear forms [35], one forms (g, K) by setting

$$g = \varphi^4 g_0, \tag{1.3}$$

$$K = \varphi^{-2}(\sigma + \widehat{\mathcal{L}}_X g_0) + \tfrac{H}{3}\varphi^4 g_0, \tag{1.4}$$

where $\widehat{\mathcal{L}}$ is the conformal Killing operator; $\widehat{\mathcal{L}}_X g_0 = \mathcal{L}_X g_0 - \frac{2}{3} div_{g_0} X g_0$. The constraint equations (1.1)-(1.2) then become a coupled system of equations for (φ, X) which take the form

$$\delta(\widehat{\mathcal{L}}_X g_0) = -\tfrac{2}{3}\varphi^6 dH, \tag{1.5}$$

for the divergence constraint while the Hamiltonian or scalar constraint takes the form of the Lichnerowicz equation

$$\Delta\varphi = \tfrac{1}{8}R_0\varphi - \tfrac{1}{8}|\sigma + \widehat{\mathcal{L}}_X g_0|^2\varphi^{-7} + \tfrac{1}{12}H^2\varphi^5. \tag{1.6}$$

Here δ and Δ are the divergence and Laplacian with respect to the fixed representative $g_0 \in [g]$ and R_0 is the scalar curvature of g_0. The metric g_0 is usually taken to be a Yamabe metric in $[g]$ so that R_0 is the Yamabe constant $Y[g]$ of $[g]$. It is well-known that the equations (1.5)-(1.6) form a determined elliptic system for (φ, X), given (g_0, σ, H), cf. also Section 2.

The basic question is then for what free data $([g], \sigma, H) \in \mathcal{D}$ are these equations solvable, or even better, uniquely solvable. This has been thoroughly answered only in the case where $H = const$, the CMC case, where necessary and sufficient conditions for the existence and uniqueness of solutions have been developed, cf. [22], based on work of [24], [8], [29] and others. This is briefly discussed further in Section 2. There has been significant recent progress on this issue also in the near CMC case, where the derivative dH is sufficiently small compared with H, cf. [23] and references therein for a recent survey. There has also been some interesting progress in the far-from-CMC regime, cf. [25], [31], [32], [34], but in general this realm remains not well understood.

The reason for the simplification in the CMC case is well-known; in this case one may set $X = 0$ in (1.5) and the system (1.5)-(1.6) reduces to the Lichnerowicz equation (1.6) for φ involving only the given data R_0, σ, H. This equation is closely related to the well-understood Yamabe equation for constant scalar curvature metrics.

In this paper, we take a somewhat different perspective from previous work on this issue, namely a global analysis perspective going back to the work of Smale [33]. Let $\mathcal{C}$ be the space of all C^∞ smooth pairs (g, K) on M satisfying the constraint equations (1.1)-(1.2). Instead of studying the solvability of (1.5)-(1.6) for fixed data $([g], \sigma, H) \in \mathcal{D}$, we consider the behavior of the natural (projection) map

$$\Pi : \mathcal{C} \to \mathcal{D}, \tag{1.7}$$

$$(g, K) \to ([g], \sigma, H).$$

The fibers of Π (if non-empty) are pairs

$$(\varphi, X)$$

satisfying the equations (1.5)-(1.6). Of course one requires $\varphi > 0$.

The main interest is the global behavior of the map Π. In particular, one would like to understand the image of Π and the injectivity of Π, corresponding to the existence and uniqueness of solutions of (1.5)-(1.6). As noted above, within the CMC class where $H = const$, the maximal domain

$$\mathcal{C}_+^{cmc} \subset \mathcal{C} \cap \{H = const\}$$

on which Π is one-to-one is exactly understood. The restriction

$$\Pi^{cmc} = \Pi|_{\mathcal{C}_+^{cmc}} : \mathcal{C}_+^{cmc} \to \mathcal{D}^{cmc} = \mathcal{D} \cap \{H = const\}$$

is not surjective onto $\mathcal{D}^{cmc}$ but its image $\mathcal{D}_+^{cmc} = \operatorname{Im}\Pi^{cmc}$ is fully understood, and recalled briefly in Section 2.

Returning to the general situation regarding (1.7), let $\mathcal{C}_0' \subset \mathcal{C}_0$ be the space of conformal classes which have no essential conformal Killing field. Thus if $M \neq S^3$, $\mathcal{C}_0' = \mathcal{C}$ while if $M = S^3$, we exclude the conformal class of the round metric $[g_{+1}]$; by the well-known Obata theorem, this is the only conformal class with a non-compact conformal group.

Let

$$\mathcal{D}' \subset \mathcal{D} \tag{1.8}$$

be the restriction of the fibration $\mathcal{D}$ to the domain $\mathcal{C}'_0$ and let

$$\mathcal{C}' = \Pi^{-1}(\mathcal{D}') \subset \mathcal{C},$$

with the induced map

$$\Pi' : \mathcal{C}' \to \mathcal{D}'. \tag{1.9}$$

Next let

$$\mathcal{D}_+ = \{([g], \sigma, H) \in \mathcal{D}' : \inf |H| > 0 \ \text{ and } \ \sup |\sigma| > 0 \text{ when } Y[g] \geq 0\}. \tag{1.10}$$

Thus $\mathcal{D}_+$ consists of data for which the mean curvature function H never vanishes (so has a definite sign) and, when the Yamabe constant of $[g]$ is non-negative, the symmetric 2-tensor σ does not vanish identically. The domain $\mathcal{D}_+$ is a connected, open domain in $\mathcal{D}'$. Let

$$\mathcal{C}_+ = \Pi^{-1}(\mathcal{D}_+);$$

more precisely $\mathcal{C}_+$ is defined to be the path component of $\Pi^{-1}(\mathcal{D}_+)$ containing the domain $\mathcal{C}_+^{cmc}$. The conditions defining $\mathcal{D}_+$ are closely related to the conditions defining $\mathcal{D}_+^{cmc}$, cf. Section 2.

The main result of this paper is the following:

Theorem 1.1. *The map*

$$\Pi_+ : \mathcal{C}_+ \to \mathcal{D}_+ \tag{1.11}$$

is surjective. Thus for any data $([g], \sigma, H) \in \mathcal{D}_+$ there is at least one smooth solution (φ, X) of the constraint equations (1.5)-(1.6).

This result considerably extends the previously known region in $\mathcal{D}$ where the constraints (1.5)-(1.6) have solutions. It gives a simple criterion for the existence of solutions in the far-from-CMC regime. There are reasons to believe it can be extended further, i.e. we do not believe Theorem 1.1 is sharp. In particular, the restriction that H have a definite sign when $Y[g] > 0$ may be superfluous; cf. Remark 4.3 for further discussion.

We are not able to address the uniqueness issue at all in this work. In fact, it is not known if Π_+ is even generically finite-to-one on $\mathcal{C}_+$, i.e. if there are only finitely many solutions of the constraint equations (in $\mathcal{C}_+$) for generic data $([g], \sigma, H) \in \mathcal{D}_+$. This remains an interesting open problem. There have been a number of recent works on non-uniqueness and bifurcation results for solutions (φ, X) of the constraint equations, cf. [11], [32], [34] for example. To date, all such results involve extra matter sources or a non-zero cosmological constant and so have not yet been applicable to the vacuum case per se. However, we expect similar phenomena hold in the vacuum case in that the global map Π_+ is likely to exhibit fold and bifurcation behavior leading to multiple solutions in numerous regions. Of course it would be very interesting to explore this further.

The methods of the paper do show that if Π_+ is finite-to-one on $\mathcal{C}_+$, (or on a suitable open domain $\mathcal{U}_+ \subset \mathcal{C}_+$), then the resulting map Π_+ (or $\Pi_+|_{\mathcal{U}_+}$) is proper and can be shown to have a corresponding $\mathbb{Z}_2$-valued Smale degree, cf. [33]. Further, the degree of Π_+ would necessarily be 1 (mod 2). Such a degree would give a parity count (even-odd) for the number of generic solutions (φ, X) of (1.5)-(1.6) in $\mathcal{C}_+$.

The method of proof of Theorem 1.1 is not based on traditional approaches, for instance via the construction of suitable barriers, i.e. sub and super solutions of the Lichnerowicz equation. Instead, the approach can be considered as a generalization of the well-known method of continuity. Note first that it is not expected that the continuity method itself can be implemented here, since it implies the absence of bifurcation or fold behavior. (Recall that the continuity method requires openness of the set of solutions along a path; this fails when there is a bifurcation in the space of solutions). On the other hand, as mentioned above, we have not been able to prove that Π_+ is proper, i.e. establish fully general apriori estimates for the set of solutions, in order to obtain a well-defined degree. (On the other hand, note that properness does allow for bifurcations in the space of solutions).

Instead, we develop an intermediate course between these two approaches. To describe this briefly here, we simplify the discussion and consider the regular regions $\mathcal{C}_+^{reg} \subset \mathcal{C}_+$ and $\mathcal{D}_+^{reg} \subset \mathcal{D}_+$ which have the structure of smooth manifolds with

$$\Pi_+^{reg} = \Pi|_{\mathcal{C}_+}^{reg} : \mathcal{C}_+^{reg} \to \mathcal{D}_+^{reg} \tag{1.12}$$

a smooth Fredholm map between them. We will apply the global analysis methods introduced by Smale [33] to analyse the surjectivity of Π_+^{reg}. The domain $\mathcal{C}_+^{reg}$ is a large open set in $\mathcal{C}_+$, conjecturally dense, and we show in Section 5 that the methods used to prove Theorem 1.1 in $\mathcal{C}_+^{reg}$ extend to the full space $\mathcal{C}_+$.

An important starting point is the fact that CMC solutions are unique, i.e. solutions in $\mathcal{C}^{cmc}$ are uniquely determined by the target data in $\mathcal{D}^{cmc}$. (Actually this uniqueness is needed only for one solution). Let $y(t)$, $t \in I = [0,1]$ be a compact path in $\mathcal{D}_+^{reg}$, transverse to Π_+^{reg}, with $y(0) \in \mathcal{D}_+^{cmc}$. Although the properness of Π_+^{reg} remains unknown, we show in Section 4 that Π_+^{reg} is *proper along paths*, i.e. if $x(\tau)$ is a path in $\mathcal{C}_+^{reg}$ with $\Pi_+^{reg}(x(\tau)) \in y(I)$, then the curve $x(\tau)$ is compact in $\mathcal{C}_+^{reg}$. As a consequence, following the basic methodology of Smale [33], the path $y(t)$ has a unique lift to a path $x(\tau) \in \mathcal{C}_+^{reg}$, $\tau \in [0,1]$ with $x(0)$ the unique lift of $y(0)$, i.e. $\Pi_+(x(0)) = y(0)$. From this we show that x covers y in that $\Pi_+(x(I)) = y(I)$ and hence $\Pi_+(x(1)) = y(1)$, i.e. the constraint equations are solvable at $y(1)$. Although overly simplified here for the sake of brevity, this is the main perspective underlying the proof of Theorem 1.1.

The contents of the paper are briefly as follows. In Section 2, we introduce background material and results needed for the work to follow. In particular, the spaces $\mathcal{C}^{reg}$ and $\mathcal{D}^{reg}$ are defined and shown to admit smooth separable Banach manifold structures, cf. Corollary 2.4. We also summarize the known existence and uniqueness results for CMC solutions and show how these relate to the definitions of $\mathcal{C}_+$ and $\mathcal{D}_+$. Some basic initial a priori estimates for the map Π_+ are derived in Section 3. Here one sees that solutions (φ, X) of the constraints (1.5)-(1.6) cannot degenerate over compact sets in $\mathcal{D}_+$ *provided* one has a fixed sup bound on φ. The definition of the domain $\mathcal{D}_+$ is largely motivated by these non-degeneration results in Section 3.

In Theorem 4.1 we prove that a sup bound on φ can be obtained from a bound on the Harnack constant $C_{Har}(\varphi) = \frac{\sup \varphi}{\inf \varphi}$, given a bound on H away from zero. Finally we show that a bound on the Harnack constant of φ along paths can be obtained from a natural gauge condition, the divergence-free gauge – away from the round conformal class of S^3. This brings in the important rôle of the action of the diffeomorphism group on the space of metrics and solutions of the constraint equations, which does not appear to have played a significant rôle in previous approaches. (The proof of Theorem 4.1 is by a blow-up argument where again the action of diffeomorphisms plays a crucial rôle). The proof of Theorem 1.1 in the regular region of $\mathcal{C}_+$, (the region where (g, K) is not Killing initial data) is then completed in Section 4. Section 5 builds on the methods of Section 4 and extends the proof of Theorem 1.1 to the singular region of $\mathcal{C}^+$ where (g, K) are Killing initial data.

2. Background and Set-up

In this section, we present background material needed for the work to follow. Throughout the paper, M denotes a compact 3-manifold, without boundary; cf. Remark 5.4 regarding the situation $dim\, M > 3$.

To begin, we discuss the topologies of the spaces $\mathcal{C}$ and $\mathcal{D}$. The C^∞ topology is a Fréchet space topology, which is not suitable for analysing nonlinear Fredholm maps, mainly due to the failure of the inverse function theorem.

The simplest Banach spaces on which elliptic operators are well-behaved are the Hölder spaces $C^{m,\alpha}$ and Sobolev spaces $W^{k,p}$ for suitable (m, α) or (k, p). For the Einstein evolution equations where energy estimates play a key role, one usually uses the Sobolev spaces $H^s = W^{s,2}$, for suitable $s \geq 2$. However, we will use the Hölder spaces $C^{m,\alpha}$ here, since the projection map Π in (1.7) is only known to be well-behaved in Hölder spaces $C^{m,\alpha}$. (It is possible one could work in the class of Morrey spaces [1], but this will not be pursued here). Throughout the paper we assume $m \geq 3$, $\alpha \in (0,1)$. (We will not be concerned here with obtaining the lowest possible regularity results).

Moreover, it is well-known that Hölder spaces $C^{m,\alpha}$ are not separable Banach spaces; they do not admit a countable basis. Since separability will be an important property, we work instead with a maximal closed separable subspace of $C^{m,\alpha}$, namely the so-called little Hölder space $c^{m,\alpha}$. This may be defined to be the completion of C^{m+1} or C^{∞} with respect to the $C^{m,\alpha}$ norm. Equivalently, functions f on smooth domains $\Omega \subset \mathbb{R}^n$ are in $c^{0,\alpha}(\Omega)$ if $f \in C^{0,\alpha}(\Omega)$ and, for $x, y \in \Omega$,

$$\lim_{r \to 0} \sup_{0 < dist(x,y) < r} \frac{|f(x) - f(y)|}{dist(x,y)^{\alpha}} = 0.$$

The space $c^{m,\alpha}(\Omega)$ consists of functions f whose partial derivatives up to order m exist and are in $c^{0,\alpha}(\Omega)$. The space $c^{m,\alpha}$ is a separable Banach space, embedded as a closed subspace of $C^{m,\alpha}$, cf. [6]. Note that $C^{m,\alpha'} \subset c^{m,\alpha}$ for all $\alpha' > \alpha$.

Let $Met^{m,\alpha}(M)$ be the space of $c^{m,\alpha}$ metrics g on M; thus in a smooth atlas for M, the coefficients of g are $c^{m,\alpha}$ functions. Similarly let $S_2^{m-1,\alpha}(M)$ be the space of $c^{m-1,\alpha}$ symmetric bilinear forms K on M. Define then

$$\mathcal{C} := \mathcal{C}^{m,\alpha} \subset Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$$

to be the subspace satisfying the constraint equations (1.1)-(1.2), with the induced topology.

Next, let $\mathcal{C}_0^{m,\alpha}$ be the space of $c^{m,\alpha}$ conformal equivalence classes of metrics in $Met^{m,\alpha}(M)$; thus $g_1 \sim g_2$ if $g_2 = \varphi^4 g_1$, for some positive function $\varphi \in c^{m,\alpha}$. Let $\mathcal{T}^{m-1,\alpha}$ be the fibration of $c^{m-1,\alpha}$ transverse-traceless tensors σ over $\mathcal{C}_0^{m,\alpha}$, i.e. the fiber over $[g_1]$ consists of σ such that $tr_{g_1}\sigma = \delta_{g_1}\sigma = 0$; (this is well-defined, cf. [26] for example). Define also

$$\mathcal{D} := \mathcal{D}^{m-1,\alpha} = \mathcal{T}^{m-1,\alpha} \times c^{m-1,\alpha}(M).$$

Thus we have the map Π as in (1.7),

$$\Pi : \mathcal{C} \to \mathcal{D}. \tag{2.1}$$

It will be important in the work to follow to have an explicit parametrization of $\mathcal{D}$, or more precisely a parametrization of the base space $\mathcal{C}_0 = \mathcal{C}_0^{m,\alpha}$ of conformal classes. It is not crucial for the following exactly which background representative g_0 is chosen for $[g]$ as long as g_0 varies in a compact set of representatives of $[g]$. For convenience, we will always choose g_0 to be a (minimizing) Yamabe metric, i.e. a metric of constant scalar curvature and unit volume realizing the Yamabe invariant $Y[g]$ of $[g]$. In the case $Y(g) \leq 0$, such Yamabe metrics are unique. This is not the case in general when $Y(g) > 0$, but by the solution to the Yamabe problem the class of Yamabe metrics of volume 1 in $[g]$ is compact when $(M, g) \neq (S^3, g_{+1})$, and a specific choice within this class is not needed. Note that this compactness does *not* hold for the case of the conformal class $[g_{+1}]$ of the round metric on S^3; this is the basic reason that the round metric on S^3 will be excluded from the target space $\mathcal{D}$, as in (1.8). This non-compactness is closely related to the Nirenberg problem and the Kazdan-Warner obstruction on S^3, cf. [21] for further details.

Given this choice, we now analyse the map Π in (2.1) in more detail. Let $\Lambda_1^{m-2,\alpha}(M)$ be the space of 1-forms on M with coefficients in $c^{m-2,\alpha}$ and consider the constraint map

$$\Phi : Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M) \to c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M), \tag{2.2}$$

$$\Phi(g, K) = \begin{pmatrix} R_g - |K|^2 + H^2 \\ \delta K + dH \end{pmatrix}.$$

A simple inspection shows that the map Φ is well-defined and is a C^{∞} smooth map of Banach spaces, (or more precisely open domains of Banach spaces). If one fixes an element $y = (g_0, \sigma, H) \in \mathcal{D}$ with representative $g_0 \in [g] \in \mathcal{C}_0^{m,\alpha}$, then it follows from the York decomposition as in (1.3)-(1.4) that the constraint map Φ takes the form

$$\Phi_y : c^{m,\alpha}(M) \times \chi^{m,\alpha}(M) \to c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M), \tag{2.3}$$

$$\Phi_y(\varphi, X) = \begin{pmatrix} \varphi^{-5} \\ \varphi^{-6} \end{pmatrix} \cdot \begin{pmatrix} -\Delta\varphi + \frac{1}{8}R_0\varphi - \frac{1}{8}|\sigma + \widehat{\mathcal{L}}_X g_0|^2\varphi^{-7} + \frac{1}{12}H^2\varphi^5 \\ \delta(\widehat{\mathcal{L}}_X g_0) + \frac{2}{3}\varphi^6 dH \end{pmatrix}.$$

Here $\chi^{m,\alpha}$ is the space of $c^{m,\alpha}$ vector fields on M. Of course (φ, X) depend on the choice of background metric $g_0 \in [g]$ while (g, K) do not. Again Φ_y is a smooth map of Banach spaces.

It is well-known and easy to see that the system (2.3) is a (non-linear, second order) elliptic system for the unknowns (φ, X). For $(g_0, \sigma, H) \in \mathcal{D}^{m-1,\alpha}$, the coefficients of the 2$^{\text{nd}}$ order derivatives of (φ, X) are in $C^{m,\alpha}$, (in fact in $c^{m,\alpha} \subset C^{m,\alpha}$), the coefficients of the 1^{st} order derivatives are in $C^{m-1,\alpha}$ while the coefficients of the 0-order terms are in $C^{m-2,\alpha}$. Basic elliptic regularity estimates, cf. [28, Theorem 6.2.5] for instance, show that

$$|(\varphi, X)|_{C^{m,\alpha}} \leq C[||D\Phi_y(\varphi, X)||_{C^{m-2,\alpha}} + ||(\varphi, X)||_{C^0}],$$

where C depends only on the Hölder norms of the coefficients above. One has the same estimate for the formal L^2 adjoint of $D\Phi_y$.

It follows from elliptic theory that the fiber map $\Phi_y = \Phi|_{\Phi^{-1}(y)}$ is Fredholm, i.e. at any (φ, X), the linearization $D_{(\varphi,X)}\Phi_y$ has finite dimensional kernel, finite dimensional cokernel, and is of closed range. It is for this reason that we choose to work with Hölder spaces; this Fredholm property does not appear to hold for Sobolev spaces.

The rows of (2.3), corresponding to the equations (1.5)-(1.6), are in general coupled, but are uncoupled and of Laplace type at leading order. Hence the Fredholm index of the map Φ_y is zero. The full constraint map Φ in (2.2) is an underdetermined elliptic operator; the linearization $D\Phi$ is semi-Fredholm, with finite dimensional cokernel but infinite dimensional kernel.

Given (g, K), let $D_{(g,K)}\Phi$ be the linearization of Φ at (g, K) and let $(D\Phi)^*$ denote the L^2 adjoint. Define the regular set

$$\mathcal{C}_*^{reg} \subset \mathcal{C} \tag{2.4}$$

to be the set of points $(g, K) \in \mathcal{C}$ such that $\mathrm{Ker}(D_{(g,K)}\Phi)^* = 0$, cf. also the discussion following (2.6). This provisional definition of the regular set will be modified slightly below in (2.14). We then have:

Proposition 2.1. *The space $\mathcal{C}_*^{reg} \subset \mathcal{C}^{m,\alpha}$ is a smooth separable Banach manifold.*

Proof: Naturally, the proof uses the implicit function theorem for Banach manifolds. To begin, one has the L^2 orthogonal splitting

$$c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M) = \overline{\mathrm{Im}\, D\Phi} \oplus \mathrm{Ker}(D\Phi)^*. \tag{2.5}$$

Since $\mathrm{Ker}(D\Phi)^* = 0$ on $\mathcal{C}_*^{reg}$, to apply the implicit function theorem, we need to show that $D\Phi$ has closed range and $\mathrm{Ker}\, D\Phi$ splits. As discussed above, the fiber map $D\Phi_y$ is of closed range with image of finite codimension. Let S be a slice to $\mathrm{Im}\, D\Phi_y$, so that S is finite dimensional. Since $D\Phi$ has dense range, one may perturb S slightly if necessary so that $S \subset \mathrm{Im}\, D\Phi$ and choose a finite collection of "vectors" (h_j, κ_j) in the domain of $D\Phi$ such that the collection $\{D\Phi(h_j, \kappa_j)\}$ span S. It then follows easily that $D\Phi$ is of closed range.

To see that $\mathrm{Ker}\, D\Phi$ splits, write

$$T(Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)) = H \oplus V,$$

where $V = \iota(T(c^{m,\alpha}(M) \times \chi^{m,\alpha}(M)))$; here $T(c^{m,\alpha}(M) \times \chi^{m,\alpha}(M))$ is the domain of $D\Phi_y$ and ι is the natural "inclusion" map $\iota(\varphi, X) = (g, K)$ as in (1.5)-(1.6), given fixed data in $\mathcal{D}$. The subspace H corresponds to $T\mathcal{D}$. Since $\mathrm{Ker}\, D\Phi_y$ is finite dimensional and splits, one has

$$V = \mathrm{Ker}\, D\Phi_y \oplus L,$$

where L closed and of finite codimension in V. By construction, $\mathrm{Ker}\, D\Phi \cap L = 0$. We claim that $\mathrm{Ker}\, D\Phi \oplus L$ is of finite codimension in $T(Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M))$. To see this, let (g', K') be any variation of (g, K) in $T(Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M))$ and let $D\Phi(g', K') = w$. Recall that $\mathrm{Im}\, D\Phi_y$ is of finite codimension. Thus if $w \in \mathrm{Im}\, D\Phi_y$, there exists unique $(\varphi', X') \in L$ such that $D\Phi(g', K') - \iota(\varphi', X')) = 0$. This proves the claim. Since any space of finite codimension splits, it follows that $\mathrm{Ker}\, D\Phi$ splits.

This shows that $D\Phi$ is a submersion on $\mathcal{C}_*^{reg}$ and the implicit function theorem (or regular value theorem) for Banach manifolds implies that the zero set

$$\mathcal{C}_*^{reg} = \Phi^{-1}(0)$$

is a smooth Banach manifold. Thus $\mathcal{C}_*^{reg}$ is an open Banach submanifold within $\mathcal{C}$.

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Remark 2.2. Proposition 2.1 is discussed in detail and proved in the C^∞ setting in [15] and [17], by working in Sobolev spaces $H^s \times H^{s-1}$ and passing to the limit $s \to \infty$. The proof of Proposition 2.1 is partly based on this method and perspective but differs in certain respects. Namely as pointed out in [4], the proof of the manifold structure given in [15] or [17] does not hold when restricting to Sobolev spaces H^s of finite differentiability. This issue is also discussed in detail in [10].

Let

$$\mathcal{C}_*^{sing} \subset \mathcal{C} \tag{2.6}$$

denote the space of solutions with $\mathrm{Ker}(D_{(g,K)}\Phi)^* \neq 0$, so that $\mathcal{C}_*^{sing} = \mathcal{C} \setminus \mathcal{C}_*^{reg}$ is closed. The structure of $\mathcal{C}$ near points $(g, K) \in \mathcal{C}_*^{sing}$ has been analysed in detail in particular by Moncrief, Fischer and Marsden. To describe this, let $(\mathcal{M}, g^{(4)})$ be the maximal vacuum Cauchy development of the initial data set (M, g, K). Let ν be the unit (future-directed) time-like normal to M in $(\mathcal{M}, g^{(4)})$. Then by [27], $(N, Y) \in \mathrm{Ker}(D\Phi)^*$ if and only if the vector field $Z = N\nu + Y \in T\mathcal{M}|_M$ extends to a space-time Killing field on $(\mathcal{M}, g^{(4)})$.

Let $\mathcal{C}^{cmc} \subset \mathcal{C}$ denote the subspace of solutions where $H = const$. It is proved in [17] that the space $\mathcal{C}$ has cone-like singularities at the locus $\mathcal{C}_*^{sing} \cap \mathcal{C}^{cmc}$. Moreover, it is proved in [14], cf. also [5], that the closed set $\mathcal{C}_*^{sing} \cap \mathcal{C}^{cmc}$ is nowhere dense in $\mathcal{C}^{cmc}$; any point in $\mathcal{C}_*^{sing} \cap \mathcal{C}^{cmc}$ is a limit of points in the open regular domain $\mathcal{C}_*^{reg} \cap \mathcal{C}^{cmc}$; cf. the discussion concerning (2.17). This basic property appears to remain unknown however when $H \neq const$, cf. [5].

Next, before introducing the analogous decomposition of $\mathcal{D}$ into its regular and singular part, we need to discuss the action of diffeomorphisms. Let $\mathrm{Diff}^{m+1,\alpha}(M)$ be the group of $c^{m+1,\alpha}$ diffeomorphisms of M. This group acts continuously, but not smoothly, on the spaces $Met^{m,\alpha}(M)$, $\mathcal{C}$ and $\mathcal{D}$ by pullback; $(\psi, (g, K)) \to (\psi^* g, \psi^* K)$ and similarly on the factors of $\mathcal{D}$. The map Π is equivariant with respect to these actions and so induces an action on the fibers of Π: $(\psi, (\varphi, X)) \to (\varphi \circ \psi, \psi^* X)$.

Now the conformal automorphism group $\mathrm{Conf}(M, [g])$ acts on the space of conformal factors φ: if $\psi \in \mathrm{Conf}(M, [g])$ and $\psi^* g_0 = \lambda^4 g_0$, then $(\psi, \varphi) \to \widetilde{\varphi} = \lambda\varphi$, since $\varphi^4 \lambda^4 g_0 = \varphi^4 \psi^* g_0 = \widetilde{\varphi}^4 g_0$. If $\mathrm{Conf}(M, [g])$ is compact, then this indeterminacy in the choice of (φ, g_0) plays no significant rôle. On the other hand, if $\mathrm{Conf}(M, [g])$ is non-compact, so that $\mathrm{Conf}(M, [g]) = \mathrm{Conf}(S^3, [g_{+1}])$ by the well-known Obata theorem, then the non-compactness of the choice of φ becomes an important issue. As discussed above, it is for this reason that $(S^3, [g_{+1}])$ is excluded from $\mathcal{D}_+$ in (1.10)

A well-known natural local slice to the action of $\mathrm{Diff}^{m+1,\alpha}(M)$ on the space of metrics $Met^{m,\alpha}(M)$ is given by the divergence-free gauge condition, the Palais-Ebin slice. Given an arbitrary fixed $g \in Met^{m+1,\alpha}(M)$, consider nearby metrics $\widetilde{g} \in Met^{m,\alpha}(M)$; then there is a diffeomorphism $\psi \in \mathrm{Diff}^{m+1,\alpha}(M)$ close to the identity and unique up to isometry, such that

$$\delta_g \psi^* \widetilde{g} = 0. \tag{2.7}$$

To prove this, the linearization of (2.7) is given by

$$\delta_g(\widetilde{g} + \delta_g^* Y) = 0,$$

equivalent to

$$\delta_g \delta_g^* Y = -\delta_g \widetilde{g}. \tag{2.8}$$

The operator $\delta_g \delta_g^*$ is formally elliptic and self-adjoint, with kernel given by the space of Killing fields on (M, g). If $g \in Met^{m+1,\alpha}(M)$ the operator $\delta_g \delta_g^* : Met^{m+1,\alpha}(M) \to \Lambda_1^{m-1,\alpha}(M)$ is Fredholm and so in particular the operator δ_g has closed range, (cf. again [28, Theorem 6.2.5] for example). One thus has the L^2 orthogonal splitting $\mathrm{Im}\,\delta \oplus \mathrm{Ker}\,\delta^* = \Lambda_1^{m-1,\alpha}(M)$. Note that $\delta_g \widetilde{g} \in \mathrm{Im}\,\delta\delta^*$. By elliptic regularity theory, the equation (2.8) is uniquely solvable for $Y \in \chi^{m+1,\alpha}$ modulo $\mathrm{Ker}\,\delta^*$, i.e. modulo Killing fields. The statement (2.7) then follows from the implicit function theorem. This shows that the condition

$$\delta_g \widetilde{g} = 0, \tag{2.9}$$

gives a local slice to the action of $\mathrm{Diff}^{m+1,\alpha}(M)$ on $Met^{m,\alpha}(M)$, cf. [12] for further details.

It is well-known that the action of $\mathrm{Diff}^{m+1,\alpha}(M)$ on $Met^{m,\alpha}(M)$ is not C^1; for X a $c^{m+1,\alpha}$ vector field on M and $g \in Met^{m,\alpha}(M)$, $\delta_g^* X \in S_2^{m-1,\alpha}(M) \neq T(Met^{m,\alpha}(M))$. This explains the need to form slices as in (2.9) based at metrics $g \in Met^{m+1,\alpha}(M)$. For the same reason, the L^2 orthogonal splitting

$$S_2^{m,\alpha}(M) = T_g Met^{m,\alpha}(M) = \mathrm{Im}\,\delta^* \oplus \mathrm{Ker}\,\delta, \tag{2.10}$$

holds for $g \in Met^{m+1,\alpha}(M)$ but not for general $g \in Met^{m,\alpha}(M)$. However, since $Met^{m+1,\alpha}(M)$ is dense in $Met^{m,\alpha}(M)$, this does not create any undue difficulties.

The gauge condition (2.9) will be important in the analysis in Section 4.

We now turn the regular and singular decomposition of $\mathcal{D}$. Let $\mathcal{C}_0^{reg} = (\mathcal{C}_0^{reg})^{m,\alpha}$ be the space of $c^{m,\alpha}$ conformal classes which have no (non-zero) conformal Killing field. Note here that we are not excluding only essential conformal Killing fields (i.e. the round conformal class on S^3) but the larger space of conformal classes containing a conformal Killing field. It is clear that $\mathcal{C}_0^{reg}$ is open in $\mathcal{C}_0$; it is also well-known that $\mathcal{C}_0^{reg}$ is connected and dense in $\mathcal{C}_0$, cf. [12] for instance. Let then

$$\mathcal{D}^{reg} \subset \mathcal{D}, \tag{2.11}$$

be the restriction of the fibration $\pi : \mathcal{D} \to \mathcal{C}_0$ to $\mathcal{C}_0^{reg}$, so that the fibers over $[g]$ are the transverse trace-free symmetric 2-tensors over $[g] \in \mathcal{C}_0^{reg}$. It is clear that $\mathcal{D}^{reg}$ is also connected, open and dense in $\mathcal{D}$. Note that $\mathcal{D}^{reg} \subset \mathcal{D}'$ for $\mathcal{D}'$ as in (1.8).

Proposition 2.3. *The space $\mathcal{D}^{reg} = (\mathcal{D}^{reg})^{m-1,\alpha}$ is a smooth separable Banach manifold and the projection map $\pi : \mathcal{D}^{reg} \to \mathcal{C}_0^{reg}$ is a smooth bundle map.*

Proof: This result is essentially well-known; the proof is based on the York decomposition [35], cf. also [15], [16]. To begin, consider the operator

$$\delta_0 = \delta + \tfrac{1}{3}dtr : Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M) \to \Lambda^{m-1,\alpha}(M), \quad (g,h) \to \delta_g h + \tfrac{1}{3}dtr_g h.$$

Note that δ_0 is the L^2 adjoint of the conformal Killing operator $\widehat{\mathcal{L}}$. We first claim that δ_0 is a submersion, so that the implicit function theorem implies that $\mathcal{Z} = \delta_0^{-1}(0)$ is a smooth separable Banach submanifold of $Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$. To see this, analogous to (2.5), one has

$$\Lambda_1^{m-1,\alpha} = \overline{\mathrm{Im}\,\delta_0} \oplus \mathrm{Ker}\,\delta_0^*.$$

By assumption, $\mathrm{Ker}\,\delta_0^* = 0$. To show that δ_0 has closed range, let $\widetilde{g}$ be a metric in $(\mathcal{C}_0^{reg})^{m+1,\alpha}$ sufficiently close to $g \in (\mathcal{C}_0^{reg})^{m,\alpha}$ and let $\delta_0 = (\delta_0)_g$, $\widetilde{\delta}_0 = (\delta_0)_{\widetilde{g}}$. As in (2.8), consider the mapping

$$\delta_0 \widetilde{\delta}_0^* : \chi^{m+1,\alpha} \to \Lambda_1^{m-1,\alpha}. \tag{2.12}$$

This is an elliptic operator and so Fredholm for $\widetilde{g}$ sufficiently near g. Also $\mathrm{Ker}\,\delta_0\widetilde{\delta}_0^* = 0$, since this operator is a small perturbation of $\delta_0\delta_0^*$ which has no kernel by definition. It follows that δ_0 is of closed range and surjective.

To see that the kernel splits, given any $h \in T(S_2^{m,\alpha}(M))$, form $\delta_0 h$. The discussion above shows that for any such $\delta_0 h$, there is a unique X such that $\delta_0\widetilde{\delta}_0^* X = \delta_0 h$. Hence $h = (h - \widetilde{\delta}_0^* X) + \widetilde{\delta}_0^* X$ is the required splitting since $\widetilde{\delta}_0^* X \in S_2^{m,\alpha}(M)$. It thus follows from the implicit function theorem that $\mathcal{Z}$ is a smooth separable Banach manifold.

Next, observe that the trace operator $tr : Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M) \to c^{m,\alpha}(M)$, $(g,h) \to tr_g h$ is clearly a smooth submersion, so that $\mathcal{V} = tr^{-1}(0)$ is a smooth separable Banach submanifold of $Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$. The intersection $\mathcal{Z} \cap \mathcal{V}$ is transverse, cf. [16] for instance, and hence $\mathcal{D}^{reg} = \mathcal{Z} \cap \mathcal{V}$ is a smooth separable Banach submanifold of $Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$.

The fact that $\pi : \mathcal{D}^{reg} \to \mathcal{C}_0^{reg}$ is a bundle projection follows then again from the transversal intersection of $\mathcal{Z} \cap \mathcal{V}$.

■

Note that since space-time Killing fields Z need not be tangent to M, it is not true that $\Pi(\mathcal{C}_*^{sing}) \subset \mathcal{D}^{sing} = \mathcal{D} \setminus \mathcal{D}^{reg}$. For the same reason, $\mathcal{C}_*^{reg}$ does not map to $\mathcal{D}^{reg}$ under Π. Thus we need to modify the definition of the regular set $\mathcal{C}_*^{reg}$ slightly and define

$$\mathcal{C}^{reg} = \mathcal{C}_*^{reg} \cap \Pi^{-1}(\mathcal{D}^{reg}), \tag{2.13}$$

with $\mathcal{C}^{sing} = \mathcal{C} \setminus \mathcal{C}^{reg}$. Since $\mathcal{D}^{reg}$ is open and dense in $\mathcal{D}$, $\mathcal{C}^{reg}$ is open and dense in the manifold $\mathcal{C}^{reg}_*$; (the dense property follows easily from the fact that Π is a projection map). Thus one has a well-defined smooth map

$$\Pi^{reg} : \mathcal{C}^{reg} \to \mathcal{D}^{reg},$$

between separable Banach manifolds.

Corollary 2.4. *The map*

$$\Pi^{reg} : \mathcal{C}^{reg} \to \mathcal{D}^{reg} \tag{2.14}$$

is a smooth Fredholm map of Banach manifolds of Fredholm index zero.

Proof: This follows directly from Propositions 2.1 and 2.3, using the well-known Fredholm alternative and the fact that $D\Phi_y$ is Fredholm of index zero.

■

While it is currently unknown in general if the closure $\overline{\mathcal{C}^{reg}} = \mathcal{C}$, it is clear from the remarks above that $\overline{\mathcal{D}^{reg}} = \mathcal{D}'$, where the closure here is taken in $\mathcal{D}'$.

As discussed in (1.12), we first prove Theorem 1.1 for the map (2.14) in Section 4.

Next we discuss briefly the results established in the CMC case where $H = const$, cf. [22], [26]. Let $\mathcal{D}^{cmc} \subset \mathcal{D}$ be the subset of $([g], \sigma, H)$ where $H = const$. Let $Y[g]$ be the Yamabe constant of $[g]$. Define a connected open set $\mathcal{D}^{cmc}_+ \subset \mathcal{D}^{cmc}$ by

$$\mathcal{D}^{cmc}_+ = \begin{cases} Y[g] < 0 : & H \neq 0, \\ Y[g] = 0 : & \sigma \neq 0 \text{ and } H \neq 0, \\ Y[g] > 0 : & \sigma \neq 0. \end{cases} \tag{2.15}$$

Thus if $Y[g] < 0$, then $([g], \sigma, H) \in \mathcal{D}^{cmc}_+$ if the constant $H \neq 0$, if $Y[g] = 0$ then $([g], \sigma, H) \in \mathcal{D}^{cmc}_+$ if $H \neq 0$ and σ is not identically zero, while if $Y[g] > 0$, then $([g], \sigma, H) \in \mathcal{D}^{cmc}_+$ if σ is not identically zero.

Let $\mathcal{C}^{cmc}_+ = \Pi^{-1}(\mathcal{D}^{cmc}_+)$. Then the map

$$\Pi^{cmc}_+ : \mathcal{C}^{cmc}_+ \to \mathcal{D}^{cmc}_+,$$

is a smooth, proper homeomorphism; in particular Π^{cmc}_+ is one-to-one and onto. In the region where $Y[g] \geq 0$, Π^{cmc}_+ is a diffeomorphism, but this may not be the case everywhere when $Y[g] < 0$.

Let $\mathcal{D}^{cmc}_- = \mathcal{D}^{cmc} \setminus \mathcal{D}^{cmc}_+$ be the complementary closed set, so that $\mathcal{D}^{cmc}_- \subset \mathcal{D}^{cmc}$ is given by

$$\mathcal{D}^{cmc}_- = \begin{cases} Y[g] < 0 : & H = 0, \\ Y[g] = 0 : & \sigma = 0 \text{ or } H = 0, \\ Y[g] > 0 : & \sigma = 0. \end{cases} \tag{2.16}$$

Correspondingly, let $\mathcal{C}^{cmc}_- = \Pi^{-1}(\mathcal{D}^{cmc}_-)$. Then

$$\Pi^{cmc}_- : \mathcal{C}^{cmc}_- \to \mathcal{D}^{cmc}_-,$$

is the empty map, i.e. $\mathcal{C}^{cmc}_- = \emptyset$, except in the exceptional, boundary, situation where $Y[g] = 0$, $\sigma = H = 0$ in which case one has the trivial solutions $(\varphi, X) = (const, 0)$ with $g = c^4 g_0$ scalar-flat metrics with $K = 0$.

This gives a very clear distinction between the regions of existence and non-existence of solutions of the constraint equations (1.5)-(1.6). The map Π^{cmc}_+ must thus degenerate essentially everywhere on approach to $\partial\mathcal{C}^{cmc}_+$. Since $X = 0$, this means that φ must degenerate, as a positive function in $c^{m,\alpha}$, on approach to essentially any point in $\partial\mathcal{C}^{cmc}_+$. This will be seen in further detail in the analysis in Section 3. Referring to the definition (1.10), one clearly has

$$\mathcal{D}^{cmc}_+ \subset \mathcal{D}_+.$$

Note also that setting $\Pi^{cmc}_1 = \pi_1 \circ \Pi^{cmc}_+$ where π_1 is projection onto the base space $\mathcal{C}_0$ of conformal classes within $\mathcal{D}$, an inspection of the form of $\mathcal{D}^{cmc}_+$ in (2.15) shows that

$$\Pi^{cmc}_1 : \mathcal{C}^{cmc}_+ \to \mathcal{C}_0$$

is surjective.

In addition, we point out that although $\mathcal{C}^{cmc} \cap \mathcal{C}^{sing} \neq \emptyset$, the intersection is meager; in fact the complement $\mathcal{C}^{cmc} \cap \mathcal{C}^{reg}$ is open and dense in $\mathcal{C}^{cmc}$. To see this, it is proved in [14] that for $H = const$, space-time Killing fields Z are necessarily tangent to M (so $N = 0$ in the context of the discussion following (2.6)), except in the trivial case where g_0 is flat, $\varphi = const$ and $N = const$. In particular, it follows that

$$\mathcal{C}_*^{reg} \cap \mathcal{C}^{cmc} = \mathcal{C}^{reg} \cap \mathcal{C}^{cmc}, \tag{2.17}$$

except in this very special case. It is well-known (cf. [12] for instance) that the space $Met_0^{m,\alpha}(M)$ of metrics without Killing fields is open and dense in $Met^{m,\alpha}(M)$. From the structure of $\mathcal{D}^{cmc}$ and $\mathcal{C}^{cmc}$ above, this proves the claim.

Next we note a few standard elliptic regularity estimates to be used below. Let (φ, X) be a solution of the constraint equations (1.5)-(1.6). Elliptic regularity applied to the divergence constraint (1.5) gives the estimate

$$|X|_{C^{1,\alpha}} \leq C|\varphi^6|_{L^\infty}|dH|_{L^\infty}, \tag{2.18}$$

where the $C^{1,\alpha}$ and L^∞ norms are with respect to g_0; the constant C depends only on M and $g_0 \in \mathcal{D}$, cf. [28, Theorm 6.2.5]. Also, observe that, modulo constants, $|X|_{C^{m,\alpha}} \leq |\delta\widehat{\mathcal{L}}_X g_0|_{C^{m-2,\alpha}} \leq |\varphi^6 dH|_{C^{m-2,\alpha}}$, so that

$$|X|_{C^{m,\alpha}} \leq c|\varphi^6|_{C^{m-2,\alpha}}|H|_{C^{m-1,\alpha}}, \tag{2.19}$$

with again c depending only on M and $g_0 \in \mathcal{D}$. The estimates (2.18) and (2.19) require that (M, g_0) has no conformal Killing fields; they hold for general $g_0 \in Met^{m,\alpha}(M)$ if one assumes that X is L^2 orthogonal to the space of conformal Killing fields on (M, g_0).

Finally, as noted above, solutions (φ, X) of (1.5)-(1.6) depend on the choice of background metric $g_0 \in [g]$. This choice has been fixed to be a choice of unit volume Yamabe (minimizing) metric in $[g]$. However, for later purposes, it will be important to understand the behavior of the data (g_0, σ, H) and the fibers (φ, X) under rescalings of g_0.

Lemma 2.5. *For fixed geometric or physical data (g, K), suppose the background metric g_0 is changed by a rescaling*

$$\widetilde{g}_0 = m^4 g_0. \tag{2.20}$$

Then

$$\widetilde{\varphi} = m^{-1}\varphi, \quad \widetilde{\sigma} = m^{-2}\sigma, \quad \widetilde{X} = m^{-6}X, \tag{2.21}$$

while H remains unchanged, $\widetilde{H} = H$.

Proof: One has $g = \varphi^4 g_0 = \widetilde{\varphi}^4\widetilde{g}_0$, which gives the first statement in (2.21). Similarly, since $K = \varphi^{-2}(\sigma + \mathcal{L}_X g_0) + \frac{1}{3}H\varphi^4 g_0$ remains invariant, one has

$$\widetilde{\varphi}^{-2}\widetilde{\sigma} = \varphi^{-2}\sigma, \ \text{ and } \ \widetilde{\varphi}^{-2}\mathcal{L}_{\widetilde{X}}\widetilde{g}_0 = \varphi^{-2}\mathcal{L}_X g_0.$$

The first term gives $\widetilde{\sigma} = m^{-2}\sigma$. For the second term, $\mathcal{L}_{\widetilde{X}}\widetilde{g}_0 = m^{-2}\mathcal{L}_X g_0$, so $\mathcal{L}_{\widetilde{X}} g_0 = m^{-6}\mathcal{L}_X g_0$, which gives the last equality in (2.21).

■

It is easy to verify that all terms in the Lichnerowicz equation (1.6) scale in the same way, namely as m^{-5}. Similarly, both sides of the divergence constraint (1.5) scale as m^{-6}.

3. Initial estimates.

In this section, we derive initial estimates on the behavior of solutions (φ, X) of the constraint equations. These will be used later in Section 4.

We assume throughout this section (and the following) that (φ, X) solve the constraint equations (1.5)-(1.6) with data $(g_0, \sigma, H) \in \mathcal{D}$. If moreover g_0 has conformal Killing fields, it is always assumed that X is L^2 orthogonal to the space of conformal Killing fields.

Lemma 3.1. *Suppose there is a constant $D < \infty$ such that*

$$0 < D^{-1} \leq \inf\varphi \leq \sup\varphi \leq D < \infty. \tag{3.1}$$

Then there is a constant C, depending only on D and the background data $(g_0, \sigma, H) \in \mathcal{D}$ such that

$$|\varphi|_{C^{m,\alpha}} + |X|_{C^{m,\alpha}} \leq C. \tag{3.2}$$

Proof: By (3.1), φ and φ^{-1} are bounded in L^∞ by a fixed constant D. In particular, the right side of (1.5) is thus bounded in L^∞, since dH is bounded in $C^{m-2,\alpha}$. Elliptic regularity applied to the divergence constraint (1.5) then gives

$$|X|_{C^{1,\alpha}} \leq C,$$

since X is L^2 orthogonal to $\mathrm{Ker}\,\delta\widehat{\mathcal{L}}g_0$. The right side of the Lichnerowicz equation (1.6) is thus bounded in C^α and elliptic regularity applied to (1.6) implies φ is bounded in $C^{2,\alpha}$. In turn, this implies the right side of the divergence equation (1.5) is bounded in $C^{k,\alpha}$, $k = \min(2, m-2)$, and so elliptic regularity again implies X is bounded in $C^{k+2,\alpha}$. Continuing this process inductively gives (3.2).

■

Proposition 3.2. *Let*

$$\sup\varphi = M_0. \tag{3.3}$$

Then there is a constant $C < \infty$, depending only on M_0, and the target data $(g_0, \sigma, H) \in \mathcal{D}$, such that

$$\sup\varphi \leq C\inf\varphi.$$

In particular, under an upper bound on φ, $\inf\varphi$ can approach 0 only if $\sup\varphi$ approaches 0.

Proof: The proof uses the well-known Moser iteration argument; we follow closely the description of this method as described in [18, pp.194-198]. All computations below are with respect to the background Yamabe metric g_0 with $R_{g_0} = Y[g]$ (or more generally a compact set of such metrics).

By rescaling the background metric g_0 by a bounded amount $m = M_0$ as in Lemma 2.5, the potential φ may be rescaled to satisfy

$$\sup\varphi = 1. \tag{3.4}$$

We assume this has been done in the following. The bound (3.4), (or (3.3)), will only be used however in the arguments after (3.9).

To begin, from (1.6) we have

$$-\varphi^{7+k}\Delta\varphi = -\tfrac{1}{8}R_0\varphi^{k+8} + \tfrac{1}{8}|\sigma + \widehat{\mathcal{L}}_X g_0|^2\varphi^k - \tfrac{1}{12}H^2\varphi^{12+k}. \tag{3.5}$$

Integrating over M and applying the divergence theorem gives

$$-\int\varphi^{7+k}\Delta\varphi = \int\langle d\varphi^{7+k}, d\varphi\rangle = (7+k)\int\varphi^{6+k}|d\varphi|^2 = \frac{7+k}{(4+(k/2))^2}\int|d\varphi^{4+(k/2)}|^2. \tag{3.6}$$

Here and throughout the following, the integration over M is with respect to the volume form of (M, g_0). Also, constants c, C, c_S, used below may change from line to line, or even inequality to inequality, but only depend on the target data (M, g_0, σ, H). The Sobolev constant c_S of g_0 is uniformly controlled, so that

$$\left(\int\varphi^6\right)^{1/3} \leq c_S\int(|d\varphi|^2 + \varphi^2).$$

Applying this to $\varphi^{4+(k/2)}$ and using (3.6), one obtains from (3.5) that

$$\frac{7+k}{(4+k/2)^2}\left(\int\varphi^{24+3k}\right)^{1/3} \leq C\left(\int|\widehat{\mathcal{L}}_X g_0|^2\varphi^k + \int\varphi^k + \int\varphi^{8+k}\right), \tag{3.7}$$

for $k+7 > 0$, where we have dropped the negative H^2 term.

If $k+7 < 0$, the sign changes; in this case we may drop the σ and $\widehat{\mathcal{L}}_X g_0$ terms and obtain

$$\frac{|7+k|}{(4+k/2)^2}\left(\int\varphi^{24+3k}\right)^{1/3} \leq C\left(\int\varphi^{8+k} + \int H^2\varphi^{12+k}\right), \tag{3.8}$$

provided $k+8 \neq 0$. The case $k+8=0$ (the log case), will be considered later.

We begin with the case $k+7>0$, (the subsolution case). First, standard elliptic estimates for the divergence constraint (1.5) imply that

$$|\widehat{\mathcal{L}}_X g_0|_{L^4} \leq c|X|_{L^{1,4}} \leq c|X|_{L^{2,2}} \leq c|\delta\widehat{\mathcal{L}}_X g_0|_{L^2} \leq c(\int \varphi^{12})^{1/2},$$

where we have used Sobolev for the second inequality. By the Hölder inequality, this gives

$$\int |\widehat{\mathcal{L}}_X g_0|^2\varphi^k \leq (\int |\widehat{\mathcal{L}}_X g_0|^4)^{1/2}(\int \varphi^{2k})^{1/2} \leq c(\int \varphi^{2k})^{1/2}\int \varphi^{12}.$$

Inserting this in (3.7) implies that

$$\frac{1}{k}(\int \varphi^{24+3k})^{1/3} \leq c(\int \varphi^{2k})^{1/2}\int \varphi^{12} + c\int \varphi^k + c\int \varphi^{k+8},$$

where c depends only on the target data (g_0, σ, H). One may then iterate these inequalities, as in the usual Moser iteration, and starting with $k=4$, obtain

$$\sup \varphi \leq C|\varphi|_{L^4} \leq C|\varphi|_{L^2}, \tag{3.9}$$

where the last inequality follows from a standard interpolation inequality, [18, p.146]. Again C depends only on $(g_0, \sigma, H) \in \mathcal{D}$. Note that the estimate (3.9) does not require the assumption (3.3).

Next, as in [18], consider the two cases $-1 < k+7 < 0$ and $k+7 < -1$. First, by (3.4), $H^2\varphi^{12+k} = H^2\varphi^{8+k}\varphi^4 \leq H^2\varphi^{8+k}$, so that (3.8) implies that

$$\frac{|7+k|}{(4+k/2)^2}(\int \varphi^{3(8+k)})^{1/3} \leq C\int \varphi^{8+k}. \tag{3.10}$$

Now first choose $k+8 = p \in (0,1)$ small. Then Moser iteration starting at p and ending at $k+8=2$ shows that

$$\int \varphi^2 \leq c(\int \varphi^p)^{2/p}, \tag{3.11}$$

for any $p>0$ small, with $c = c(p)$.

Next one may perform the same Moser iteration for $k+8<0$ to obtain, for $p \in (0,1)$ as in (3.11),

$$(\int \varphi^{-p})^{-1/p} \leq c\inf \varphi, \tag{3.12}$$

with again $c = c(p)$. To connect the estimates (3.11) and (3.12), we claim that there is a constant $C = C(g_0, \sigma, H)$ and $p_0 \in (0,1)$ such that

$$\int \varphi^{p_0}\int \varphi^{-p_0} \leq C. \tag{3.13}$$

For this, the log case, we return to the Lichnerowicz equation (1.6) and write it as

$$\varphi^{-1}\Delta\varphi = \tfrac{1}{8}R_0 - \tfrac{1}{8}|\sigma + \widehat{\mathcal{L}}_X g_0|^2\varphi^{-8} + \tfrac{1}{12}H^2\varphi^4. \tag{3.14}$$

Integration, the divergence theorem and the estimate (3.4), together with the control on R_0 and H imply that

$$\int |d\log\varphi|^2 \leq C,$$

Next, still following [18, p.198], given any $p \in M$ and r small, let $\eta = \eta(p,r)$ be a cutoff function satisfying $\eta = 1$ on the geodesic ball $B_p(r)$, $\eta = 0$ on $M \setminus B_p(2r)$ with $|d\eta| \leq C/r$. One has

$$\int_{B_p(r)} |d\log\varphi| \leq c(\int_{B_p(r)} |d\log\varphi|^2)^{1/2}r^{3/2} \leq cr^{3/2}(\int_{B_p(2r)} |d\eta\log\varphi|^2)^{1/2}. \tag{3.15}$$

Multiplying (3.14) by η^2 and integrating by parts in the same way, using also the Cauchy-Schwarz inequality and the scale change $r \to 2r$, gives for r small,

$$\int_{B_p(2r)} |d\log\varphi|^2 \leq cr,$$

and hence by (3.15)

$$\int_{B_p(r)} |d\log\varphi| \leq Cr^2.$$

It then follows from the John-Nirenberg estimate [18, p.166], as in [18, p.198], that

$$\int \varphi^{p_0} \int \varphi^{-p_0} \leq C,$$

for some $p_0 \in (0,1)$, which proves (3.13).

Combining then (3.4), (3.9), (3.11)-(3.13) shows that

$$1 = \sup\varphi \leq C(\int \varphi^{p_0})^{1/p_0} \leq C(\int \varphi^{-p_0})^{-1/p_0} \leq C\inf\varphi,$$

which proves the result.

■

Proposition 3.2 shows that an upper bound on $\sup\varphi$ gives control of the Harnack constant

$$C_{Har}(\varphi) = \frac{\sup\varphi}{\inf\varphi}, \tag{3.16}$$

of φ, given control of the target data in $\mathcal{D}$. A converse of this result is proved in Theorem 4.1 below. As an application of Proposition 3.2, we prove the following:

Proposition 3.3. *Continuing under the assumption* (3.3)*, suppose there is a constant* $s_0 > 0$ *such that* $\inf|\sigma| \geq s_0 > 0$*. Then there is a constant* $\kappa_0 > 0$*, depending only on* M_0*,* s_0 *and* $(g_0, \sigma, H) \in \mathcal{D}$*, such that*

$$\inf\varphi \geq \kappa_0 > 0. \tag{3.17}$$

Moreover, if $Y(g) \leq -y_0 < 0$*, then*

$$\inf\varphi \geq \kappa_0 > 0, \tag{3.18}$$

where κ_0 *depends only on* M_0*,* y_0 *and* $(g_0, \sigma, H) \in \mathcal{D}$*.*

Proof: To prove (3.17), by Proposition 3.2 it suffices to obtain a lower bound on $m_0 = \sup\varphi$. Namely if $m_0 \leq 1$ then the bound on C_{Har} from Proposition 3.2 shows that a lower bound on $\inf\varphi$ and $\sup\varphi$ are equivalent. Now integrating the Lichnerowicz equation (1.6) over (M, g_0) gives

$$\int_M |\sigma + \widehat{\mathcal{L}}_X g_0|^2 \varphi^{-7} \leq \tfrac{1}{8}|R_0| \int_M \varphi + \tfrac{1}{12}\sup H^2 \int_M \varphi^5 \leq cm_0,$$

for a fixed constant c. We assume here without loss of generality that $m_0 \leq 1$. Since $\varphi^{-7} \geq m_0^{-7}$, it follows that

$$m_0^{-7}\int_M |\sigma|^2 \leq m_0^{-7}\int_M |\sigma + \widehat{\mathcal{L}}_X g_0|^2 \leq \int_M |\sigma + \widehat{\mathcal{L}}_X g_0|^2 \varphi^{-7} \leq cm_0,$$

so that

$$\int_M |\sigma|^2 \leq cm_0^8.$$

Now $|\sigma|^2$ is controlled in $c^{m-1,\alpha}$ and so the bound $\inf|\sigma| := |\sigma|(p) \geq s_0 > 0$ implies there is a fixed r_0 such that $|\sigma|(x) \geq s_0/2$ for all $x \in B_p(r_0)$. It follows that

$$r_0^4 s_0^2 \leq \int_M |\sigma|^2 \leq cm_0^8.$$

This gives a lower bound for m_0 in terms of σ and s_0, which thus proves (3.17).

For (3.18), evaluating the Lichnerowicz equation (1.6) at $\min\varphi = \inf\varphi$ gives

$$0 \leq \tfrac{1}{12}H^2(\inf\varphi)^5 + \tfrac{1}{8}R_0\inf\varphi,$$

(regardless of the behavior of σ and $\widehat{\mathcal{L}}_X g_0$). Recall that R_0 is the Yamabe constant $Y[g]$ of $[g]$. If $R_0 < 0$, then $H^2(\inf\varphi)^4 \geq \frac{3}{2}|R_0|$, which proves (3.18).

■

Remark 3.4. Simple examples show that (3.17) is not true without the assumption on σ. Thus, suppose g_0 is the standard product metric on $S^1(1) \times S^2(1)$, so that $R_{g_0} = 2$. Choose

$$\sigma = \kappa(-d\theta^2 + \tfrac{1}{2}g_{S^2(1)}),$$

for some constant κ. The form σ is transverse-traceless with respect to g_0 and has constant norm $|\sigma|^2 = \frac{3}{2}\kappa^2$. Let also $H = c$, an arbitrary constant. Then the divergence constraint (1.5) is satisfied by setting $X = 0$ while the Lichnerowicz equation (1.6) holds if $\varphi = \varepsilon = const$ and

$$0 = \tfrac{1}{4}\varepsilon - \tfrac{1}{8}|\sigma|^2\varepsilon^{-7} + \tfrac{1}{12}H^2\varepsilon^5.$$

This holds by choosing κ so that $\frac{3}{2}\kappa^2 = 2\varepsilon^8 + \frac{2}{3}H^2\varepsilon^{12}$.

This example shows that one may have $Y(g) > 0$ with H an arbitrary constant, with $\varphi \to 0$ uniformly as $\sigma \to 0$ uniformly.

Summarizing briefly, the results of this section show that, given control on the target data in $\mathcal{D}'$, there is no degeneration of the fiber data (φ, X) when $Y(g) < 0$ is bounded away from 0, or when $Y(g) \geq 0$ and σ bounded away from 0 at some point, *provided* one has a sup bound on φ. This shows that control of the target data in $\mathcal{D}_+$ and control of $\sup\varphi$ implies control of the fiber data (φ, X). This behavior is one of the main ingredients in the definition of $\mathcal{D}_+$.

4. Control on φ.

In this section, we address the existence of a sup bound on φ and prove the main part of Theorem 1.1. We begin with a general discussion of convergence or blow-up of metrics in the current context.

Let $(g^i, K^i) \in \mathcal{C}$ be a sequence of vacuum Einstein initial data with target data controlled in $\mathcal{D}'$, so that

$$g_0^i \to g_0 \ \text{ in } \ Met^{m,\alpha}(M), \tag{4.1}$$

as representatives of $[g_i] \to [g]$, and similarly,

$$\sigma_i \to \sigma \ \text{ in } \ S_2^{m-1,\alpha}(M), \text{ and } H_i \to H \ \text{ in } \ c^{m-1,\alpha}(M). \tag{4.2}$$

Let p_i be any sequence of points in M such that

$$m_i = \varphi_i(p_i) \to \infty, \ \text{ as } \ i \to \infty. \tag{4.3}$$

(The points p_i may or may not realize the maximum of φ_i). We first perform a rescaling of the background metrics g_0^i as in (2.20). Let

$$\widetilde{g}_0^i = m_i^4 g_0^i, \tag{4.4}$$

so that, by (2.21),

$$\widetilde{\varphi}_i = m_i^{-1}\varphi_i.$$

Thus $\widetilde{\varphi}_i > 0$ with $\widetilde{\varphi}_i(p_i) = 1$. As discussed in Section 2, the physical data (g, K) do not change under such rescalings. One has also the rescaled data $(\widetilde{\sigma}_i, \widetilde{X}_i)$ as in (2.21). The rescaled, tilded data satisfy the constraint equations (1.5)-(1.6).

Now if $\varphi_i \to \infty$ at a sequence of points $p_i \in M$, then of course the sequence $\{\widetilde{g}_0^i\}$ (as well as $\{g_i\}$) is also blowing up and has no limit. As originally made clear by Gromov [19], this can be corrected or compensated by diffeomorphisms which blow up or dilate the coordinates around p_i. Thus, the metrics g_0^i have uniformly bounded curvature by (4.1) and hence the metrics $\widetilde{g}_0^i$ have curvature $|Rm_{\widetilde{g}_0^i}|_{\widetilde{g}_0^i}$ converging to 0 as $i \to \infty$. By the Cheeger-Gromov convergence theorem, cf. [30] and references therein, the control (4.1) implies that the sequence $\{\widetilde{g}_0^i\}$ based at p_i converges to the tangent space $T_pM = \mathbb{R}^3$ to M at $p = \lim p_i$ (in a subsequence), endowed with a flat Euclidean metric $g_{Eucl} = g_0|_{T_p(M)}$. The convergence is locally in $c^{m,\alpha}$ *modulo diffeomorphisms*.

This means the following in this context. Let x^k, $k = 1, 2, 3$ be a fixed system of local coordinates around p (and so also around p_i for i sufficiently large). Define new coordinates

$$y^k = m_i^2 x^k.$$

This gives a sequence of local diffeomorphism from a domain in $\mathbb{R}^3$ (with y-coordinates) to a domain in M (with x-coordinates) containing p. A unit ball in the y-coordinates maps to a very small ball in the x-coordinates. These mappings may be extended to a sequence of global embeddings

$$\psi_i : \mathbb{R}^3 \to M, \tag{4.5}$$

which locally, in domains of bounded $\mathbb{R}^3$-distance to p, are of the form $\psi_i(y^k) = x^k$. The dilations ψ_i pull back very small balls to balls of unit size; the coordinates are being blown-up. Note that

$$dy^k = m_i^2 dx^k, \quad \text{while} \quad \frac{\partial}{\partial y^k} = m_i^{-2}\frac{\partial}{\partial x^k}.$$

By (4.4), one has $(\widetilde{g}_0^i)_{k\ell}$ in the x-coordinates is of order $\sim m_i^4$, but $\psi_i^*(\widetilde{g}_0^i)_{k\ell}$, now in the y-coordinates, is on the order of 1. The convergence theory above implies that, in a subsequence,

$$\psi_i^* \widetilde{g}_0^i \to g_{Eucl}, \tag{4.6}$$

locally in $C^{m,\alpha}$, uniformly on compact subsets of $\mathbb{R}^3$. Thus, when restricted to any ball $B(R)$ of finite radius in $\mathbb{R}^3$, the metrics $\psi_i^* \widetilde{g}_0^i$ converge in $C^{m,\alpha}$ to the Euclidean metric on $B(R) \subset \mathbb{R}^3$.

As in Section 2, diffeomorphisms ψ act on the physical data (g, K) in the usual way by pullback,

$$(g, K) \to (\psi^* g, \psi^* K),$$

and similarly on the target data $([g], \sigma, H) \in \mathcal{D}$ and fiber data (φ, X),

$$([g], \sigma, H) \to ([\psi^* g], \psi^*\sigma, H \circ \psi)., \quad (\varphi, X) \to (\varphi \circ \psi, \psi^* X)$$

Since the physical data (g, K) are generally covariant, the map Π in (1.7) is equivariant with respect to these actions of $\mathrm{Diff}^{m+1,\alpha}(M)$ on the domain and target of Π. The constraint equations (1.5)-(1.6) are invariant under this action.

Let $\bar{g}_0^i = \psi_i^* \widetilde{g}_0^i$ and similarly let $\bar{\sigma}_i = \psi_i^* \widetilde{\sigma}_i$, $\bar{X}_i = \psi_i^* \widetilde{X}_i$, $\bar{H}_i = H_i \circ \psi_i$. We begin with an analysis of $\bar{\sigma}_i$. One has

$$\bar{\sigma}_i = \bar{\sigma}^i_{k\ell} dy^k dy^\ell = \widetilde{\sigma}^i_{k\ell} dx^k dx^\ell.$$

Since $y^k = m_i^2 x^k$, $\bar{\sigma}^i_{k\ell} = m_i^{-4} \widetilde{\sigma}^i_{k\ell} \to 0$. In addition, by (2.21), $\widetilde{\sigma}_i = m_i^{-2}\sigma_i$. Since $\bar{g}_0^i(\partial_{y_k}, \partial_{y_\ell}) \to \delta_{k\ell}$, it follows that

$$|\bar{\sigma}_i|_{\bar{g}_0^i} = O(m_i^{-6}) \to 0. \tag{4.7}$$

(An equivalent calculation is as follows. One has $|\bar{\sigma}_i|^2_{\bar{g}_0^i} = |\widetilde{\sigma}_i|^2_{\widetilde{g}_0^i}$. Since $\widetilde{g}_0^i = m_i^4 g_0^i$, orthornormal bases scale as $\widetilde{e}_k = m_i^{-2} e_k$ and hence $|\widetilde{\sigma}_i|^2_{\widetilde{g}_0^i} = m_i^{-8}|\widetilde{\sigma}_i|^2_{g_0^i} = m_i^{-12}|\sigma_i|^2_{g_0^i}$. Since $|\sigma_i|^2_{g_0^i}$ is uniformly bounded, one has $|\bar{\sigma}_i|^2_{\bar{g}_0^i} = O(m_i^{-12})$ as in (4.7)).

Next, dropping the index i from the notation, note that

$$|(\bar{g}_0)_{k\ell} - \delta_{k\ell}| = O(m^{-4}). \tag{4.8}$$

To see this, $\bar{g}_0(\partial_{y_k}, \partial_{y_\ell})$ on $B(R) \subset \mathbb{R}^3$ equals $g_0(\partial_{x_k}, \partial_{x_\ell})$ on $B(Rm^{-2}) \subset (M, g_0)$. The metrics $g_0 = g_0^i$ are uniformly controlled in $C^{m,\alpha}$ and one has then the uniform estimate $|(g_0)_{k\ell} - \delta_{k\ell}| = r^2|Rm|_{g_0} + o(r^2)$. The $|Rm|_{g_0}$ term is uniformly bounded, and since $r \sim m^{-2}$, (4.8) follows. Similar estimates hold up to order $m + \alpha$.

At this point, to simply the discussion and notation, we assume that the sequence p_i in (4.3) satisfies

$$m_i = \varphi_i(p_i) \geq c_0 \sup \varphi_i, \tag{4.9}$$

for some fixed constant $c_0 > 0$. The constants appearing below depend then on c_0. (It is simple to generalize the discussion below to the general case of (4.3), where (4.9) may not hold; however one then loses the uniform bound (4.11) below).

Next, (2.21) gives $\bar{X} = m^{-6}X$ while $|X|_{g_0} \leq Cm^6$ by (2.18) and (4.9), so that

$$|\bar{X}|_{\bar{g}_0} = m^2 |\bar{X}|_{g_0} \leq Cm^2. \tag{4.10}$$

Thus, apriori, $|\bar{X}_i|_{\bar{g}_0^i}$ could be unbounded as $i \to \infty$. On the other hand, since for any vector field Y, $|\bar{\nabla} Y|_{\bar{g}} = m^{-4}m^4 |\nabla Y|_g$, (since the covariant derivative ∇ is scale-invariant), it follows that

$$|\bar{\nabla}\bar{X}|_{\bar{g}_0} = m^{-6}|\nabla X|_{g_0} \leq C, \tag{4.11}$$

again by (2.18) and (4.9).

Finally, observe that $\bar{H}_i = \psi_i^*(H_i)$ converges in $C^{m-1,\alpha}_{loc}$ to the constant function $\bar{H} = H(p)$ on $\mathbb{R}^3$.

In sum, in such blow-ups, the target data converges to "trivial data",

$$(M, [g_0^i], \sigma_i, H_i) \to (\mathbb{R}^3, [g_{Eucl}], 0, H(p)), \tag{4.12}$$

where $[g_{Eucl}]$ is the conformal class of the Euclidean metric on $\mathbb{R}^3$ and $H(p) = \lim H_i(p_i)$ is a constant function on $\mathbb{R}^3$. Moreover,

$$(M, \bar{g}_0^i, p_i) \to (\mathbb{R}^3, g_{Eucl}, 0), \tag{4.13}$$

The convergence in (4.12) and (4.13) is in $C^{m,\alpha}$, uniformly on compact subsets of $\mathbb{R}^3$.

Using the analysis above, we first, we prove a sup bound on φ in the special case where φ satisfies an apriori global Harnack inequality.

Theorem 4.1. *Suppose there is a constant $H_0 > 0$ such that $|H(x)| \geq H_0$, for all $x \in M$. If φ satisfies a uniform Harnack inequality, i.e. there is a constant C (depending only on the target data in $\mathcal{D}'$) such that*

$$C_{Har} = \frac{\sup \varphi}{\inf \varphi} \leq C, \tag{4.14}$$

then there is a constant K, depending only on H_0, C and the target data in $\mathcal{D}'$ such that

$$\sup \varphi \leq K. \tag{4.15}$$

Proof: The proof is by contradiction, using a blow-up argument. Thus, if (4.15) fails under the given hypotheses, then there is a sequence $(g_i, K_i) \in \mathcal{C}'$, with $\Pi(g_i, K_i) = ([g_i], \sigma_i, H_i)$ bounded in the metric on $\mathcal{D}'$, and points $p_i \in M$ such that

$$\varphi_i(p_i) \to +\infty.$$

As above, we fix background metrics $g_0^i \to g_0^\infty$, representing the convergence of the conformal classes. We then blow-up or rescale the background metrics g_0^i, based at p_i, to the metrics $\bar{g}_0^i$ as described above with

$$m_i = \varphi_i(p_i).$$

Observe that by the assumption (4.14), there is a uniform constant c such that

$$0 < c \leq \bar{\varphi}_i \leq c^{-1}, \tag{4.16}$$

globally on M. Throughout the rest of this section, we will often use the relation

$$\widehat{\mathcal{L}}_X g = 2\delta_0^* X,$$

where δ_0^* is the trace-free part of the adjoint δ^* of δ with respect to (M, g).

Consider then the divergence constraint (1.5) in bar data, i.e.

$$\bar{\delta}\bar{\delta}_0^* \bar{X}_i = -\tfrac{1}{3}\bar{\varphi}_i^6 d\bar{H}_i, \tag{4.17}$$

where $\bar{\delta}$ and $\bar{\delta}_0^*$ are taken with respect to $\bar{g}_0^i$. As noted above, $\bar{\varphi}_i$ is uniformly bounded, while $\bar{H}_i \to$ *const* and $|d\bar{H}_i| = O(m_i^{-2}) \to 0$.

Although $\bar{X}_i$ may be unbounded in the $\bar{g}_0^i$ norm, by (4.11) the covariant derivative $\bar{\nabla}\bar{X}_i$ and hence $\bar{\delta}_0^*\bar{X}_i$ remains bounded as $i \to \infty$. It follows then from (4.17) and the analysis above that

$$\bar{\delta}\bar{\delta}_0^*\bar{X}_i \to 0,$$

in $C^{m-2,\alpha}_{loc}(\mathbb{R}^3)$.

Consider first the limit operator $\bar{\delta}\bar{\delta}_0^*$ on $\mathbb{R}^3$. We temporarily drop the bar notation in the following. First, a standard Weitzenbock formula on $\mathbb{R}^3$ gives $\beta\delta^* = \frac{1}{2}D^*D$, where β is the Bianchi operator $\beta = \delta + \frac{1}{2}dtr$. One has $\beta\delta^* = \delta\delta^* - \frac{1}{2}d\delta$, so that $\delta\delta^* = \frac{1}{2}D^*D + \frac{1}{2}d\delta$. Also, $\delta_0^*W = \delta^*W + \frac{1}{3}\delta W g$ so that $\delta\delta_0^* = \delta\delta^* - \frac{1}{3}d\delta$. This gives on $\mathbb{R}^3$,

$$\delta\delta_0^*W = \tfrac{1}{2}D^*DW + \tfrac{1}{6}d\delta W.$$

On the blow-up sequence, although $\bar{X}_i$ may be unbounded, $\bar{D}\bar{X}_i$ remains bounded, and hence the limit DX on $\mathbb{R}^3$ satisfies

$$D^*DX + \tfrac{1}{3}d\delta X = 0. \tag{4.18}$$

Next, take the divergence δ of (4.18); since δ commutes with D^*D on $\mathbb{R}^3$, one obtains

$$-\Delta\delta X - \tfrac{1}{3}\Delta\delta X = -\tfrac{4}{3}\Delta\delta X = 0.$$

Since by (4.11), δX is bounded, the Liouville theorem on $\mathbb{R}^3$ implies $\delta X = const$ and hence $d\delta X = 0$. It follows that

$$D^*DX = 0. \tag{4.19}$$

Formally then, the components of X (in a Cartesian frame) are thus harmonic functions and since DX is bounded, it follows again from the Liouville theorem on $\mathbb{R}^3$ that the components of X are affine functions, and DX is parallel on $\mathbb{R}^3$. In particular, $|DX| = const$ and so $|\delta_0^* X| = const$.

We now return to the analysis of the blow-up sequence. The analysis above shows that

$$\bar{D}^*\bar{D}\bar{X}_i = O(m_i^{-2}), \tag{4.20}$$

since as in (4.17), $|d\bar{H}_i| = O(m_i^{-2})$. Here $\bar{D}^*\bar{D}$ is the rough Laplacian with respect to $\bar{g}_0^i$. It follows then from (4.8) that

$$D_e^*D_e\bar{X}_i = O(m_i^{-2}), \tag{4.21}$$

where $D_e^*D_e$ is now the Euclidean Laplacian on $\mathbb{R}^3$. This holds for any choice of base points p_i, given the uniform behavior in (4.16). At this point, we specialize the choice of the base points p_i by requiring

$$|\bar{X}_i|(p_i) = \max|\bar{X}_i|,$$

where the norms are with respect to $\bar{g}_0^i$. By (4.8) and (4.10),

$$||\bar{X}_i| - |\bar{X}_i|_{g_{Eucl}}| \leq C, \tag{4.22}$$

for some constant C and so in particular

$$|\max|\bar{X}_i| - \max|\bar{X}_i|_{g_{Eucl}}| \leq C, \tag{4.23}$$

Around $p_i \to p \in M$, (in a subsequence), choose flat Cartesian coordinates y^j (depending on i and centered at p_i) so that, dropping the index i from the notation,

$$\bar{X} = \sum a^j \partial_{y^j},$$

and, at p, i.e. p_i,

$$\bar{X}(p) = a^1(p)\partial_{y^1},$$

$a^1(p) > 0$, so that $a^2 = a^3 = 0$ at p. In particular by (4.22)-(4.23),

$$|\max|\bar{X}|_{g_{Eucl}} - a^1(p)| \leq C. \tag{4.24}$$

Of course we may have $a^j = a_i^j \to \infty$ uniformly as $i \to \infty$ in general.

Now let

$$\widehat{X} = \bar{X} - a^1(p)\partial_{y^1}, \tag{4.25}$$

with coordinates $\widehat{a}^1 = a^1 - a^1(p)$, $\widehat{a}^k = a^k$ for $k = 2,3$. The point p_i realizes the maximum of $|\bar{X}|$ and by (4.24),

$$\widehat{a}^1 \leq C.$$

By (4.21),

$$\Delta_{Eucl}\widehat{a}^1 = O(m^{-2}),$$

i.e. $\Delta_{Eucl}\widehat{a}_i^1 = O(m_i^{-2})$ on $\mathbb{R}^3$; here Δ_{Eucl} is the scalar Euclidean Laplacian. By the Liouville theorem, any harmonic function on $\mathbb{R}^3$ which is bounded above is constant. Since $\widehat{a}^1(p) = 0$, it follows that

$$\widehat{a}^1 \to 0,$$

uniformly on compact sets in $\mathbb{R}^3$ and so in $C_{loc}^{m,\alpha}(\mathbb{R}^3)$. It then follows that also

$$\widehat{a}^k \to 0,$$

in $C_{loc}^{m,\alpha}(\mathbb{R}^3)$, for $k = 2,3$ and hence $\widehat{X} \to 0$ in $C_{loc}^{m,\alpha}(\mathbb{R}^3)$.

Since in (4.25) we have just subtracted a constant vector field $a_i^1(p)\partial_{y_i^1}$ from $\bar{X}_i$, using (4.8) once more, it follows that

$$\widehat{\mathcal{L}}_{\bar{X}_i}\bar{g}_0^i = 2\bar{\delta}_0^*\bar{X}_i \to 0. \tag{4.26}$$

This completes the analysis of the rescaling blow-up of the divergence constraint equation.

Consider now the same rescaling of the Lichnerowicz equation (1.6), i.e.

$$\bar{\Delta}\bar{\varphi}_i = \tfrac{1}{8}\bar{R}_0\bar{\varphi}_i - \tfrac{1}{8}|\bar{\sigma} + \widehat{\mathcal{L}}_{\bar{X}_i}\bar{g}_0|^2\bar{\varphi}^{-7} + \tfrac{1}{12}H^2\bar{\varphi}_i^5, \tag{4.27}$$

where again we have dropped the index i from $\bar{\Delta}$, $\bar{\sigma}$ and H. Here all terms are bounded and converge (in a subsequence) to a limit on $\mathbb{R}^3$. One has $\bar{R}_0 = R_{\bar{g}_0^i} \to 0$, $\bar{\sigma}_i \to 0$ and (4.26) then implies that

$$\Delta\bar{\varphi} = \tfrac{1}{12}H^2\bar{\varphi}^5, \tag{4.28}$$

on $\mathbb{R}^3$ with $\bar{\varphi}$ uniformly bounded and $H = \lim H(p_i) = const.$ Now the standard Bochner-Weitzenbock formula on $\mathbb{R}^3$ states

$$\tfrac{1}{2}\Delta|du|^2 = |D^2u|^2 + \langle d\Delta u, du\rangle.$$

Setting $u = \bar{\varphi}$ and using (4.28) gives

$$\tfrac{1}{2}\Delta|d\bar{\varphi}|^2 = |D^2\bar{\varphi}|^2 + \tfrac{5}{12}H^2\bar{\varphi}^4|d\bar{\varphi}|^2 \geq 0. \tag{4.29}$$

Since $|d\bar{\varphi}|^2$ is uniformly bounded, evaluating (4.29) on a maximizing sequence for $|d\bar{\varphi}|^2$ shows that $d\bar{\varphi} = 0$, so that $\bar{\varphi} = const.$ Then (4.28) implies necessarily $H = 0$. However, the hypothesis implies that H is uniformly bounded away from 0. This contradiction proves the result.

■

Remark 4.2. It does not appear straightforward to generalize the proof of Theorem 4.1 to the situation where the hypothesis (4.14) is removed. The essential difficulty is obtaining a bound on $|\nabla\bar{X}|_{\bar{g}_0}$, or a bound on δX, without use of a Harnack inequality. Such a generalization of the proof would make the different arguments to follow below unnecessary.

Remark 4.3. It is possible that a more careful analysis of the proof of Theorem 4.1, in particular regarding the limiting behavior of the equation (4.27), would allow one to remove the hypothesis $\inf|H| \geq H_0 > 0$ when $Y[g] = R_{g_0} \geq r_0 > 0$. If so, this would give a generalization of Theorem 4.1 (and thus of Theorem 1.1) in the region where $Y[g] > 0$. Note that this is consistent with the behavior of Π_+^{cmc} on $\mathcal{D}_+^{cmc}$ defined as in (2.15). We will not pursue this issue further here however.

Remark 4.4. There are numerous examples of curves (g_t, K_t), $t \in [0,\infty)$ with $Y(g_t) \leq -c < 0$ where $H_t \to 0$, $\sigma_t \to 0$ and $\varphi_t \to \infty$ pointwise as $t \to \infty$. The simplest examples are the Milne universe or hyperbolic cone metric

$$g^{(4)} = -dt^2 + t^2g_{-1},$$

where (M, g_{-1}) is a hyperbolic 3-manifold. This is a flat (and hence Ricci-flat) Lorentz metric on $\mathbb{R}^+ \times M$. One easily sees that on the slices $M = M_t = \{t = const\}$, $\sigma = 0$, $\varphi_t = \sqrt{t} \to \infty$ and $H_t = \frac{3}{t} \to 0$ as $t \to \infty$.

Similar behavior occurs in the long-time future behavior of vacuum space-times near the flat hyperbolic cone space-time by the work of Andersson-Moncrief [2], as well as in the $U(1)$-symmetric space-times of Choquet-Bruhat-Moncrief [9].

Theorem 4.1, together with the results of Section 3, shows that it suffices to obtain a bound on the global Harnack constant

$$C_{Har}(\varphi) = \frac{\sup\varphi}{\inf\varphi},$$

to obtain control on the fiber data (φ, X).

The remainder of the argument (for the proof of Theorem 1.1 below) does not use the constraint equations (1.5)-(1.6) per se at all. It just involves the space of metrics $Met^{m,\alpha}(M)$ (away from the round metric (S^3, g_{+1})).

To begin, recall from Section 2 that the group $\mathrm{Diff}^{m+1,\alpha}(M)$ acts on the fiber data (φ, X) by pullback. It is clear that the basic invariants of φ used above, in particular, $\sup\varphi$, $\inf\varphi$, and $C_{Har}(\varphi)$ etc, are invariant under this action when $\mathrm{Conf}(M,[g]) = \{id\}$. When $\mathrm{Conf}(M,[g])$ is compact, there is a bounded indeterminacy in the choice of φ which does not effect the arguments below. However, to avoid an unbounded indeterminacy, we exclude the round conformal class $(S^3, [g_{+1}])$.

To understand the local behavior of φ, in particular $C_{Har}(\varphi)$ on $\mathcal{C}'$, it suffices to understand this behavior transverse to the $\mathrm{Diff}^{m+1,\alpha}(M)$ action, i.e. on slices satisfying the divergence-free gauge condition (2.9).

We first express the divergence-free condition in terms of the target data in $\mathcal{D}$. The following Lemma is a standard calculation; for completeness we include the proof.

Lemma 4.5. *On a Riemannian 3-manifold (M, g), if $g = \varphi^4 g_0$, then*

$$\delta_{\varphi^4 g_0} h = \varphi^{-4}\delta_{g_0} h - \varphi^{-6} h(\nabla_{g_0}\varphi^2, \cdot) + 2\varphi^{-4}(tr_{g_0} h) d\log\varphi. \tag{4.30}$$

Proof: By definition, for Y a vector field on M, one has

$$\int \widetilde{\delta} h(Y) dv_{\widetilde{g}} = \int \langle h, \widetilde{\delta}^* Y\rangle_{\widetilde{g}} dv_{\widetilde{g}}.$$

Now $\widetilde{\delta}^* Y = \frac{1}{2}\mathcal{L}_Y \varphi^4 g = \frac{1}{2}[\varphi^4 \mathcal{L}_Y g + 4\varphi^4 Y(\log\varphi) g]$ so that

$$\int \langle h, \widetilde{\delta}^* Y\rangle_{\widetilde{g}} dv_{\widetilde{g}} = \int \langle h, \varphi^4 \delta^* Y + 2\varphi^4 Y(\log\varphi) g\rangle_{\widetilde{g}} dv_{\widetilde{g}}$$

$$= \int \langle h, \varphi^{-4}\delta^* Y + 2\varphi^{-4} Y(\log\varphi) g\rangle_g dv_{\widetilde{g}} = \int \langle h, \varphi^2 \delta^* Y + 2\varphi^2 Y(\log\varphi) g\rangle_g dv_g.$$

This gives

$$\int \varphi^6 \widetilde{\delta} h(X) dv = \int \langle \delta(\varphi^2 h)(X) + 2\varphi^2 (tr_g h) d\log\varphi)(X).$$

Since Y is arbitrary, it follows that

$$\widetilde{\delta} h = \varphi^{-6}\delta(\varphi^2 h) + 2\varphi^{-4}(tr_g h) d\log\varphi,$$

which then easily leads to (4.30).

∎

Proposition 4.6. *Let $g \in Met^{m,\alpha}(M)$ and suppose $\widetilde{g}$ is a nearby metric in the divergence-free slice through g, so that (2.9) holds. Let $D = diam_{g_0}(M)$. Then there is a fixed numerical constant c such that if*

$$|\widetilde{g}_0 - g_0|_{C^1} \leq \mu, \tag{4.31}$$

then

$$C_{Har}(\widetilde{\varphi}) \leq c e^{c\mu D}[C_{Har}(\varphi)]^{1+c\mu}, \tag{4.32}$$

Proof: Set $h = \widetilde{g} = \widetilde{\varphi}^4 \widetilde{g}_0$ in Lemma 4.5. Then (4.30) becomes

$$\delta_{\varphi^4 g_0}(\widetilde{\varphi}^4 \widetilde{g}_0) = (\frac{\widetilde{\varphi}}{\varphi})^4 \delta_{g_0}(\widetilde{g}_0) - \varphi^{-4}\widetilde{g}_0(\nabla_{g_0}\widetilde{\varphi}^4, \cdot) - \varphi^{-6}\varphi^4 \widetilde{g}_0(\nabla_{g_0}\varphi^2, \cdot) + 2(\frac{\widetilde{\varphi}}{\varphi})^4 (tr_{g_0}\widetilde{g}_0) d\log\varphi,$$

which gives

$$\delta_{\varphi^4 g_0}(\widetilde{\varphi}^4 \widetilde{g}_0) = (\frac{\widetilde{\varphi}}{\varphi})^4 \delta_{g_0}(\widetilde{g}_0) - (\frac{\widetilde{\varphi}}{\varphi})^4 \widetilde{g}_0(\nabla_{g_0}\log\widetilde{\varphi}^4, \cdot) - (\frac{\widetilde{\varphi}}{\varphi})^4 \widetilde{g}_0(\nabla_{g_0}\log\varphi^2, \cdot) + 2(\frac{\widetilde{\varphi}}{\varphi})^4 (tr_{g_0}\widetilde{g}_0) d\log\varphi,$$

By (2.9), $\delta_{\varphi^4 g_0}(\widetilde{\varphi}^4 \widetilde{g}_0) = 0$, so that dividing by $(\frac{\widetilde{\varphi}}{\varphi})^4$, gives

$$\delta_{g_0}(\widetilde{g}_0) - \widetilde{g}_0(\nabla_{g_0}\log\widetilde{\varphi}^4, \cdot) - \widetilde{g}_0(\nabla_{g_0}\log\varphi^2, \cdot) + 2(tr_{g_0}\widetilde{g}_0) d\log\varphi = 0.$$

Next write $\widetilde{g}_0(\nabla_{g_0}\log\varphi^2, \cdot) = 2d\log\varphi + 2(\widetilde{g}_0 - g_0)(\nabla_{g_0}\log\varphi, \cdot)$, $\widetilde{g}_0(\nabla_{g_0}\log\widetilde{\varphi}^4, \cdot) = 4d\log\widetilde{\varphi} + 4(\widetilde{g}_0 - g_0)(\nabla_{g_0}\log\widetilde{\varphi}, \cdot) = 4d\log\varphi + 4d\log\frac{\widetilde{\varphi}}{\varphi} + 4(\widetilde{g}_0 - g_0)(\nabla_{g_0}\log\varphi, \cdot)$ and $tr_{g_0}\widetilde{g}_0 = 3 + tr_{g_0}(\widetilde{g}_0 - g_0)$, $\delta_{g_0}\widetilde{g}_0 = \delta_{g_0}(g_0 - \widetilde{g}_0)$. This implies that

$$4d\log\frac{\widetilde{\varphi}}{\varphi} = \delta_{g_0}(\widetilde{g}_0 - g_0) + 2tr_{g_0}(\widetilde{g}_0 - g_0) d\log\varphi - 6(\widetilde{g}_0 - g_0)(\nabla_{g_0}\log\varphi, \cdot). \tag{4.33}$$

It follows that if (4.31) holds, then

$$|d\log\frac{\widetilde{\varphi}}{\varphi}| \leq c\mu|d\log\varphi| + c\mu,$$

where the norm is taken with respect to g_0, so that

$$|d\log\widetilde{\varphi}| \leq (1+c\mu)|d\log\varphi| + c\mu, \tag{4.34}$$

The quantity $|d\log\varphi|$ is the infinitesimal Harnack constant and a bound on $|d\log\varphi|$ gives a bound on the Harnack constant by integration. Thus (4.34) shows that the growth of the infinitesimal Harnack constant is controlled by the distance of the background metrics g_0 in C^1 norm. In particular, integration of (4.34), or more precisely (4.33), along a minimizing geodesic joining points realizing $\sup\widetilde{\varphi}$ and $\inf\widetilde{\varphi}$ shows that

$$C_{Har}(\widetilde{\varphi}) \leq ce^{c\mu D}[C_{Har}(\varphi)]^{1+c\mu},$$

for $\widetilde{g}$ and g satisfying (4.31). This proves the result.

■

The Harnack constant $C_{Har}(\varphi)$ is a continuous function of $\varphi \in c^{m,\alpha}(M)$, but of course a priori it may not be a smooth or even differentiable function. Nevertheless, one may linearize the argument above to obtain a bound on the upper derivative

$$\bar{D}_{\varphi}C_{Har}(\varphi') := \limsup_{t\to 0}\frac{1}{t}[C_{Har}(\varphi+t\varphi') - C_{Har}(\varphi)].$$

Namely let $g_0(t)$ be a curve in $Met^{m,\alpha}(M)$ of the form $g_0(t) = g_0 + t\mu'$. Setting then $\mu(t) = t\mu'$, and correspondingly $\varphi_t = \varphi + t\varphi'$ in (4.32), it follows easily that

$$\bar{D}_{\varphi}C_{Har}(\varphi') \leq c|\mu'|_{C^1}[[C_{Har}(\varphi)+1]. \tag{4.35}$$

Thus an infinitesimal bound on the metric variation in $\mathcal{D}'$ gives a bound for the infinitesimal variation of the Harnack constant C_{Har}.

Theorem 4.7. *Let $x(t) = (g(t), K(t))$ be a smooth curve in $\mathcal{C}_+$ such that the image curve $y(t) = \Pi_+(x(t)) = (g_0(t), \sigma(t), H(t))$ has bounded length in $\mathcal{D}_+$:*

$$\ell(g_0(t), \sigma(t), H(t)) \leq L < \infty.$$

Then there is a constant c, depending only on $\inf_t dist(y(t), \partial\mathcal{D}_+) > 0$, such that the length of $x(t)$ is bounded by

$$\ell(g(t), K(t)) \leq cL.$$

Proof: By the results of Section 3 and Theorem 4.1, it suffices to obtain a bound on the Harnack constant $C_{Har}(g(t))$. Without loss of generality we may assume that t is arclength parameter for the curve $y(t) \in \mathcal{D}_+$, so that $|\frac{dg_0(t)}{dt}|_{C^1} \leq 1$.

Let $h = dg(t)/dt$. As in (2.12) and (2.10), for any fixed t, we may write

$$h = \delta^*_{\widetilde{g}}Z + h_0,$$

where $\delta_g h_0 = 0$ and $\widetilde{g} \in Met^{m+1,\alpha}(M)$ is arbitrarily close to g, i.e. $g(t)$. The derivative of the Harnack constant $C_{Har}(g(t))$ in the direction $\delta^*_{\widetilde{g}}Z$ may be made arbitrarily small since its derivative in the direction $\delta^*_g Z$ vanishes; recall here that C_{Har} is diffeomorphism invariant, cf. the discussion preceding Lemma 4.5. On the other hand, by (4.35) the variation of C_{Har} in the h_0 direction is bounded above by the variation $\mu' = |dg_0(t)/dt|_{C^1}$ of $g_0(t)$. This gives

$$\frac{\bar{d}}{dt}C_{Har}(g(t)) \leq C|\frac{dg_0(t)}{dt}|_{C^1}[C_{Har}(g(t))+1] \leq C[C_{Har}(g(t))+1],$$

where $\frac{\bar{d}}{dt}$ denotes the upper derivative. It follows that the Harnack constant C_{Har} of $g(t)$ can increase at most exponentially in t. This proves the result.

■

Note that given the results in Section 3 and Theorem 4.1, the proof of Theorem 4.7 itself depends only on the curve $g(t)$ of metrics in $Met^{m,\alpha}$ and the image curve $[g(t)] = g_0(t)$ under Π', i.e. the constraint equations are not used. The proof above breaks down when curves $g(t)$ approach metrics g which have an essential conformal Killing field, i.e. g is conformal to the round metric on S^3.

We now combine the results above to prove Theorem 1.1. We first prove the result in the regular region

$$\mathcal{C}^{reg}_+ = \mathcal{C}_+ \cap \mathcal{C}^{reg},$$

cf. (2.13). More precisely, $\mathcal{C}_+^{reg}$ is defined to be the collection of components of $\mathcal{C}_+ \cap \mathcal{C}^{reg}$ intersecting $\mathcal{C}_+^{cmc}$. In the following section, the proof will be adapted to include the complementary singular region $\mathcal{C}_+^{sing} = \mathcal{C}_+ \cap \mathcal{C}^{sing}$. We note that as discussed in Section 2, it is not known whether

$$\overline{\mathcal{C}_+^{reg}} = \mathcal{C}_+, \tag{4.36}$$

i.e. whether the closure of $\mathcal{C}_+^{reg}$ in the $C^{m,\alpha} \times C^{m-1,\alpha}$ topology is all of $\mathcal{C}_+$. Apriori, it is possible that the singular set $\mathcal{C}_+^{sing}$ has non-empty interior so that $\partial \mathcal{C}_+^{reg} \neq \mathcal{C}_+^{sing}$.

While one may treat the two cases $\mathcal{C}_+^{reg}$ and $\mathcal{C}_+^{sing}$ together as a whole, we discuss them separately for clarity of the exposition.

Note also that we prove Theorem 1.1 in the more general setting $(g, K) \in C^{m,\alpha} \times C^{m-1,\alpha}$, $m \geq 3$, and not directly in the C^∞ setting.

Proof of Theorem 1.1: Regular region of $\mathcal{C}_+$.

We begin by recalling some basic facts from global analysis on separable Banach manifolds first developed by Smale [33]. Let $F : X \to Y$ be a smooth Fredholm map of index zero between connected separable Banach manifolds X, Y. A point $x \in X$ is a regular point of F if the linearization $D_x F : T_x X \to T_{f(x)} Y$ of F at x is a surjective bounded linear map. Since the Fredholm index is zero, $D_x F$ is then an isomorphism of Banach spaces. A point is a singular point if it is not a regular point. By the inverse function theorem, the map F is locally a diffeomorphism onto its image in a neighborhood of a regular point. A point $y \in Y$ is a regular value of F if every point in the inverse image $F^{-1}(y)$ is a regular point. It follows again from the inverse function theorem that $F^{-1}(y)$ is then a discrete, countable collection of points in X. A point $y' \in Y$ is a singular value of F if it is not a regular value, i.e. some point $x' \in F^{-1}(y')$ is a singular point of F. Note that by definition, any point $y \notin \operatorname{Im} F$ is a regular point of F.

By the Sard-Smale theorem [33], the regular values of F are of second category in Y, so given as the intersection of a countable collection of open and dense sets in Y. Moreover, let V be a compact connected finite dimensional manifold, possibly with boundary, of dimension at least one. Then for any $\varepsilon > 0$, any smooth embedding $g : V \to Y$ admits a smooth perturbation $g' : V \to Y$, ε-close to g, such that g' is transverse to F; this means that for any $(x, v) \in X \times V$ such that $F(x) = g'(v) = y$, $T_y Y$ is spanned by the image of $D_x F$ and $D_v g'$. In addition, for such maps g' transverse to F, the inverse image $F^{-1}(g'(V))$ is a smooth embedded submanifold of X of dimension equal to $dim\, V$.

The results above do not require that F is a proper Fredholm map. If F is proper, then the regular values are open and dense. The restriction $F|_K : K \to Y$ of F to any compact domain $K \subset X$ is of course a proper map; in particular, for any regular value y of $F|_K$, the cardinality of the inverse image $(F_K)^{-1}(y) = F^{-1}(y) \cap K$ is finite.

We apply the perspective above to the separable Banach manifold $X = \mathcal{C}_+^{reg}$ and the Fredholm map

$$F = \Pi_+^{reg} := \Pi_+|_{\mathcal{C}_+^{reg}} : \mathcal{C}_+^{reg} \to \mathcal{D}_+^{reg}.$$

Pick an arbitrary base point $y = y(0) \in \mathcal{D}_+^{reg}$ and assume $y(0)$ is a regular point of Π_+^{reg}. (Note that the notion of regular point of Π_+^{reg} is quite different than the notion of regular point of $\mathcal{C}$ in the sense of (2.13)). Choose any other regular point of Π_+^{reg} in $\mathcal{D}_+^{reg}$ and let $y(t) = ((g_0(t), \sigma(t), H(t))$, $t \in I = [0, 1]$ be a smooth compact curve in $\mathcal{D}_+^{reg}$ with endpoints $y(0)$, $y(1)$. As discussed above, without loss of generality, we will assume that $y(t)$ is transverse to Π_+^{reg}. The inverse image $(\Pi_+^{reg})^{-1}(y(I))$ is a countable collection of curves in $\mathcal{C}_+^{reg}$, $(\Pi_+^{reg})^{-1}(y(I)) = \cup_n x_n(I)$.

Let $x(\tau) \subset (\Pi_+^{reg})^{-1}(y(I))$ be any component curve in the inverse image. The exact parametrization $\tau \in I = [0, 1]$ chosen for x does not play any significant role; the image curve $x(I)$ is the primary object. Theorem 4.7 implies that $x(\tau)$ is a curve of bounded length in $\mathcal{C}_+^{reg}$. This implies that $x(\tau)$ cannot "run off to infinity" in $\mathcal{C}_+^{reg}$. However, there may be points τ_i such that $x(\tau_i) \to \partial \mathcal{C}_+^{reg} \subset \mathcal{C}_+^{sing}$. This will be analysed in Section 5 (using essentially the same methods) and thus we assume here that any such component $x(\tau)$ is a compact curve in $\mathcal{C}_+^{reg}$:

$$x(I) \subset\subset \mathcal{C}_+^{reg}. \tag{4.37}$$

This is equivalent to the statement that the closure

$$\overline{(\Pi_+^{reg})^{-1}(y(I))} \cap \mathcal{C}_+^{sing} = \emptyset.$$

Given any finite collection x_j, $j = 1, \dots, \ell$ of such curves, one may choose a large connected compact submanifold $K \subset \mathcal{C}_+^{reg}$ such that $K \cap (\Pi_+^{reg})^{-1}(y(I)) = \cup_{j=1}^{\ell} x_j(I)$ and consider the proper map $\Pi_+^{reg}|_K : K \to \mathcal{D}_+$. By transversality, the images $x_j(I)$ of the curves x_j are compact embedded 1-manifolds over $y(I)$ in C_+^{reg}. Hence each such curve $x(I) \in \{x_j(I)\}$ is either an embedded circle, or a curve with endpoints $x(0)$ and $x(1)$ in the discrete fibers $(\Pi_+^{reg})^{-1}(\partial y(I)) \subset \mathcal{C}_+^{reg}$. Thus either $\partial x(I) = \emptyset$ or $\partial x(I) \subset (\Pi_+^{reg})^{-1}(\partial y(I))$, cf. [13, p.74] for further discussion. Note here that it is not asserted that $x(0) \in (\Pi_+^{reg})^{-1}(y(0))$ or $x(1) \in (\Pi_+^{reg})^{-1}(y(1))$. Simple examples based on fold maps $x \to x^2 = y : \mathbb{R} \to \mathbb{R}$ show that this may not be the case. We will say the component curve $x(\tau)$ covers $y(I)$ if $\Pi_+^{reg}(x(I)) = y(I)$, so that $\Pi_+^{reg} \circ x$ surjects onto $Im\, y$. This is the case exactly when the endpoints $x(0)$ and $x(1)$ are in distinct fibers of $\Pi_+^{reg}(\partial y(I))$.

Now choose the initial regular value point $y(0) \in \mathcal{D}_+^{reg}$ of $y(t)$ to be a point with mean curvature $H(0) = const$. As discussed in Section 2, since $y(0) \in \mathcal{D}_+^{cmc}$, there is a *unique* solution $(\varphi(0), X(0)) = (\varphi(0), 0)$ of the constraint equations (1.5)-(1.6), i.e. the fiber over $y(0)$ consists of a single point

$$(\Pi_+)^{-1}(y(0)) = \mathbf{x}(0).$$

It is then easy to see that there are many initial regular value points $y(0) \in \mathcal{D}^{cmc}$.

Consider an arbitrary regular point $y = y(1)$ which may be joined to $y(0)$ in $\mathcal{D}_+^{reg}$ by a compact curve (of finite length) transverse to Π_+^{reg} and satisfying (4.37). There is then a unique lift, i.e. a smooth curve $\mathbf{x}(\tau)$ with $\Pi_+^{reg}(\mathbf{x}(I)) \subset y(I)$, with $\mathbf{x}(\tau)$ starting at the point $\mathbf{x}(0)$. If $\mathbf{x}(1) = \mathbf{x}(0)$ then the curve $\mathbf{x}(\tau)$ is not an embedded submanifold-with-boundary, giving a contradiction. An equivalent argument is as follows. For any compact n-dimensional ball $B \subset \mathcal{C}_+^{reg}$ containing $\mathbf{x}(0)$, for t small (depending on B), the values $y(t)$ are regular values of $(\Pi_+^{reg})|_B$ and the intersection of the fiber $(\Pi_+^{reg})^{-1}(y(t))$ with B consists of a unique point $\mathbf{x}(t)$ near $\mathbf{x}(0)$. If $\mathbf{x}(0) = \mathbf{x}(1)$ then for $t > 0$ small, the inverse image $(\Pi_+^{reg})^{-1}(y(t)) \cap B$ would consist of two distinct points $\mathbf{x}(\tau)$, $\mathbf{x}(\tau')$, $\tau \sim 0$, $\tau' \sim 1$, which contradicts the uniqueness above.

It follows that one must have $\mathbf{x}(1) \in (\Pi_+^{reg})^{-1}(y(1))$ and hence $y(1) \in \mathrm{Im}\,\Pi_+^{reg}$. This proves that Π_+^{reg} is surjective onto the regular values in $\mathcal{D}_+^{reg}$, provided (4.37) holds. It is then also surjective onto the singular values in $\mathcal{D}_+^{reg}$, since any singular value in $\mathcal{D}_+^{reg}$ may be joined by a short curve in $\mathcal{D}_+^{reg}$ to a regular value in $\mathcal{D}_+^{reg}$ and one may apply the argument above to this extended curve. This completes the proof of Theorem 1.1 in the regular region C_+^{reg}.

∎

Remark 4.8. Observe that the proof above proves Theorem 1.1 in the case that $\mathcal{C}_+^{reg}$ is dense in $\mathcal{C}_+$, i.e. (4.36) holds. As in the last paragraph of the proof of Theorem 1.1 above, this follows from the compactness of the lifts $x(\tau)$ from Theorem 4.7, which hold equally well for curves $x(\tau) \in \overline{\mathcal{C}_+^{reg}}$ and not only $x(\tau) \in \mathcal{C}_+^{reg}$.

Recall that $\mathcal{D}_+^{reg}$ is dense in $\mathcal{D}_+$, cf. the discussion following (2.11), and it is conjectured that $\mathcal{C}^{reg}$ is dense in $\mathcal{C}$, cf. [5].

Recall further that $\mathcal{C}_+^{reg}$ is dense in $(\mathcal{C}_*^{reg})_+ = \mathcal{C}_+ \cap \mathcal{C}_*^{reg}$, cf. (2.13) and the discussion following (2.11).

5. Singular locus of $\mathcal{C}$.

In this section we extend the results of Sections 3-4 to the singular region $\mathcal{C}^{sing}$ of $\mathcal{C}_+$, i.e. remove the assumption (4.37) and thus complete the proof of Theorem 1.1. As noted above in Remark 4.8, the significant part of the singular set for this purpose is $\mathcal{C}_*^{sing}$ as in (2.6).

The singular region $\mathcal{C}_*^{sing}$ of $\mathcal{C}_+$ is the locus of $(g, K) \in \mathcal{C}_+$ such that

$$\mathrm{Ker}(D_{(g,K)}\Phi)^* \neq 0.$$

In the following let (γ, κ) denote a typical element in $\mathcal{C}_*^{sing}$; we use (g, K) to denote general points (regular or singular) in $\mathcal{C}_+$. If $(N, Y) \in \mathrm{Ker}(D_{(\gamma,\kappa)}\Phi)^*$ then the vector field $Z = N\nu + Y$ extends to a space-time Killing field of the maximal Cauchy development $(\mathcal{M}, g^{(4)})$ of (M, γ, κ). It is then well-known, cf. [5] for instance, that the data (γ, κ) and (N, Y) satisfy the Killing initial data equations

$$\tfrac{1}{2}\mathcal{L}_Y \gamma = -N\kappa, \tag{5.1}$$

$$D^2N = N(Ric + H\kappa - 2\kappa^2) - \mathcal{L}_Y\kappa,$$

where the Hessian D^2 and Ricci curvature are taken with respect to (M,γ), $H = tr_\gamma\kappa$.

Following [15], the structure of the singular locus $\mathcal{C}_*^{sing}$ is analysed by means of the Lyapunov-Schmidt method based on the splitting (2.5), i.e.

$$c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M) = \operatorname{Im} D_{(\gamma,\kappa)}\Phi \oplus \operatorname{Ker}(D_{(\gamma,\kappa)}\Phi)^*. \tag{5.2}$$

Thus, fix a given $(\gamma,\kappa) \in \mathcal{C}_*^{sing}$ with the associated space of space-time Killing fields $K^* = \operatorname{Ker}(D_{(\gamma,\kappa)}\Phi)^*$ and let $k = dim\operatorname{Ker}(D_{(g,K)}\Phi)^*$. While it is most common to work with the slice K^* to $\operatorname{Im} D\Phi$ as in (5.2), it will be more convenient in the following to work with the space

$$\widehat{K} = \{(0,Y) : (N,Y) \in K^*\}, \tag{5.3}$$

i.e. we only consider the shift component Y of a Killing field Z, setting the lapse N to 0. To prove $\widehat{K}$ gives a slice to $\operatorname{Im} D\Phi$ as in (5.2), we need the following:

Lemma 5.1. *One has $dim\widehat{K} = dimK^*$ and $\widehat{K}$ is a complementing space to $\operatorname{Im} D\Phi$, so that*

$$c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M) = \operatorname{Im} D_{(\gamma,\kappa)}\Phi \oplus \widehat{K}. \tag{5.4}$$

The splitting (5.4) is a direct, but not L^2 orthogonal, sum of closed subspaces.

Proof: Since the splitting (5.2) is L^2 orthogonal, it suffices to show that for any $0 \neq (N,Y) \in K^*$, $(0,Y)$ is not L^2 orthogonal to K^*. This implies in particular that if (N_i,Y_i), $i = 1,\ldots,k$ is a basis for K, then $(0,Y_i)$ is a basis for $\widehat{K}$.

Suppose instead that $(0,Y)$ is L^2 orthogonal to K^*, so that in particular $\langle(0,Y),(N,Y)\rangle = 0$ for the L^2 pairing with respect to γ. This implies that

$$\int_M |Y|^2 = 0,$$

so that $Y = 0$ and so the Killing vector field $Z = N\nu$. It follows from the first equation in (5.1) that $N\kappa = 0$, so that either $N = 0$ or $\kappa = 0$. If $N = 0$, then $Z = (N,Y) = 0$ which is ruled out. Since then $\kappa = 0$, $H = 0$ which is also ruled out since $(\gamma,\kappa) \in \mathcal{C}_+$. Note also that if $\kappa = 0$, the second equation in (5.1) implies that $D^2N = NRic$. Taking the trace implies that $\Delta N = NR_\gamma$. Since $\kappa = 0$, the Hamiltonian constraint (1.2) gives $R_\gamma = 0$, so that $\Delta N = 0$ and hence $N = const$. It follows again from the second equation in (5.1) that $Ric_\gamma = 0$, so that (M,γ) is a flat 3-manifold. This corresponds to the exceptional case discussed following (2.16).

■

For (g,K) near (γ,κ), consider the local "enhanced" constraint domain

$$\mathcal{C}_\star^{enh} = \Phi^{-1}(\widehat{K}). \tag{5.5}$$

Thus the local space $\mathcal{C}_\star^{enh}$ consists of pairs (g,K) near (γ,κ) such that

$$\Phi(g,K) = (0,Y),$$

for $(N,Y) \in K^*$. Let π_1 be the projection onto the first factor in (5.4) along the slice $\widehat{K}$. The splitting (5.4) implies that the linearization $\pi_1 \circ \Phi$ is a submersion onto $\operatorname{Im} D\Phi$ at (γ,κ) and it follows from the implicit function theorem that $(\pi_1 \circ \Phi)^{-1}(0) = \mathcal{C}_\star^{enh}$ is a smooth separable Banach manifold near (γ,κ). The space

$$\mathcal{C} \cap \mathcal{C}_\star^{enh} \subset \mathcal{C}_\star^{enh}$$

is then given locally near (γ,κ) by the zero set of a smooth function

$$F = \pi_2 \circ \Phi|_{\mathcal{C}_\star^{enh}} : \mathcal{C}_\star^{enh} \to \mathbb{R}^k, \tag{5.6}$$

i.e. $\mathcal{C}_+ \cap \mathcal{C}_\star^{enh} = F^{-1}(0)$. Here π_2 is projection onto the second factor $\widehat{K}$ in (5.3). We will assume in the following that the local space $\mathcal{C}_\star^{enh}$ is maximally extended in $Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$ as a smooth connected Banach submanifold. Roughly speaking, $\mathcal{C}_\star^{enh}$ represents a k-dimensional manifold thickening of $\mathcal{C}_+$ near (γ,κ).

Now the projection map

$$\Pi^{enh} : \mathcal{C}_\star^{enh} \to \mathcal{D},$$

is defined just as before, $\Pi^{enh}(g, K) = ([g], \sigma, H) \in \mathcal{D}$. However, only the open-dense region $\mathcal{D}^{reg} \subset \mathcal{D}$ (cf. (2.11)) is a smooth Banach manifold. Since we need to work in the context of smooth Banach manifolds, throughout the following we work with the restriction

$$\mathcal{C}^{enh} = \mathcal{C}_{\star}^{enh} \cap \Pi^{-1}(\mathcal{D}^{reg}), \tag{5.7}$$

as in (2.13). Note that the singular region $\mathcal{C}_{*}^{sing} \cap \mathcal{C}^{enh}$ are exactly vacuum Killing initial data with no non-tangential conformal Killing field; in particular the lapse N is typically non-zero. As in Remark 4.8, the main point is that $\mathcal{C}^{enh}$ is open and dense in $\mathcal{C}_{\star}^{enh}$ so that the restriction to $\mathcal{C}^{enh}$ does not cause any significant problems.

It now follows that the projection map

$$\Pi^{enh} : \mathcal{C}^{enh} \to \mathcal{D}^{reg},$$

is a smooth Fredholm map between smooth separable Banach manifolds. Note that the Fredholm index of Π^{enh} is now k. The inverse image of a regular point $y \in \mathcal{D}^{reg}$ of Π^{enh} is a collection (possibly infinite) of connected k-dimensional submanifolds $V^k = (\Pi^{enh})^{-1}(y) \subset \mathcal{C}^{enh}$. Since $\widehat{K}$ is non-compact, one cannot expect the components V^k to be compact. The inverse image of a curve $y(t)$, $t \in I = [0, 1]$, in $\mathcal{D}^{reg}$ transverse to Π^{enh} is a collection of connected $(k+1)$-dimensional submanifolds $W^{k+1} \subset (\Pi^{enh})^{-1}(y(I)) \subset \mathcal{C}^{enh}$ again typically non-compact.

Given such a path $y(t)$, consider the map F as in (5.6),

$$F : W^{k+1} \to \mathbb{R}^k. \tag{5.8}$$

The components of $F^{-1}(0)$ give the components of the inverse image $\Pi_{+}^{-1}(y(I)) \subset \mathcal{C}_{+} \cap W^{k+1}$. The basic compactness property analogous to Theorem 4.7 that is needed is given by the following result.

Proposition 5.2. *let $q \in \mathbb{R}^k$ be a regular value of F. Then $F^{-1}(q)$ is a countable collection of compact connected manifolds of codimension k in W^{k+1}, i.e. a collection of compact 1-manifolds $\{x_q\} \subset W^{k+1}$.*

Proof: The (finite dimensional) regular value theorem implies that the components $\{x_q\}$ are 1-manifolds, so one needs to prove compactness. For this, it suffices to show that all the estimates and results of Sections 3-4 hold for (g, K) in the local enhanced spaces

$$\mathcal{C}_{+}^{enh} = (\Pi^{enh})^{-1}(\mathcal{D}_{+}^{loc}) \supset \mathcal{C}_{+}^{loc}.$$

Observe that when $k = 0$, Proposition 5.2 is essentially equivalent to Theorem 4.7, cf. the proof of Theorem 1.1 in Section 4. To prove that the components x_q are compact, fix the data $\Pi(\gamma, K) = (\gamma_0, \sigma_0, H_0) \in \mathcal{D}_{+}^{reg}$ and consider a neighborhood $\mathcal{D}_{+}^{loc} \subset \mathcal{D}_{+}^{reg}$ of $(\gamma_0, \sigma_0, H_0)$. Given fixed data $(N, Y) \in K^* = \mathrm{Ker}(D\Phi_{(\gamma,\kappa)})^*$, elliptic regularity implies that $(N, Y) \in c^{m,\alpha} \times \chi^{m,\alpha}$, so that in particular $Y \in \chi^{m,\alpha}(M)$. We consider all fiber data (φ, X) over $\mathcal{D}_{+}^{loc}$ such that

$$\Phi(\varphi, X) \in \widehat{K},$$

for $\widehat{K}$ as in (5.3).

To begin, the divergence and Lichnerowicz equations (1.5)-(1.6) now take the form

$$\delta(\widehat{\mathcal{L}}_X g_0) = (-\tfrac{2}{3}dH + Y)\varphi^6, \tag{5.9}$$

$$\Delta\varphi = \tfrac{1}{8}R_0\varphi - \tfrac{1}{8}|\sigma + \widehat{\mathcal{L}}_X g_0|^2\varphi^{-7} + \tfrac{1}{12}H^2\varphi^5. \tag{5.10}$$

Thus the only change is to the divergence constraint. (This is the reason for working with $\widehat{K}$ in place of K^*). For such fixed $Y \in \widehat{K}$ consider the set of solutions $(\varphi, X) \in \mathcal{C}_{+}^{enh}$ of (5.9)-(5.10). We then list the changes needed to the proofs of the results in Sections 3-4.

To begin, given the control on Y, a simple and straightforward inspection shows that the proof of Lemma 3.1 and proofs of Propositions 3.2 and 3.3 remain valid with the presence of Y, without any further changes besides constants depending on $|Y|_{C^{m,\alpha}}$.

The proof of Theorem 4.1 does require some non-trivial changes. First note that the shift vector fields Y in $\widehat{K}$ (and $(N, Y) \in K^*$) are independent of the scale chosen for the background metric g_0; thus they remain the same when passing from g_0^i to the rescaled background metrics $\bar{g}_0^i = m_i^4 g_0^i$. As before, one passes to the Euclidean limit g_{Eucl} by means of a sequence of blow-up diffeomorphisms. Under this convergence, Y converges to a constant or parallel vector field on $\mathbb{R}^3$, (namely the field $Y(p)$ parallel translated to $\mathbb{R}^3$).

Now perform the same analysis on the rescaled divergence equation (5.9) as in (4.17)-(4.20). The equation (4.18), on the sequence $\bar{X}_i$ and $\bar{g}_0^i$, takes the component form

$$\Delta a^j + \tfrac{1}{3}\partial_{y^j}(div\bar{X}) = \varphi^6\langle Y, \partial_{y^j}\rangle + O(m_i^{-2}),$$

where all metric data are with respect to the Euclidean metric and we have dropped the subscript i. One has $\partial_{y^j} div\bar{X} = div(\partial_{y^j}\bar{X})$ and $\langle Y, \partial_{y^j}\rangle = c^j$ for some constant c^j. This gives

$$\Delta a^j + \tfrac{1}{3}div(\partial_{y^j}\bar{X}) = c^j\bar{\varphi}^6 + O(m_i^{-2}). \tag{5.11}$$

Recall from Section 4, in particular (4.11), that although $\{a^j\} = \{a_i^j\}$ may be unbounded, the derivatives da^j are uniformly bounded and pass to the limit $\mathbb{R}^3$, remaining uniformly bounded. This gives the equation

$$\Delta a^j + \tfrac{1}{3}div(\partial_{y^j}\bar{X}) = c^j\bar{\varphi}^6, \tag{5.12}$$

on the limit $\mathbb{R}^3$. Now choose r large and integrate (5.12) over the r-ball $B(r)$. The divergence theorem implies that

$$\int_{S(r)}(\partial_r a^j + \tfrac{1}{3}\langle\partial_{y^j}\bar{X}, \partial_r\rangle) = c^j\int_{B(r)}\bar{\varphi}^6. \tag{5.13}$$

Since $\partial_r a^j$ and $\langle\partial_{y^j}\bar{X}, \partial_r\rangle$ are uniformly bounded, it follows that the left side of (5.13) is bounded by Cr^2, for some constant C, independent of r large. On the other hand, $\bar{\varphi}$ is uniformly bounded away from 0 and infinity, so the right side of (5.13) grows as r^3 unless $c^j = 0$. It follows that $Y = 0$ on the limit $\mathbb{R}^3$. The remaining part of the proof of Theorem 4.1 now carries forward, without any further changes.

Finally, the results of Section 4 after Theorem 4.1 do not depend on the constraint equations (1.5)-(1.6) and so remain valid here also. This proves the result.

■

The 1-manifolds $\{x_q\}$ of Proposition 5.2 may be embedded S^1's, so have empty boundary, or are embedded intervals with boundary in $\partial W^{k+1} \subset (\Pi^{enh})^{-1}(\partial y(I))$. These regular curves $\{x_q\}$ are local analogs of the curves x in Section 4. In particular they satisfy the conclusion of Theorem 4.7.

Now let $q = q_\varepsilon$ be a regular value of F with $|q_\varepsilon| \le \varepsilon$, with inverse image a collection of curves $\{x_\varepsilon(\tau)\}$. The compactness of each x_ε and continuity of F implies that for any given $\tau \in I$ and for any given sequence $\varepsilon_i \to 0$, a subsequence of $x_{\varepsilon_i}(\tau)$ converges to a limit point $x(\tau) \in \mathcal{C}_+ \cap W^{N+1} = F^{-1}(0)$ with $x(\tau) \in \Pi_+^{-1}(y(I))$. Thus as $\varepsilon \to 0$, the q_ε-level sets $x_\varepsilon(I)$ of F converge in the Hausdorff topology on W^{k+1} to the zero level set $F^{-1}(0) = \mathcal{C}_+ \cap W^{k+1}$. Without further information, $F^{-1}(0)$ may be an arbitrary closed set in W^{k+1} and so $\{x_\varepsilon\}$ is a smooth ε-Hausdorff approximation to $F^{-1}(0)$.

Remark 5.3. Given $(\gamma, \kappa) \in (\mathcal{C}_*^{sing})_+$, let L be any finite dimensional space of vector fields in $\chi^{m,\alpha}(M)$ containing the kernel $\widehat{K}$, i.e. $(0, L) \supset \widehat{K}$. In analogy to (5.7), define the local space

$$\mathcal{C}_L^{enh} = \Phi^{-1}(L) \cap \Pi^{-1}(\mathcal{D}_+^{reg}).$$

It is then easy to see that all the discussion above, including Lemma 5.1 and Proposition 5.2 hold for the larger space $\mathcal{C}_L^{enh} \supset \mathcal{C}^{enh}$. Namely to obtain a splitting for $c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M)$ as in (5.4), write

$$L = \widehat{K} \oplus \widehat{K}^\perp,$$

and let $dim\widehat{K}^\perp = p$. The space $\widehat{K}^\perp$ intersects $\mathrm{Im}\, D_{(\gamma,\kappa)}\Phi$ in a closed subspace $H \subset \mathrm{Im}\, D_{(\gamma,\kappa)}\Phi$ of codimension p and in analogy to (5.4), one has

$$c^{m-2,\alpha}(M) \times \Lambda_1^{m-2,\alpha}(M) = H \oplus L. \tag{5.14}$$

As before, let π_1 be the projection onto the first factor in (5.14) along the slice L. It follows that the linearization $\pi_1 \circ \Phi$ is a submersion onto H at (γ, κ). Hence, the implicit function theorem implies that $(\pi_1 \circ \Phi)^{-1}(0) \cap \Pi^{-1}(\mathcal{D}_+^{reg}) = \mathcal{C}_L^{enh}$ is a smooth separable Banach manifold near (γ, κ). Again, the space

$$\mathcal{C} \cap \mathcal{C}_L^{enh} \subset \mathcal{C}_L^{enh}$$

is then given locally near (γ, κ) by the zero set of a smooth function

$$F = \pi_2 \circ \Phi|_{\mathcal{C}_L^{enh}} : \mathcal{C}_L^{enh} \to \mathbb{R}^k, \tag{5.15}$$

i.e. $\mathcal{C} \cap \mathcal{C}_L^{enh} = F^{-1}(0)$, where π_2 is projection onto the second factor L in $(0, L) \supset \widehat{K}$.

Similarly, the proof of Proposition 5.2 does not require the vector fields Y to be in $\widehat{K}$. The same proof holds for any finite dimensional vector space $L \subset \chi^{m,\alpha}$. As above, we assume $\mathcal{C}_L^{enh}$ is chosen to be the maximal open domain in $Met^{m,\alpha}(M) \times S_2^{m-1,\alpha}(M)$ on which it is a smooth connected submanifold.

We will use these larger spaces $\mathcal{C}_L^{enh}$ for the following reason. Consider a pair of local spaces $\mathcal{C}_1^{enh}$ and $\mathcal{C}_2^{enh}$ built from points (γ_1, κ_1) and (γ_2, κ_2) with kernels $\widehat{K}_1$ and $\widehat{K}_2$. Suppose, as will be the case generically, that $\widehat{K}_1 \cap \widehat{K}_2 = 0$. Then either $\mathcal{C}_1^{enh} \cap \mathcal{C}_2^{enh} = \emptyset$ or the intersection is given by $\mathcal{C}_1^{enh} \cap \mathcal{C}_2^{enh} = \mathcal{C}_+ \cap \mathcal{C}_1^{enh} \cap \mathcal{C}_2^{enh}$. Thus it is not possible to simply glue together such local spaces to obtain a manifold thickening of $\mathcal{C}_+$ globally.

Completion of Proof of Theorem 1.1.

The proof of Theorem 1.1 is now completed much as in the regular case in Section 4 but with the extra technical substructure developed above. For each $(\gamma, \kappa) \in \mathcal{C}_+ \cap \Pi^{-1}(\mathcal{D}_+^{reg}) = (\mathcal{C}_+^{reg} \cup \mathcal{C}_+^{sing}) \cap \Pi^{-1}(\mathcal{D}_+^{reg})$ we have constructed above an open manifold neighborhood, either $\mathcal{C}_+^{reg}$ itself or a local space $\mathcal{C}^{enh}$, with $\mathcal{C}^{enh} = \mathcal{C}^{reg}$ when $dim\widehat{K} = 0$. The projection maps $\Pi_+ : \mathcal{C}_+^{reg} \to \mathcal{D}_+$ or $\Pi^{enh} : \mathcal{C}^{enh} \to \mathcal{D}_+$ are defined and consistent on intersections.

Let $y(t)$, $t \in I = [0,1]$ be a smooth curve of finite length in $\mathcal{D}_+^{reg}$ as in the proof of Theorem 1.1 in Section 4 with $y(0)$ a regular value of Π_+^{reg} with $H = const$. We will inductively construct over y a finite collection of local enhanced spaces $\mathcal{C}^{enh}$ and then using Remark 5.3 assemble these to a single, global enhanced space $\widetilde{\mathcal{C}}^{enh}$ over y. The methods used in Section 4 will then be applied to $\widetilde{\mathcal{C}}^{enh}$.

To begin, as discussed in Section 4, there is a maximal first interval $I_0 = [0, t_0) \subset I$ for which $y(t)$, $t \in I_0$, has a unique lift to a curve

$$\mathbf{x}(\tau) \subset \Pi_+^{-1}(y(I_0)) \cap \mathcal{C}_+^{reg} \tag{5.16}$$

starting at the unique lift $\mathbf{x}(0) = \Pi_+^{-1}(y(0))$ of $y(0)$. The parametrization τ is chosen so that $\Pi_+(\mathbf{x}(\tau_0 - \delta)) = y(t_0 - \delta)$ for δ small. We may assume that there is a $\delta > 0$ such that $\mathbf{x}(\tau_0 - \delta) \in \mathcal{C}_1^{enh}$, for some local enhanced space $\mathcal{C}_1^{enh}$ based at $(\gamma_1, \kappa_1) \in \mathcal{C}^{sing}$. (For otherwise, if there is no such $t_0 < 1$ then $\mathbf{x}$ satisfies (4.37) and the result follows from the proof in Section 4).

Consider then the path $y(I) \subset \mathcal{D}_+^{reg}$. By a slight perturbation of y if necessary, we may assume that y is transverse to $\Pi^{enh} : \mathcal{C}_1^{enh} \to \mathcal{D}_+^{reg}$. Let $W_1 = W_1^{k_1+1} \subset \mathcal{C}_1^{enh}$ be the component of $\mathcal{C}_1^{enh}$ over y containing the point $\mathbf{x}(\tau_0 - \delta)$ for some δ sufficiently small. The image $\Pi^{enh}(W_1)$ then contains a neighborhood of $y(t_0 - \delta)$ in $\mathcal{D}_+^{reg}$. If the intersection of $\Pi_+^{-1}(y(I)) \subset \mathcal{C}_+$ with W_1 is compactly contained in the interior of W_1, the inductive construction terminates.

If not, then there exist points $(\gamma_2, \kappa_2) \in W_1 \cap \Pi_+^{-1}(y(I))$ arbitrarily near ∂W_1. For a fixed choice of such (γ_2, κ_2), let $\mathcal{C}_2^{enh}$ be the local enhancement associated to (γ_2, κ_2) with $\Pi_2^{enh} : \mathcal{C}_2^{enh} \to \mathcal{D}_+^{reg}$. As before, perturb y slightly if necessary so that y is transverse to Π_2^{enh} and choose the component $W_2 = W_2^{k_2+1}$ of $\mathcal{C}_2^{enh}$ intersecting W_1, (so containing (γ_2, κ_2)). We note that it is possible that $k_2 = 0$ so that $\mathcal{C}_2^{enh} = \mathcal{C}_+^{reg}$. This process is then repeated inductively, giving a collection of spaces $W_j = W_j^{k_j+1}$ and maps $\Pi_j^{enh} : W_j \to y(I)$. By construction and the compactness results of Theorem 4.7/Proposition 5.2, each image $\Pi_j^{enh}(W_j)$ contains a subpath of definite length within $y(I)$. Hence this process terminates at some finite $j \leq j_{max} = m$.

Now for each (γ_j, κ_j), we have the kernels K_j^* and $\widehat{K}_j$. Form then

$$L := \widehat{K}_1 \oplus \widehat{K}_2 \oplus \cdots \oplus \widehat{K}_m \simeq \mathbb{R}^N,$$

where $N = \sum k_j$. (Recall that it is possible that $k_j = 0$ for some collection of j's). We now use the construction in Remark 5.3 to produce a larger enhanced space containing the local spaces $W_j = W_j^{k_j+1}$ defined above. Thus at each j, consider the splitting (5.4). Write

$$L = \widehat{K}_j \oplus (\widehat{K}_j)^c,$$

so that $dim(\widehat{K}_j)^c = N - k_j$. As in Remark 5.3, $\mathrm{Im}\, D_{\gamma_j,\kappa_j)}\Phi \cap (\widehat{K}_j)^c$ is a closed subspace H_j of codimension $N - k_j$ in $\mathrm{Im}\, D_{(\gamma_j,\kappa_j)}\Phi$ and the projected map $\pi_{H_j} \circ \Phi$ is a submersion onto H_j at (γ_j, κ_j). The implicit function theorem then gives a local manifold structure to

$$\widetilde{\mathcal{C}}_j^{enh} = (\pi_{H_j} \circ \Phi)^{-1}(0) \cap \Pi^{-1}(\mathcal{D}_+^{reg}) \tag{5.17}$$

near each (γ_j, κ_j). Thus $\widetilde{\mathcal{C}}_j^{enh}$ consists of all $(g, K) \in \Pi^{-1}(\mathcal{D}^{reg})$ near (γ, κ) such that $\Phi(g, K) \in L$. As before, we assume $\widetilde{\mathcal{C}}_j^{enh}$ is maximally extended as a smooth connected Banach submanifold satisfying (5.17). By construction, the previously defined spaces $\mathcal{C}_j^{enh}$ satisfy $\mathcal{C}_j^{enh} \subset \widetilde{\mathcal{C}}_j^{enh}$. The intersection of consecutive spaces $\widetilde{\mathcal{C}}_j^{enh} \cap \widetilde{\mathcal{C}}_{j+1}^{enh}$ is an open subset of each. Hence the union

$$\widetilde{\mathcal{C}}^{enh} = \cup_{j=1}^{m} \widetilde{\mathcal{C}}_j^{enh},$$

is a smooth, path connected Banach manifold.

As before, one has the projection map

$$\widetilde{\Pi} : \widetilde{\mathcal{C}}^{enh} \to \mathcal{D}_+^{reg},$$

and again by a small perturbation of $y : I \to \mathcal{D}_+^{reg}$, we may assume $\widetilde{\Pi}$ is transverse to y. Let W^{N+1} be the component of $\widetilde{\Pi}^{-1}(y(I)) \subset \widetilde{\mathcal{C}}^{enh}$ containing the initial curve $\mathbf{x}(I_0)$ as in (5.16). Thus

$$\widetilde{\Pi} : W^{N+1} \to y(I).$$

The boundary $\partial W^{N+1} = V^N(0) \cup V^N(1)$ is contained in $\widetilde{\Pi}^{-1}(\partial y(I))$. Here $V^N(i)$ denotes the components (possibly empty) of ∂W^{N+1} over $\widetilde{\Pi}^{-1}(y(i))$, $i = 0, 1$.

Now consider the map

$$F = \pi_2 \circ \Phi : W^{N+1} \to L \simeq \mathbb{R}^N.$$

As in the discussion preceding Remark 5.3, choose a regular value $Y_\varepsilon \in L$ of F, ε-close to 0, so that $|Y_\varepsilon|_{C^{m,\alpha}} < \varepsilon$. The inverse image $F^{-1}(Y_\varepsilon)$ is a collection of regular curves $\{x_\varepsilon\}$ in W^{N+1} with $\Phi(x_\varepsilon) = O(\varepsilon)$. As above, the boundary of each curve $x_\varepsilon(\tau)$ is either empty (giving an embedded S^1) or gives a cobordism between its endpoints in the fibers $\widetilde{\Pi}^{-1}(\partial y(I)) \supset \partial W^{N+1} = V^N(0) \cup V^N(1)$.

Now among the collection $\{x_\varepsilon\}$ of curves in $F^{-1}(Y_\varepsilon)$, we choose a component $\mathbf{x}_\varepsilon$ such that

$$\mathbf{x}_\varepsilon(0) \to \mathbf{x}(0),$$

on some sequence $\varepsilon = \varepsilon_i \to 0$ where $\mathbf{x}(0)$ is defined as in (5.16). Without loss of generality, assume $\mathbf{x}_\varepsilon(0) \in V^N(0)$. The endpoint $\mathbf{x}_\varepsilon(1)$ must lie in one of the boundary component regions $V^N(0)$ or $V^N(1)$. If $\mathbf{x}_\varepsilon(1) \in V^N(0)$, one has exactly the same contradiction to uniqueness as in the proof of Theorem 1.1 in Section 4. Hence $\mathbf{x}_\varepsilon(1) \in V^N(1)$. As before, this implies that $\mathbf{x}_\varepsilon$ covers $y(I)$. Letting $\varepsilon = \varepsilon_i \to 0$ and using the compactness from Proposition 5.2 (as discussed preceding Remark 5.3) gives a point $\mathbf{x}(1) \in \mathcal{C}_+$ such that $\Pi_+(\mathbf{x}(1)) = y(1)$. Since $y(1)$ is arbitrary in $\mathcal{D}_+^{reg}$, this proves that

$$\Pi_+|_{\mathcal{C}_+ \cap \Pi^{-1}(\mathcal{D}_+^{reg})} : \mathcal{C}_+ \cap \Pi^{-1}(\mathcal{D}_+^{reg}) \to \mathcal{D}_+^{reg} \tag{5.18}$$

is surjective onto $\mathcal{D}_+^{reg}$.

Finally, $\mathcal{C}_+ \cap \Pi^{-1}(\mathcal{D}_+^{reg})$ is dense in $\mathcal{C}_+$ and $\mathcal{D}_+^{reg}$ is dense in $\mathcal{D}_+$. As in Remark 4.8, the compactness results Theorem 4.7/Proposition 5.2 imply that the continuous extension Π_+ of (5.18) to $\mathcal{C}_+$ is surjective onto $\mathcal{D}_+$. This completes the proof of Theorem 1.1.

■

It is possible that the proof above could be streamlined by using the structure of polyfolds introduced and developed by Hofer-Wysocki-Zehnder [20]. While it may be of interest to bring in polyfold structures into the study of the constraint space $\mathcal{C}$, we will not pursue this further here.

We conclude the paper with a few remarks.

Remark 5.4. Although the analysis here has been restricted to the physically most interesting dimension $dim\, M = 3$, it is not difficult to verify that all of the results, and in particular Theorem 1.1, remain valid for M a compact n-dimensional manifold.

Similarly, we expect the methods used here can be extended to the situation of complete asymptotically flat initial data sets, as well as the vacuum constraint equations with non-zero cosmological constant Λ, provided one has the uniqueness result for CMC solutions used in the proof of Theorem 1.1. We will not pursue this further here however.

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