# The Compactification of a Minimal Submanifold in Euclidean Space by the Gauss Map

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## §1. Introduction.

In a series of papers [14], [15], [6], Osserman and Chern-Osserman proved a number of fundamental results on the structure of complete minimal surfaces $M^2$ immersed in $\mathbb{E}^3$ and $\mathbb{E}^N$. These results center about properties of the Gauss map $G$ and its relation to the geometry of $M$. Recall that $G$ associates to $p \in M$ the tangent plane $T_pM$ translated to the origin, considered as an element of the Grassmannian $G_{2,N}$. For instance, in [15] Osserman shows that the Gauss map of a complete, non-planar minimal surface $M^2 \subset \mathbb{E}^3$ either

- attains every value $a \in G_{2,3} = S^2$ infinitely many times, with the possible exception of a set $Z \subset S^2$ of logarithmic capacity zero, or

- attains every value $a \in G_{2,3}$ finitely many times, omitting at most three values.

The single intrinsic invariant $\int_M (-K)\,dA$ of the geometry of $M$, namely the total Gaussian curvature, distinguishes between the two modes of behavior. In the former case, the total curvature is infinite while in the latter it is finite. In fact, a basic consequence of the theory is the quantization condition

$$(1.1) \qquad \int_M (-K)\,dA = 4N\pi \quad ,$$

$N \in \mathbb{Z}^+$, on the total curvature. There are similar results that hold for minimal surfaces in $\mathbb{E}^N$; c.f. [6]. Xavier has recently sharpened these results on the value distribution of the Gauss map considerably : the image omits at most six points [19]. It is a beautiful open question whether in fact the Gauss map omits at most four points in general.

These structure theorems, reminiscent of the Picard theorem in one complex variable, are based on the connection between minimal surfaces in $\mathbb{E}^N$ and holomorphic curves in the Grassmanian $G_{2,N}$ inherent in the Weierstrass representatic

More precisely, the Grassmaninan is realized as the quadric $z_1^2+z_2^2+\cdots+z_N^2 = 0$ in complex projective space $\mathbb{C}\mathbb{P}^{N-1}$. The Gauss map of $M^2$ gives rise to a holomorphic curve in $G_{2,N} \subset \mathbb{C}\mathbb{P}^{N-1}$ and one studies the value distribution of this curve in the sense of Weyland Ahlfors. The two modes of behavior above correspond exactly to the two coarsest characteristics of holomorphic curves in $G_{2,N}$, namely transcendental or algebraic. The Gauss map is algebraic if there is a compact Riemann surface $\overline{M^2}$ and a finite number of points $\{p_i\}_1^r \in \overline{M^2}$ such that $M^2$ is conformally equivalent to $\overline{M^2} \setminus \{p_i\}_1^r$ and the Gauss map $G : M^2 \to G_{2,N}$ extends to a holomorphic curve $\overline{G} : \overline{M^2} \to G_{2,N}$; thus the Gauss map compactifies the Riemann surface $M^2$. The Gauss map is transcendental if the curve $G(M^2)$ is not an open subvariety of a closed projective curve in $G_{2,N}$.

This paper is concerned with establishing the beginning of such a theory for higher dimensional minimal submanifolds $M^k$ in $\mathbb{E}^N$. There are a number of difficulties in carrying out such a program. First, one no longer has the tools of complex function theory. More importantly, the topological structure of $M$, particularly regarding the asymptotic behavior, is more complicated. For example, let $V^n$ be an n-dimensional smooth algebraic variety in $\mathbb{C}^N$. Under an affine inclusion $\mathbb{C}^N \subset \mathbb{C}\mathbb{P}^N$, $V^n$ is an open subvariety of a compact algebraic variety $\overline{V^n} \subset \mathbb{C}\mathbb{P}^N$ and the complement $\overline{V^n} \setminus V^n$ may be an arbitrary algebraic variety in $\mathbb{C}\mathbb{P}^{N-1}$. Similar behavior is exhibited by the Gaussian image $G(V^n) \subset G_{n,N}(\mathbb{C})$ and the complement $\overline{G(V^n)} \setminus G(V^n)$. Moreover, the immediate generalization of Osserman's theorem on the density of the Gaussian image to higher dimensions is false. This is a consequence of the failure of the Bernstein conjecture in dimensions greater than eight [2].

Finally, there are a number of notions of total curvature for submanifolds in $\mathbb{E}^N$. Perhaps the most natural are the total Gauss-Bonnet-Chern curvature $\int_{M^n} \Omega$, where $\Omega$ is the Gauss-Bonnet-Chern integrand, and the total absolute curvature $\int_{M^n} K\, dV$, in the sense of Chern-Lashof [5]. However, both may vanish on non-planar minimal submanifolds; further, in odd dimensions, $\Omega \equiv 0$. The notion we concentrate on in this paper is the integral

$$\mathcal{A} = \int_{M^n} |A|^n dV$$

where $A$ is the second fundamental form of $M$ in $\mathbb{E}^N$. It is easy to see that for any submanifold $M^n \subset \mathbb{E}^N$,

$$\int_{M^n} \Omega \leq c_1 \int_{M^n} |K| dV \leq c_2 \int_{M^n} |A|^n dV ,$$

for universal constants $c_1, c_2$ depending only on $n$. Moreover, $\mathcal{A} = 0$ if and only if $M^n$ is an affine n-plane. It is important to note that all three integrals are scale invariant quantities. The Gauss-Bonnet-Chern integral is clearly an intrinsic isometric invariant. For minimal submanifolds of $\mathbb{E}^N$, $\mathcal{A}$ is also; in fact, $\mathcal{A} = \int_{M^n} (-\tau)^{n/2}$, where $\tau$ is the scalar curvature of $M$, suitably normalized. For this reason, we call $\mathcal{A}$ the total scalar curvature of $M$.

The main theorem of this paper is the following generalization of the Chern-Osserman theorem on minimal surfaces in $\mathbb{E}^N$ of finite total curvature.

<u>Theorem A.</u> <u>Let</u> $M^n$ <u>be a complete, connected minimally immersed submanifold of</u> $\mathbb{E}^N$ <u>of finite total scalar curvature. Then</u> $M^n$ <u>is</u> $C^\infty$ <u>diffeomorphic to a compact</u> $C^\infty$ <u>manifold</u> $\overline{M^n}$ <u>punctured at a finite number of points</u> $\{p_i\}_1^r \in \overline{M^n}$. <u>The Gauss map</u> $G : M^n \to G_{n,N}$ <u>extends to a</u> $C^{n-2}$ <u>map</u> $\overline{G} : \overline{M^n} \to G_{n,N}$ <u>of the compactification. Further, the metric</u> $ds^2$ <u>on</u> $M^n$ <u>extends conformally to a</u> $C^{n-2}$ <u>complete Riemannian metric on</u> $\overline{M^n}$.

<u>Conversely, if</u> $M^n$ <u>is diffeomorphic to</u> $\overline{M^n} \setminus \bigcup_1^r p_i$ <u>and the Gauss map has a</u> $C^1$ <u>extension to</u> $\overline{M^n}$, <u>then</u> $M^n$ <u>has finite total scalar curvature.</u>

Thus, one obtains an intrinsic characterization of minimal immersions

$M^n \to \mathbf{E}^N$ whose Gauss map achieves a limiting value on each end of $M$ : namely, the total scalar curvature $A$ is finite.

We note that the proof gives a new proof of the Chern-Osserman theorem, without the use of Huber's theorem [9] on Riemann surfaces of finite total curvature.

An interesting consequence of the theorem is an integrality condition on the total Gauss-Bonnet-Chern curvature of $M$ , similar to (1.1).

Theorem B. Let $M^n$ be a complete, oriented minimally immersed submanifold of $\mathbf{E}^N$ , of finite total scalar curvature. Then

$$\int_{M^n} \Omega = \chi(M^n) - \sum_{i=1}^{r} m_i \, , \tag{1.2}$$

where $m_i \in \mathbb{Z}^+$ is the multiplicity of the end $\{p_i\}$ . If $n \geq 3$ , then $m_i = 1$ , for all $i$ .

We refer to §5 for the definition of $\chi(M^n)$ , in case $n$ is odd, and of $m_i$ . Eqn. (1.2) implies that $\int_{M^n} \Omega < \chi(M^n)$ , so that a generalization of the Cohn-Vossen inequality holds for this class of minimal submanifolds of $\mathbf{E}^N$ . Clearly, equality is never achieved as is the case for compact manifolds. We also refer to §5 for further applications of the above theorems.

We now outline the contents of the paper. In §2, using P.D.E. and scaling arguments, we obtain an estimate on the curvature decay of minimal submanifolds in $\mathbb{E}^N$ with $\mathcal{A} < \infty$:

$$\sup_{M \cap S(r)} |A|^2 \leq \frac{1}{r^2}\mu(r) \tag{1.3}$$

where $\mu(r) \to 0$ as $r \to \infty$. Thus, M locally becomes Euclidean near infinity. This result easily implies that M is of finite topological type (Corollary 2.5).Section 3 contains the proof of Theorem A. The curvature estimate (1.3) is shown to imply that each end of $M^n$ is diffeomorphic to $S^{n-1} \times [0,\infty)$. Using the condition $\mathcal{A} < \infty$, we show that the Gauss map extends continuously over each point at infinity. The smooth extension of the Gauss map then follows from P.D.E. arguments.

In §4, we prove Theorem B by applying the Gauss-Bonnet formula to the domains $M \cap B(r)$; the control over the boundary terms again comes from (1.3). Finally, in §5, we present several applications of the results above. The Bernstein-type result, Theorem 5.2 is of particular interest.

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## §2. Estimate of curvature decay

Throughout this paper, $M^n$ will denote a complete, connected, minimally immersed submanifold of Euclidean space $\mathbb{E}^N$. We let $i: M^n \to \mathbb{E}^N$ denote the immersion. The metric, or first fundamental form, of M is that induced from $\mathbb{E}^N$. The second fundamental form $A: TM \odot TM \to NM$ is given by

$$A(X,Y) = (\bar{\nabla}_X Y)^N.$$

Occasionally, we will view A as a map $TM \to \mathrm{Hom}(TM, NM)$, or $NM \to \mathrm{Hom}(TM, TM)$.The notation $\bar{\nabla}$ (respectively $\nabla$), is used for covariant differentiation on $\mathbb{E}^N$ (M). One has $\bar{\nabla}^T = \nabla$ where T denotes tangential projection. Finally, let B(r) denote the open ball of radius r about the origin 0 in $\mathbb{E}^N$, $S(r) = \partial B(r)$ and $A(r.s) = B(s) - B(r)$.

The following theorem will be of importance in the work to follow.

Theorem 2.1. (Smooth Compactness Theorem) Let $\{M_i^n\}$ be a sequence of connected, minimally immersed submanifolds in $B^N(1)$, such that $\partial M_i^n \cap B^N(1) = \emptyset$. Suppose there is a constant $C$ such that $\sup|A_i|(x) \leq C$, for all $i$. Then there is a subsequence of $(M_i)$, denoted by $(M_i)$, that converges in the $C^\infty$ topology on compact sets in $B^N(1)$ to a smooth minimally immersed submanifold $M_\infty$ in $B^N(1)$ with $\sup |A_\infty| \leq C$.

By $C^\infty$ convergence to $M_\infty$ we mean the following: for any $p \in M_\infty$, there is a neighborhood $U \subset\subset B^N(1)$ of $p$ such that each component of $M_i \cap U$, for $i$ sufficiently large, may be graphed over $U \cap T_pM_\infty$ by a function $F_i$: $U \cap T_pM_\infty \longrightarrow N_pM_\infty$. One requires the functions $F_i$ to converge, in the usual $C^\infty$ topology, to $F_\infty$, the graphing function for $M_\infty$.

Proof: This theorem is rather well-known and we will only sketch the proof, c.f.[7] for further details. The curvature bound $\sup|A_i|(x) \leq C$ implies there is an $\epsilon_0 > 0$, depending only on $C$, $n$, $N$ such that the components of $M_i \cap B_{p_i}(\epsilon)$, with $p_i \in M_i$, $B_{p_i}(\epsilon) \subset\subset B^N(1)$ and $\epsilon < \epsilon_0$ may be graphed over $T_{p_i}M_i \cap B_{p_i}(\epsilon)$ by functions $f_i$. Each $f_i$ satisfies the elliptic system $\mathcal{M}f_i = 0$, where $\mathcal{M}$ denotes the operator of the minimal surface system. Further, the curvature estimate immediately implies a uniform $C^{1,\alpha}$ bound on $\{f_i\}$, for $\alpha < 1$. The regularity theory of the operator $\mathcal{M}$ implies that a subsequence of $\{f_i\}$ converges in the $C^\infty$ topology to a solution $f_\infty$. Now using an elementary covering argument ,[7] , it is not difficult to show that a subsequence of $\{M_i\}$ converges to an $M_\infty$ in the $C^\infty$ topology and that $M_\infty$ has the required properties.

$\square$

We now return to the minimal immersion $i : M^n \longrightarrow E^N$. We assume, without loss of generality, that $0 \in i(M)$ and choose $0 \in M$ such that $i(0) = 0$. Let $D(r)$ be the geodesic ball of radius $r$ about $0$ in $M$, $D(r,s) = \{x \in M : r < \mathrm{dist}_M(x,0) < s\}$, $L(r) = \{x \in M : \mathrm{dist}_M(x,0) = r\}$, the geodesic sphere of radius $r$ about $0$. Finally, $D_p(r)$ denotes the geodesic $r$-ball about $p \in M$.

<u>Propostion 2.2</u>: *Let* $M^n \to \mathbb{E}^N$ *be a complete minimal immersion (not necessarily proper) in* $\mathbb{E}^N$, $n \geq 2$, *of finite total scalar curvature. Then there is a constant* $R_0 = R_0(M)$ *such that*

(2.1)
$$\sup_{x \in L(R)} |A|^2(x) \leq \frac{1}{R^2}\, \mu\left(\int_{D(\frac{R}{2},\, 2R)} |A|^n dV\right),$$

*for all* $R \geq R_0$, *where* $\mu(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

<u>Proof</u>: First it is useful to recale the metric. Let $ds_R^2 = \frac{1}{R^2} ds^2$ be the metric induced on $M$ by the immersion $i_R = \delta_R \circ i$, where $\delta_R$ is the dilation of $\mathbb{E}^N$ about $0$ by the factor $\frac{1}{R}$. Metric quantities on $M$ measured with respect to $ds_R^2$ will be denoted by a subscripted $R$. Thus for example, $|A|_R^2 = R^2\,|A|^2$; $D_R(s)$, the geodesic ball of radius $s$ about $0$ w.r.t. $ds_R^2$, satisfies $D_R(s) = D(sR)$.

Now it is easily seen that (2.1) is equivalent to the estimate

(2.2)
$$\sup_{X \in L_R(1)} |A|_R^2(X) \leq \mu\left(\int_{D_R(\frac{1}{2},\, 2)} |A|_R^n dV_R\right)$$

for the minimal immersion $i_R$. Note that since $M$ has finite total scalar curvature, for all $\varepsilon > 0$ there is an $R_0$ such that

$$\int_{D_R(\frac{1}{2},\, \infty)} |A|_R^n dV_R = \int_{D(\frac{R}{2},\, \infty)} |A|^n dV < \varepsilon ,$$

for all $R \geq R_0$. Thus, to prove (2.2), it suffices to prove the following statement.

(S) There is an $\varepsilon_0 > 0$ such that if $h: X^n \to \mathbb{E}^N$ is any minimal immersion with $D_x(1) \cap \partial X = \phi$ for some $x \in X$ and

$$\int_{D_x(1)} |A|^n \, dV = \varepsilon \leq \varepsilon_0,$$

then
$$\sup_{p \in D_x(\frac{1}{2})} |A|^2(p) < \delta,$$

where $\delta \to 0$ as $\varepsilon \to 0$.

We will first prove the statement (S') below, from which (S) will follow easily.

(S') There is an $\varepsilon_0 > 0$ such that if $h: X \to \mathbb{E}^N$ is any minimal immersion with $D_x(1) \cap \partial X = \phi$ for some $x \in X$ and

$$\int_{D_x(1)} |A|^n \, dV = \varepsilon \leq \varepsilon_0,$$

then

$$\sup_{t \in [0,1]} \left[t^2 \sup_{D_x(1-t)} |A|^2\right] \leq 4. \tag{2.3}$$

To prove (2.3) we argue by contradiction. If (2.3) were false, there must exist a sequence of minimal immersions $h_i = X_i \to \mathbb{E}^n$ with $h_i(x_i) = 0$ and $D_{x_i}(1) \cap \partial X_i = \phi$ such that

$$\int_{D_{x_i}(1)} |A|_i^n \, dV_i \to 0$$

but
$$\operatorname*{Sup}_{t} \left[t^2 \sup_{D_{x_i}(1-t)} |A|_i^2\right] > 4$$

for all $i$. Choose $t_i \in [0,1]$ such that

$$t_i^2 \sup_{D_{x_i}(1-t_i)} |A|_i^2 = \sup_{t \in [0,1]} \left[t^2 \sup_{D_{x_i}(1-t)} |A|_i^2\right]$$

and choose $Y_i \in D_{X_i}(1-t_i)$ such that

$$|A|_i^2(Y_i) = \sup_{D_{X_i}(1-t_i)} |A|_i^2 .$$

One easily sees that

$$\sup_{D_{Y_i}(\frac{t_i}{2})} |A|_i^2 \leq 4|A|_i^2(Y_i) \tag{2.4}$$

and by assumption

$$|A|_i^2(Y_i) > \frac{4}{t_i^2} . \tag{2.5}$$

Now again it is useful to rescale the metric. Let $\widetilde{ds}_i^2 = |A|_i^2(Y_i) \cdot ds_i^2$ be the metric on $X_i$ induced by the minimal immersion $\delta_i \cdot h_i$, where $\delta_i$ is the dilation of $\mathbb{E}^N$ about $h_i(Y_i)$ by the factor $|A|_i^2(Y_i)$. (by translation, we may assume $h_i(Y_i) = 0$). Metric quanties on $X_i$ measured with respect to $\widetilde{ds}_i^2$ will be superscripted with a $\sim$ . Thus, for $p \in X_i$, $|\tilde{A}|_i\,(p) = [|A|_i(Y_i)]^{-1}\ |A|_i(p)$, $\tilde{D}_p(s) = D_p([|A|_i(Y_i)]^{-1} \cdot s)$, and so on. In particular, (2.5) implies that $\tilde{D}_{Y_i}(1) \cap \partial X_i = \phi$, while (2.4) implies that

$$\sup_{\tilde{D}_{Y_i}(1)} |\tilde{A}|_i^2 \leq 4$$

and $|\tilde{A}|_i(Y_i) = 1$. Thus the sequence $\tilde{h}_i = \tilde{D}_{Y_i} \to \mathbb{E}^N$ is a sequence of minimal immersions of open geodesic balls of radius 1, of uniformly bounded curvature, translated so that $\tilde{h}_i(Y_i) = 0$. By the smooth compactness theorem, Theorem 2.1, a subsequence converges in the $C^2$ topology on compact subsets, to a smooth minimal immersion $\tilde{h}_\infty = \tilde{D}_{Y_\infty}(1) \to \mathbb{E}^N$.

We have
$$\int_{\tilde{D}_{Y_i}(1)} |\tilde{A}|_i^n\, d\tilde{V}_i \leq \int_{D_{X_i}(1)} |A|_i^n\, dV_i \to 0$$

so that $\tilde{D}_{Y_\infty}(1)$ is isometric to a domain in a flat n-plane. However, the fact that $|\tilde{A}|_i(Y_i) = 1$, for all $i$, implies that $|\tilde{A}|_\infty(Y_\infty) = 1$, which gives a contradiction. This proves (S').

We now prove statement (S) from (S') and thus complete the proof of Proposition 2.2. Once more, we argue by contradiction. If (S) were false, there must exist a sequence of minimal immersions $g_i : Z_i \to \mathbb{E}^N$ with $g_i(z_i) = 0$ and $D_{z_i}(1) \cap \partial Z_i = \phi$ such that

$$\int_{D_{z_i}(1)} |A|_i^n \, dV_i \to 0$$

but

$$\sup_{D_{z_i}(\frac{1}{2})} |A|_i^2 \geq C^2 > 0$$

for some constant $C$. For $i$ sufficiently large, we may apply (2.3) with $t = \frac{1}{2}$ to obtain

$$\sup_{D_{z_i}(\frac{1}{2})} |A|_i^2 \leq 16 \; .$$

As above, a subsequence of $g_i : D_{z_i}(\frac{1}{2}) \to \mathbb{E}^N$ converges smoothly on compact subsets to a minimal immersion $g_\infty : D_{z_\infty}(\frac{1}{2}) \to \mathbb{E}^N$ and we have

$$\int_{D_{z_\infty}(\frac{1}{2})} |A|_\infty^n \, dV_\infty = 0 \; .$$

Since $\sup_{D_{z_\infty}(\frac{1}{2})} |A|^2 \geq C^2 > 0$, this contradiction establishes (S).

■

Remark: Part of the proof of statement (S'), namely the estimates (2.3) - (2.5), are due to Choi-Schoen, and used in their proof of Proposition 2 of [7] ; c.f. also the related results [17: Thm 2]. We note that the rest of the proof of Prop. 2.2 gives an elementary proof of Prop. 2 of [7]|.

We mention explicitly the following "gap phenomemon"; this is related to recent work of Kasue [11].

<u>Corollary 2.3</u>: Let $M^n \to \mathbb{E}^N$ <u>be a complete minimal immersion. Then there is a constant</u> $\varepsilon_0 = \varepsilon_0(n, N)$ <u>such that if</u>

$$\int_{M^n} |A|^n \, dV \le \varepsilon_0,$$

<u>then</u> $M^n$ <u>is an affine n-plane</u>.

<u>Proof</u>: This follows easily from statement (S) of Prop. 2.2. Let $\varepsilon_0$ be the quantity given by statement (S). Applying (S) to the immersions $\delta_R \circ i$ where $\delta_R$ is the dilation of $\mathbb{E}^N$ by $\frac{1}{R}$, we have, since

$$\int_{D_R(1)} |A|_R^n \, dV_R \le \varepsilon_0$$

that $\sup\limits_{P \in D_R(\frac{1}{2})} |A|_R^2(P) \le \delta$ or $\sup\limits_{P \in D(R/2)} |A|^2 < \frac{\delta}{R^2}$. If we let $R \to \infty$, we obtain the result.

■

<u>Remark</u>: Another proof of Proposition 2.2. for $n>2$ can be given using Simons' equation for $|A|$ and a Moser type iteration argument, c.f.[1].

We use Proposition 2.2 to study the behavior of M at infinity. This is done in a sequence of elementary lemmas.

Lemma 2.4. Let $M$ be as in Proposition 2.1. Then $M$ is properly immersed in $\mathbf{E}^N$ and there exists $R_o$ such that

$$1 \geq |\nabla r| > \frac{1}{2} \tag{2.6}$$

on $M \setminus B(R_o)$ .

Proof. Let $\gamma(\tau)$ be any length-minimizing geodesic ray in $M$ , starting at $0$ and let $T = \gamma'(\tau)$ , $X = \frac{1}{2}\,\overline{\nabla} r^2$ the position vector field. We have

$$T\,\langle X,T\rangle = 1 + A^X(T,T) \geq 1 - |A|\cdot|X| \quad .$$

Evaluating this at $\tau$ , using (2.1) and the fact that $|X(\tau)| \leq \tau$ , we have

$$T\langle X,T\rangle(\tau) \geq 1 - \mu(\tau) \quad . \tag{2.7}$$

Choose $R$ such that $\mu(\tau) \leq \frac{1}{4}$ , $\forall \tau \geq R$ . Integration of (2.7) from $R$ to $\tau$ gives

$$\langle X,T\rangle(\tau) \geq \frac{3}{4}(\tau - R) + \langle X,T\rangle(R) \;.$$

Since $r = |X| \geq \langle X,T\rangle$ , this implies that $M$ is properly immersed. Also,

$$|\nabla r|(\tau) \geq \frac{\langle X,T\rangle(\tau)}{\tau} \geq \frac{3}{4} - \frac{1}{4}\frac{R}{\tau} \quad ,$$

which implies (2.6) $\square$

Using elementary Morse theory, (2.6) gives the following Corollary.

Corollary 2.5. Let $M$ be as in Proposition 2.1. Then for $R_o$ sufficiently large, there is a diffeomorphism

$$\Phi : M \setminus B(R_o) \xrightarrow[\approx]{} [M \cap S(R_o)] \times [0,\infty) \;.$$

In particular, $M$ has only finitely many ends, each of finite topological type.

Proof. Let $U = \frac{\nabla r}{|\nabla r|}$ be the unit vector field defined on $M \setminus B(R_o)$ and let $\varphi_s$ denote its local 1-parameter group. Note that for $s \geq 0$, $\varphi_s : M \setminus B(R_o) \to M \setminus B(R_o)$. Define

$$\Phi(p) = (\varphi_{-t}(p), t) ,$$

where $t$ is the unique number such that $\varphi_{-t}(p) \in S(R_o)$. It is clear that $\Phi$ is a diffeomorphism.

Finally, the following Lemma will be of importance in the next section.

Lemma 2.6. *Let $M$ be as in Proposition 2.1. Let $B^r$ denote the second fundamental form of $M \cap S(r) \subset M$, with respect to the inward unit normal. Then given $\varepsilon > 0$, there is an $R_o$ such that*

$$\|B^r - \frac{1}{r} I\| < \frac{\varepsilon}{r} , \tag{2.8}$$

*for all $r \geq R_o$, where $I$ denotes the identity matrix.*

Proof. One easily computes that, for $Y \in T_p(M \cap S(r))$,

$$(D^2 r^2)(Y,Y) = (\bar{D}^2 r^2)(Y,Y) + \langle A(Y,Y), (\bar{\nabla} r^2)^N \rangle .$$

Now $\bar{D}^2 r^2 = 2I$, so by Proposition 2.1, one has $\|\frac{1}{2} D^2 r^2 - I\| \leq \mu(r)$, or

$$\|D^2 r - \frac{1}{r}\| \leq \frac{\mu(r)}{r} .$$

Since $B^r = \frac{1}{|\nabla r|} D^2 r$, the result now follows by (2.6).

□

## §3. Behavior at Infinity.

In this section, we complete the proof of Theorem A. Recall by §2 that if $i : M^n \to \mathbb{E}^N$ is a complete minimal immersion of finite total scalar curvature, then $M$ has a finite number of ends, each of finite topological type In particular, for $R_o$ sufficiently large,

$$i^{-1}(M \smallsetminus B(R_o)) = \bigcup_{k=1}^{r} V_k \quad ,$$

where $V_k$ are disjoint, smooth domains in $M$. We will identify $V_k$ with its image $i(V_k) \subset \mathbb{E}^N$ when there is no danger of confusion. In the discussion to follow, we work on each end separately, and so let $V$ denote one element in $\{V_j\}_1^r$.

Consider the submanifolds

$$(3.1) \qquad \Sigma_r = \frac{1}{r}(V \cap S(r)) \subset S(1) \ .$$

For each $r \geq R_o$, $\Sigma_r$ is a compact immersed $(n-1)$-manifold in $S(1)$ and there is a natural isotopy between $\Sigma_r$ and $\Sigma_s$, $\forall r,s \geq R_o$. The behavior of $\Sigma_r$ as $r \to \infty$ reflects the asymptotic properties of the end $V$ in $\mathbb{E}^N$. Let $A_r$ denote the second fundamental form of $V_r = \frac{1}{r}(V \cap B(r))$ in $\mathbb{E}^N$; then

$$(3.2) \qquad |A_r|^2(x) = r^2|A|^2(rx) \leq \mu(r.|x|) \quad ,$$

where $\mu(r)$ is defined by (2.1)'. In particular, $|A_r|^2 \to 0$ uniformly as $r \to \infty$ on any fixed annulus $A(\delta,1-\delta) \subset B(1)$. Also, let $C_r$ denote the second fundamental form of $\Sigma_r \subset S(1)$. Using the fact that $|\nabla r|(x) \to 1$ as $|x| \to \infty$, one easily deduces that

$$(3.3) \qquad |C_r|^2 < \varepsilon(r) \ ,$$

for $r \geq R_o$, where $\varepsilon(r) \to 0$ as $r \to \infty$.

We claim there is a uniform bound on the number of components $n_r$ of $V_r \cap B_x\left(\frac{1}{4}\right)$, for any $x \in S(1)$, $r \geq 0$. To see this, first note that by the monotonicity formula for volume of minimal varieties in $E^N$, we have

$$\mathrm{vol}\left(V_r \cap B_x\left(\tfrac{1}{4}\right)\right) \geq c \cdot n_r \left(\tfrac{1}{4}\right)^n.$$

Thus, it suffices to show the volume of $V_r \cap A\left(\frac{1}{2}, \frac{3}{2}\right)$ has a uniform upperbound. Using the fact that $|\nabla r| \to 1$ and for instance the coarea formula, one sees it is sufficient to bound $\mathrm{vol}(\Sigma_r)$ from above. (Since V is properly immersed, we need only estimate $n_r$, for r large.)

First, suppose $n \geq 3$. Then $\Sigma_r \subset S(1)$ has dimension $\geqslant 2$ and by (3.3), $|C_r| \to 0$ as $r \to \infty$. It follows that for r sufficiently large, the Ricci curvature $\mathrm{Ric}_{\Sigma_r}$ of $\Sigma_r$ satisfies $\mathrm{Ric}_{\Sigma_r} \geq c > 0$, for some constant c. The well-known comparison theorems of Rauch and Myer's theorem then imply $\mathrm{vol}(\Sigma_r)$ is bounded from above.

For $n = 2$, we use the Gauss-Bonnet theorem on the end $V$. We have

$$\int_{V \cap S(r)} \kappa - \int_{V \cap S(r_0)} \kappa = 2\pi\chi(V \cap B(r)) - \int_{V \cap B(r)} K$$

where $\kappa$ and K are the geodesic and Gaussian curvatures respectively. By the results above, we know that the right-hand side is uniformly bounded. Further, setting $v(r) = \mathrm{vol}(V \cap S(r))$, one easily computes, using the fact that $|\nabla r| > 0$ on V, that

$$v'(r) = \int_{V \cap S(r)} \kappa .$$

Thus, $v'(r) < c$, for some c, so that $v(r) \leq c' \cdot r$ and so $\mathrm{vol}(\Sigma_r) = \frac{v(r)}{r} \leq c'$.

We may now apply the smooth compactness theorem (Theorem 2.1) together with the estimate (3.2) on each component of $V_r \cap B_x\left(\frac{1}{4}\right)$. It follows that, for any point $x \in S(1)$, there are sequences $\{r_i\} \to \infty$ such that $V_{r_i} \cap B_x\left(\frac{1}{4}\right)$ converges, in the $C^\infty$ topology on compact subsets, to a finite collection of planes, each possibly with multiplicity, in $B_x\left(\frac{1}{4}\right)$. We may allow x to vary on S(1) and, by (3.2), find there are sequences $\{r_i\} \to \infty$ such that $V_{r_i}$ converges to a finite collection of planes with multiplicity, smoothly on compact subsets of $B(1) - \{0\}$. Finally, since $\sum_r \subset S(1)$ are connected, immersed submanifolds, by (3.2) or (3.3) again, we see that in fact $V_{r_i}$ converges to a single plane T with multiplicity.

The collection of linear planes $\{T_\alpha\}$ spanned by the equatorial spheres $S_\alpha^{n-1} \subset S^{N-1}(1)$ which are limits of sequences $\{\Sigma_{R_i}\}$ are called the <u>tangent</u> <u>planes</u> <u>at</u> <u>infinity</u> of V. The discussion above implies that the manifolds $\frac{1}{R_{\alpha_i}}[V \cap B(R_{\alpha_i})]$ converge to $m \cdot T_\alpha$ in the $C^k$- topology on compact subsets of $B(1) - \{0\}$.We need to prove there is a unique tangent plane at infinity.

<u>Theorem 3.1.</u> <u>Let $M^n$ be a complete minimally immersed submanifold of $E^N$ with finite total scalar curvature. Then each end V of M has a unique tangent plane at infinity.</u>

<u>Proof:</u> Consider the normal Gauss map $G : V \to G_{N-n,N}$ , $G(x) = N_xM$. If T is a tangent plane at infinity of V, determined by the sequence $\{r_i\}$ say, then it is easily seen that $X_i = \mathrm{Im}\, G|_{V \cap A(c^{-1}r_i,\, cr_i)}$ is contained in a small neighborhood of $T^\perp$ in $G_{N-n,N}$ for any $c > o$ fixed,and i sufficiently large. Further the sets $X_i$ shrink to T as $i \to \infty$. Thus, for R large , G maps $V - B(R)$ into a small tubular neighborhood of a curve $\sigma$ in $G_{N-n,N}$ .

We claim that $\sigma$ is either a point or a geodesic in $G_{N-n,N}$. If $\sigma$ is a point, then clearly T is the unique tangent plane at infinity to V. Suppose then that $\sigma$ is neither a point nor a geodesic. We may choose an arc $\alpha \subset \sigma$ such that the geodesic curvature of $\alpha$ in $G_{N-n,N}$ is bounded away from zero. Let U be a small tubular neighborhood of $\alpha$ and $\Omega = G^{-1}\left(U \cap A(r_i, r_{i+1})\right)$, for i large.

Now it is well known that $G : \Omega \to G_{N-n,N}$ is a harmonic map. We claim the image of a harmonic map cannot be contained in a small tubular neighborhood of $\alpha$. To see this, we may write $U = D \times \alpha$, where D is a small normal disc to $\alpha$. Consider the vector field $Z = f \cdot \tau$, where f is a smooth funtion, $0 \leq f \leq 1$, of compact support in U and $\tau$ is the geodesic curvature field of the curves $\alpha_p = p \times \alpha$ in U. Clearly $\tau$ is smooth and we assume $\{ x \in U : f(x) < 1 \}$ is a small neighborhood of $\partial U$. Let $\phi_t$ be the flow of Z. It is not difficult to verify that, for vectors T tangent to the $\alpha_p$ curves in U, $\frac{d}{dt}\|(\phi_t)_*(T)\|_{t=0} < 0$, while for vectors Y tangent to the D factor in U, $\frac{d}{dt}\|(\phi_t)_*(Y)\|_{t=0} = 0$. In other words, to first order in t, $\phi_t$ decreases the lengths of the $\alpha_p$ curves, keeping lengths in the D factor constant.

It follows that $\frac{d}{dt}E(\phi_t \circ G)_{t=0} < 0$, where E is the energy. This contradicts the fact that G is harmonic. Thus, G maps V—B(R) into a small neighborhood of a geodesic $\sigma$ in $G_{N-n,N}$, for R large.

Since ImG is recurrent, either $\sigma$ is a finite geodesic arc or $\sigma$ is a closed geodesic. First suppose $\sigma$ is a geodesic arc. Let $p_0$ be one of the endpoints of $\sigma$ and choose a point $p_1$ on $\sigma$ close to $p_0$. Consider the function $r_1 \circ G: V \longrightarrow R$, where $r_1(x) = \mathrm{dist}(x,p_1)$ in $G_{N-n,N}$. If $p_1$ is sufficiently close to $p_0$, $r_1$ is a convex function in a neighborhood W of $p_0$ so that $r_1 \circ G$ is subharmonic on $G^{-1}(W)$. However, $r_1 \circ G$ achieves a local maximum in $G^{-1}(W)$, which gives a contradiction.

Thus, $\sigma$ is a closed geodesic in $G_{N-n,N}$. The preceeding argument shows that G maps V roughly monotonically onto $\sigma$, i.e. all integral curves $\rho(t)$ of the vector field $\nabla r$ are mapped almost monotonically onto $\sigma$. In particular, if U is a small tubular neighborhood of $\sigma$ and $\pi: U \longrightarrow \sigma$ is the nearest point retraction, then the map $\pi \circ G$ taking $\rho(t)$ to $\sigma$ is homotopic ( rel endpoints ) to the universal covering p: $R \longrightarrow S^1$. Thus, there is a sequence $r_i \longrightarrow \infty$ such that G maps $S(r_i) \cap V$ into a small neighborhood $T_\delta$ of $T^\perp \in G_{N-n,N}$ and $\pi \circ G$ restricted to $A(r_i,r_{i+1}) - G^{-1}(T_\delta)$ is of degree 1 as a map of $\rho(t)$ into $\sigma - T_\delta$.

We now complete the proof using an approach inspired by a technique of Brian White [18]. Let C be the Chern-Lashof-Gauss map [5],

$$C: \$N(V) \longrightarrow S^{N-1},$$

where \$N(V) is the unit normal sphere bundle of M in $E^N$ : $C(\nu)$ is the parallel translate of $\nu$ to the origin. By the argument above, C maps into a small tubular neighborhood of a closed curve $\tau(t)$ of totally geodesic (N-n-1) spheres $S_{\tau(t)}$ in $S^{N-1}$ corresponding to the geodesic $\sigma$ in the Grassmannian $G_{N-n,n}$. The sequence $r_i$ above determines maps $C_i = C|_{A(r_i,r_{i+1})}$, which for i large, map $\partial\big[A(r_i\,,\,r_{i+1})\big]$ into a small tubular neighborhhod $T(\delta)$ of $S_T = S^{N-1} \cap T^\perp$. Further, for any $\nu \in \mathrm{Im}\, C_i - T(\delta)$,

$$(3.4) \qquad \deg_{C_i} \nu = \sum_{x \in C_i^{-1}(\nu)} \mathrm{sgn}\big[\det C_*(x)\big] = 1.$$

However,

$$\int_{S^{N-1}} \deg_{C_i} \nu = \int_{A(r_i,r_{i+1})} \kappa$$

where $\kappa$ is the Lipschitz-Killing curvature [5]. One easily sees that $|\kappa| \leq c_{n,N} |A|^n$ for a universal constant $c_{n,N}$ depending only on dimensions. Thus, given $\epsilon > 0$, there is an i large, such that

$$\int_{S^{N-1}} \deg_{C_i} \nu \quad < \epsilon.$$

Since $\deg_{C_i} \nu$ is constant on connected components of $S^{N-1} - C_i\big(\partial(A(r_i,r_{i+1}))\big)$ and $C_i\big(\partial(A(r_i,r_{i+1}))\big)$ is contained in $T(\delta)$, it follows that $\deg_{C_i} \nu = 0$, for $\nu \in S^{N-1} - T(\delta)$. This contradicts ( 3.4 ), showing that C cannot map into $S^{N-1} - T(\delta)$, which completes the proof.

□

The discussion above leads to the main result.

<u>**Theorem 3.2.**</u> <u>Let $M^n \to E^N$ be a complete minimally immersed submanifold of finite total scalar curvature. Then $M^n$ is $C^\infty$ diffeomorphic to a compact manifold $\overline{M^n}$ punctured at a finite number of points $\{p_i\}_{i=1}^r$. The Gauss map</u>

$$G : M^n \to G_{n,N}$$

<u>extends to $C^{n-2}$ map $\bar{G} : \overline{M^n} \to G_{n,N}$ of the compactification. Further the metric on $M^n$ conformally extends to a complete $C^{n-2}$ Riemannian metric on $\overline{M^n}$.</u>

<u>Proof</u> By the results above, $M^n$ has a finite number of ends, each diffeomorphic to a punctured n-ball. Let $I : R^N - \{0\} \longrightarrow R^N - \{0\}$ be the inversion through the origin, $I(x) = \frac{x}{|x|^2}$. For an end V of M, let $W = I(V) \subset B^N(1) - \{0\}$. By Theorem 3.1, one sees that $\bar{W} = W \cup \{0\}$ is a $C^1$ submanifold of $B^N(1)$ diffeomorphic to the n-ball $B^n$, with $T_0\bar{W} = T$, the tangent plane at infinity of V.
In this way, we obtain a $C^1$ compactification $\overline{M^n}$ of $M^n$. Since on each end V, $\lim_{|x|\to 0} G\circ I(x) = T$, the Gauss map has a $C^0$ extension to $\overline{M^n}$. Further, since I is a conformal map of $R^N$, the metric $d\bar{s}^2$ induced on $\bar{W}$ is a $C^0$ metric which is conformal to the metric on V.

Suppose first $n = 2$. Then $G \circ I : (W, \tilde{ds}^2) \to G_{2,N}$ is a harmonic map with continuous extension to $\bar{W}$. It is then well known (e.g.[7] ) that $G \circ I$ extends analytically to $\bar{W}$.

Now assume $n > 2$. We need to estimate the rate of decay of V to its tangent plane T at infinity. Let $\{x_i\}_1^N$ be standard coordinates on $E^N$ and assume $T = \mathrm{span} < x_1, \cdots, x_n >$. We may extend V to a complete manifold $\tilde{V}$ in $E^N$ by gluing on a ball $B^n$ to $\partial V = V \cap S(R_0)$. Similarly, we may extend the metric $ds^2{}_V$ to a complete metric on $\tilde{V}$ which is smoothly quasi-isometric to the flat metric on $E^n$.

Now let u be one of the coordinate functions $\{x_i\}_{i=n+1}^N$ on V. By Theorem 3.1, we see $|du| \to 0$ as $|x| \to \infty$. Recall that u is a harmonic function on V. Thus we may extend u to a smooth function on $\tilde{V}$ so that

$$\Delta u = f \tag{3.5}$$

on $\tilde{V}$, where f has support in $\overline{\tilde{V}-V}$. Let G denote the (negative) Green's function for the Laplace-Beltrami operator $\Delta$ on $\tilde{V}$. It is well-known [12] that G satisfies an estimate of the form

$$|G(x,y)| \le \frac{c}{[\mathrm{dist}(x,y)]^{n-2}}$$

for $n > 2$. Recall that the metric on V approximates the flat metric at a rate $o(r^{-2})$. Using the Schauder estimates for $\Delta$ on V, see [8], it is easily seen that

$$|D_y^k G|(x,y) \le \frac{c(k)}{[\mathrm{dist}(x,y)]^{n-2+k}}, \tag{3.6}$$

first for $k = 1,2$, and by iteration for any k. We may define

$$w(x) = \int_{\tilde{V}} G(x,y) f(y) dy$$

on $\tilde{V}$. Note that w is well defined since f has compact support, and $|D^i w(x)| = O(r^{-n-i+2})$ as $|x| \to \infty$. Let $z = u - w$ so that z is harmonic on V. By the Bochner-Lichnerowicz formula [3], we have

$$(3.7) \qquad \tfrac{1}{2}\Delta|\nabla z|^2 = |\nabla^2 z|^2 + \mathrm{Ric}(\nabla z,\nabla z).$$

We estimate (3.7) on V. Combining the above estimate for $|D^i w|$ with Young's inequality, one finds

$$|\nabla^2 z|^2 \geq (1-\epsilon)|\nabla^2 u|^2 - (1+\epsilon^{-1})\frac{c_1}{r^{2n}},$$

for any $\epsilon$ in (0,1) and some constant $c_1 > 0$. On V, one calculates that $|\nabla^2 u|^2 = |A^{\nabla^N u}|^2$ and $\mathrm{Ric}(\nabla z, \nabla z) = -|A(\nabla z)|^2 \geq -|A|^2|\nabla z|^2$. Expanding $|\nabla z|^2$ and using Young's inequality again gives

$$(3.8) \qquad \tfrac{1}{2}\Delta|\nabla z|^2 = (1-\epsilon_2)\Big(|A^{\nabla^N u}|^2 - |A|^2|\nabla^T u|^2\Big) - (1+\epsilon_2^{-1})\frac{c_1}{r^{2n}}$$

Recall that $|\nabla^T u| \to 0$ and $|\nabla^N u| \to 1$ as $|x| \to \infty$. Summing over $u = x_i$, $i = n+1, \ldots, N$, it follows that there are constants $c_2$, $c_3$ such that

$$(3.9) \qquad \sum_{i=n+1}^{N} \Delta|\nabla z_i|^2 \geq c_1|A|^2 - \frac{c_2}{r^{2n}}$$

on V, where $z_i = u_i - w_i$. On the other hand, one computes on V that

$$\Delta\Big(\frac{1}{r^{p-2}}\Big) = -\frac{(p-2)[n-p|\nabla r|^2]}{r^p}$$

so that

$$\Delta\Big(\sum|\nabla z_i|^2 - \frac{c_3}{r^{p-2}}\Big) \geq c_1|A|^2 - \frac{c_2}{r^{2n}} + \frac{c_3}{r^p}[n - p|\nabla r|^2] > 0,$$

provided $c_3$ is chosen sufficiently large and $p = (n-1) + \alpha$, for $0 < \alpha < 1$. Thus $\sum|\nabla z_i|^2 - \frac{c_3}{r^{n-3+\alpha}}$ is subharmonic on V, negative on $\partial V$ and converges to zero at infinity. By the maximum principle,

$$|\nabla z|^2 \leq \frac{c_3}{r^{n-3+\alpha}},$$

for $z = z_i$, any i.

If $n>3$, it follows by integration that $|z|$ grows slower than any positive power of $r$. By the DeGiorgi-Nash-Moser theory [13], $z$ must be a constant. If $n = 3$, we note that for $p = 1+\epsilon$,

$$\Delta|\nabla z|^p \geq p|\nabla z|^{p-2}[\tfrac{1}{2}\Delta|\nabla z|^2 + (p-2)|A^{\nabla^N z}|^2]. \tag{3.10}$$

Substituting (3.8) in (3.10) and summing over $i$ as before leads to the estimate

$$|\nabla z|^p \leq \frac{c_3}{r^{n-3+\alpha}},$$

so that the argument above shows $z$ is constant. By translating $V$ in $\mathbb{E}^N$, we may assume the constant functions are zero.

Thus, for $k\geq 0$, $r = |x|$, we have

$$|D^k u|(x) \leq \frac{c(k)}{r^{n-2+k}}. \tag{3.11}$$

Noting that $A$, or equivalently $DG$, can be expressed in terms of $D^2u$, as $u$ ranges over $\{x_k\}_{n+1}^N$, we obtain

$$|D^k|A|| \leq \frac{c(k)}{r^{n+k}} \qquad |D^k G| \leq \frac{c(k)}{r^{n+k-1}}. \tag{3.12}$$

We use these estimates to study $W$ near $0$. We may write $V = \{(x,F(x)) : x\in T-B(R)\}$, as the graph of a function $F\colon T \longrightarrow T^{\perp}$. Then $W = I(V)$ is described by $\frac{1}{R^2}(x,F(x))$, where $R^2= |x|^2 + |F(x)|^2$. Let $r^2=|x|^2$ and $y=I(x)$, so $y=\frac{x}{r^2}$. Then

$$\frac{1}{R^2}(x,F(x)) = \left(y,|y|^2F(I(y))\right)\cdot\frac{r^2}{R^2}.$$

By (3.11), $F(x) = O(|x|^{n-2})$, so that $\frac{r^2}{R^2} = 1 + O(|y|^n)$ and $|y|^2F(I(y)) = O(|y|^n)$ as $|y|\to 0$. Similarly, $D^k(|y|^2F(I(y)) = O(|y|^{n-k})$.

Thus $W$ behaves near $0$ as the graph of the function $|y|^n$. It follows that $\overline{W} = W\cup\{0\}$ is a $C^{n-1}$ submanifold of $B^n(1)$. Further, the Gauss map $G$ and the metric $d\tilde{s}^2$ have $C^{n-2}$ extensions to $\overline{W}$.

□

The converse of Theorem ~~4~~3.2 is considerably easier to prove.

*Theorem* ~~4~~3.3. *Let* $M^n \to E^N$ *be a complete minimal immersion such that* $M^n$ *is* $C^1$ *diffeomorphic to a compact manifold* $\overline{M^n}$ , *punctured at a finite number of points* $\{p_i\}$ . *Suppose the Gauss map of* $M^n$ *extends to a* $C^1$ *map of* $\overline{M^n}$ . *Then* $\int_{M^n} |A|^n dvol_M < \infty$ .

*Proof*. The hypotheses imply that the metric $g$ on $M$ extends conformally to a continuous Riemannian metric $\tilde{g}$ on the compactification $\overline{M^n}$ , as in Theorem 4.2. Note also that there is a natural identification of $A$ with the derivative $DG$ . Thus

$$\int_{M^n} |A|^n dvol_M = \int_{M^n} |DG|^n dvol_M \quad .$$

These integrals are conformally invariant, so that

$$\int_{M^n} |A|^n dvol_M = \int_{\overline{M^n}} |\widetilde{DG}|^n d\widetilde{vol}_{\overline{M^n}}$$

The fact that the Gauss map has a $C^1$ extension to $\overline{M^n}$ implies the latter integral is finite.

□

## §4. The Gauss Bonnet Theorem on $M^n$.

In this section, we prove a preliminary version of Theorem B of the Introduction; the final proof is given in §5 (Theorem 5.1). Once again, the results of this section follow basically from the estimate (1.3).

Theorem 4.1. *Let* $M^n \to \mathbf{E}^N$ *be a complete minimal immersion of a connected, oriented manifold* $M^n$ *of finite total scalar curvature. Then*

$$\chi(M^n) = \int_{M^n} \Omega + \frac{1}{\sigma_{n-1}} \cdot \lim_{r\to\infty} \frac{\mathrm{Vol}(M \cap S(r))}{r^{n-1}}, \tag{4.2}$$

*where* $\Omega$ *is the Gauss-Bonnet-Chern form on* $M$ *and* $\sigma_{n-1} = \mathrm{vol}\, S^{n-1}(1)$ .

Remark. In case $n$ is odd, we define $\chi(M) = \sum_i \mathrm{Ind}_{p_i}(\nabla r^2)$ , where $r(x) = |x|$ and $p_i$ are the zeros of $\nabla r^2$ on $M$ . By Lemma 2.4 the sum is finite.

Proof. We apply the Gauss-Bonnet-Chern theorem to the domains

$$U(r) = i^{-1}(M^n \cap B(r)) \subset M^n ,$$

and consider the limiting behavior as $r \to \infty$ . First, the structure equations on $M^n$ are given by

$$\begin{aligned} d\theta^i &= \sum_{j=1}^{n} \omega_{ij}\theta^j \\ d\omega^{ij} &= \sum_{k=1}^{n} \omega_{ik}\wedge\omega_{kj} + \Omega_{ij} \end{aligned} \tag{4.3}$$

where $\omega_{ij}$ and $\Omega_{ij}$ are the connection 1-forms and curvature 2-forms respectively of $M$ , with respect to the orthonormal coframing $\{\theta^i\}_1^n$ . One may express $\Omega_{ij}$ in terms of the second fundamental form $A = \{A^\mu\}$ , $\mu = 1,\dots N-n$ by

$$\Omega_{ij} = \sum_{k,\ell} [\sum_\mu A^\mu_{jk}A^\mu_{i\ell} - A^\mu_{ik}A^\mu_{j\ell}]\theta^k \wedge \theta^\ell \quad . \tag{4.4}$$

The Gauss-Bonnet-Chern form is the $n$-form on $M$ defined by

$$
(9.5) \qquad \Omega = \begin{cases} \dfrac{(-1)^m}{2^m \pi^m m!} \cdot \Sigma \varepsilon_{i_1 \dots i_n} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{n-1} i_n} & n = 2m \\[2ex] 0 & n = 2m+1 \end{cases}
$$

$2^n$

where the sum is over all permutations $\sigma = (i_1, \dots i_n)$ of $(1, \dots n)$ and $\varepsilon_{i_1 \dots i_n} = \mathrm{sgn}(\sigma)$ . For $0 \leq k \leq \frac{1}{2}[n-1]$ , define $(n-1)$ forms $Q_k$ on the tangent sphere bundle $SM$ of $M$ by

$$
(9.6) \qquad Q_k(V) = c_{k,n} \cdot \Sigma \varepsilon_{i_1 \dots i_{n-1}} \Omega_{i_1 i_2} \wedge \dots \wedge \Omega_{i_{2k-1} i_{2k}} \wedge \omega_{i_{2k+1} v} \wedge \dots \omega_{i_{n-1} v}
$$

for $V \in SM$ and constants given by

$$
c_{k,n} = \begin{cases} \dfrac{(-1)^k}{\pi^m 2^{m+k} k!} \cdot \dfrac{1}{1.3 \dots (n-2k-1)} & n = 2m \\[2ex] \dfrac{(-1)^{k+1}}{\pi^m 2^n m!} \dbinom{m}{k} & n = 2m+1 \end{cases}
$$

Now note that by Corollary 2.5 the domains $U(r)$ , for $r$ sufficiently large, have $C^\infty$ boundary in $M$ and are diffeomorphic to $M$ . Further, the smooth vector field $\nabla r^2$ on $M$ has only finitely many zeros : define

$$
\chi(U(r)) = \sum_i \mathrm{Ind}_{p_i} (\nabla r^2) .
$$

Note also that $V \equiv \dfrac{\nabla r}{|\nabla r|}$ is the unit outward normal to $\partial U(r)$ , so that, in case $n$ is even, $\chi(U(r))$ is a topological invariant. The Gauss-Bonnet-Chern formula [4] then reads

$$
(9.8) \qquad \chi(U(r)) = \int_{U(r)} \Omega - \sum_{k=o}^{[\frac{1}{2}(n-1)]} \int_{\partial U(r)} V^* Q_k \quad .
$$

We now examine the boundary integrals more closely. For $p \in \partial U(r)$ frame the tangent spaces $T_p(U(r))$ by eigenvectors of the second fundamental form

$B_r$ of $\partial U(r)$ in $M$. Thus $\{e_i\}_1^{n-1} \in T_p(\partial U(r))$ satisfy

$$B_r(e_i) = -\lambda_i e_i \quad .$$

We have $V^*\omega_{i,v} = \omega_{i,v} = -\lambda_i\theta^i$, where $\{\theta^i\}$ are the 1-forms dual to $\{e_i\}$.
This gives for $Q_o$,

$$V^*Q_o = (-1)^{n-1}(n-1)!\cdot c_{o,n}\cdot\left(\prod_1^{n-1}\lambda_i\right)\theta^1\wedge\cdots\wedge\theta^{n-1} .$$

Lemma 2.6 implies that $\lambda_i$ is asymptotic to $\frac{1}{r}$ as $r \to \infty$, i.e. $\lambda_i = \frac{1}{r} + o(\frac{1}{r})$. Thus we obtain the estimate

$$V^*Q_o = \frac{-(n-1)!|c_{o,n}|}{r^{n-1}}\, dV_r + \frac{o(1)}{r^{n-1}}\, dV_r \quad , \tag{4.9}$$

where $dV_r$ denotes the volume form on $\partial U(r)$.

On the other hand, if $k > 0$, each summand of $V^*Q_k$ is of the form

$$\Phi_k^I = \Omega_{i_1 i_2}\wedge\cdots\wedge\Omega_{i_{2k-1}i_{2k}}\wedge\omega_{i_{2k+1}v}\wedge\cdots\wedge\omega_{i_{n-1}v}$$

and is thus small compared to $V^*Q_o$. In fact, using (4.4), (2.1 ) and Lemma 2.6, we see that on $\partial U(r)$, there is a fixed constant $c$ such that

$$|\Phi_k^I| < c\cdot\frac{\mu^{2k}(r)}{r^{n-1}}\cdot|dV| \quad , \tag{4.10}$$

for all $I = (i_1,\ldots,i_{n-1})$, $k > 0$. Substituting this into (4.8) gives

$$\chi(U(r)) = \int_{U(r)}\Omega + [(n-1)!|c_{o,n}|+o(1)]\,\frac{\mathrm{Vol}(\partial U(r))}{r^{n-1}} \tag{4.11}$$

We have already remarked that $\chi(U(r))$ is independent of $r$, for $r$ sufficiently large. In particular, $\lim\limits_{r\to\infty}\chi(U(r)) = \chi(M)$. Also, the pointwise inequality

$$|\Omega|_{(x)} \leq c \cdot \|A\|^n_{(x)} \quad ,$$

together with the assumption of finite total scalar curvature implies by the dominated convergence theorem that

$$\lim_{r\to\infty} \int_{U(r)} \Omega = \int_M \Omega$$

and also $\left|\int_M \Omega\right| < +\infty$ . As a consequence, $\lim_{r\to\infty} \frac{\mathrm{vol}(\partial U(r))}{r^{n-1}}$ exists and is finite. The theorem follows by taking the limit of (4.11) as $r \to \infty$ and evaluation of the constants. □

## §5. Applications.

In this section, we prove Theorem 5.1 referred to in the introduction. This formula is known in the case $n = 2$ (e.g. [10]) ; however certain novel features appear in higher dimensions. Several consequences of this are deduced; in particular, we prove a Bernstein-type theorem for submanifolds $M^n$ of finite total scalar curvature with $n \geq 3$ .

Recall from §3 that if $M^n$ is a complete minimal submanifold of finite total scalar curvature, then $\chi(M) = \sum_i \mathrm{Ind}_{p_i}(\nabla r^2)$ is well defined. If $\{V_i\}_{i=1}^r$ are the collection of ends of $M^n$ , the multiplicity $m_i$ of $V_i$ is given by

$$m_i = \lim_{r\to\infty} \frac{\mathrm{vol}(V_i \cap S(r))}{\sigma_{n-1} r^{n-1}} \quad ,$$

where $\sigma_{n-1}$ is the volume of the unit (n-1) sphere.

Theorem 5.1. *Let* $M^n \to \mathbb{E}^N$ *be a complete minimal immersion of an oriented manifold of finite total scalar curvature. Then*

(5.1) $$\int_{M^n} \Omega = \chi(M^n) + \sum_{i=1}^{r} m_i \ ,$$

*where* $m_i \in \mathbb{Z}^+$ *is the multiplicity of the end* $p_i \in M^n$ . *If* $n \geq 3$ , *then* $m_i = 1$ , $\forall i$ , *so that*

(5.2) $$\int_{M^n} \Omega = \chi(M^n) + r \ .$$

Proof. Eqn. (5.1) is an immediate consequence of Theorem 3.1 together with the result from §3 that the manifolds $\frac{1}{r}(V_i \cap S(r))$ converge smoothly to an equatorial $(n-1)$-sphere with multiplicity $m_i \in \mathbb{Z}^+$ . Further for $r$ sufficiently large, $\frac{1}{r}(V_i \cap S(r))$ represents an $m_i$-fold covering of an equator $S^{n-1} \subset S^{N-1}(1)$ If $n > 2$ , it follows that $m_i = 1$ , for all $i$ . This proves (5.2). □

Remarks. 1) One should consult the paper of Jorge-Meeks [10] for a discussion of related topics.

2) The smooth convergence of the manifolds $\frac{1}{r}(M \cap S(r))$ of course implies that all ends of $M$ are embedded if $n > 2$ .

We now present a Bernstein-type theorem, valid however only in dimensions greater than two.

Theorem 5.2. *Let* $M^n \to \mathbb{E}^N$ *be a complete minimal immersion of finite total scalar curvature. If* $n > 2$ *and* $M^n$ *has one end, then* $M^n$ *is an affine* $n$-*plane*.

Proof. Let $v(r) = \mathrm{vol}(M_r) = \dfrac{\mathrm{vol}(M \cap B(r))}{r^n}$ . Then, since $M$ has only one end, of multiplicity one by Theorem 5.1, one has

$$\lim_{r\to\infty} v(r) = \lim_{r\to\infty} \frac{\mathrm{vol}(M \cap B(r))}{r^n} = \omega_n \ ;$$

where $\omega_n$ is the volume of the unit ball in $\mathbf{E}^N$. On the other hand, it is well known that $v(r)$ is monotonically non-decreasing in $r$, with $v(0) = \omega_n$. Thus, $v(r) \equiv \omega_n$. It follows from standard methods that $M^n$ must be an affine n-plane.

Remarks. 1) The theorem is clearly false for $n = 2$, as is demonstrated for instance by Enneper's surface.

2) R. Schoen [16] has recently proved a certain analogue of Theorem 5.2 for embedded minimal hypersurfaces in $\mathbf{E}^N$ having either one or two ends. He proves that the only such submanifolds which are regular at infinity are the plane and the higher dimensional catenoid. Theorem 5.2 generalizes these results in the case of one end.

3) Theorem 5.2 may be used to derive a local pointwise curvature estimate for minimally immersed n-discs in $B(1)$, $n \geq 3$; c.f. [1].

## REFERENCES


[1] M. Anderson, Local estimates for minimal submanifolds in dimensions greater than two, Proc. Symp. Pure Math. vol 44, (1986),

[2] E.Bombieri, E.deGiorgi, and E.Giusti, Minimal cones and the Bernstein problem, Inventiones Math. 7 (1968), 243-269.

[3] M.Berger, P.Gauduchon, and E.Mazet, Le spectre d'une variete riemannienne, Springer Lecture Notes 194 (1971).

[4] S.-S.Chern, On the curvatura integra in a Riemannian manifold, Ann. of Math. 46 (1945), 674-684.

[5] S.-S.Chern and R.Lashof, On the total curvature of immersed manifolds, Amer. J. Math. 79 (1957), 306-313.

[6] S.-S.Chern and R.Osserman, Complete minimal surfaces in $E^N$, J. d'Analyse Math. 19 (1967), 15-34.

[7] H.-I.Choi and R.Schoen, The space of minimal embeddings of a surface into a 3-manifold of positive Ricci curvature, Inventiones Math. 81 (1985),387-394.

[8] D.Gilbarg and N.Trudinger, Elliptic Partial Differential Equations of Second Order, Springer Verlag, New York, 1977.

[9] A.Huber, On subharmonic functions and differential geometry in the large, Comm. Math. Helv. 32 (1957), 13-72.

[10] L.P.de Melo Jorge and W.Meeks, The topology of complete minimal surfaces of finite total Gaussian curvature, Topology, 22 No.2, (1983), 203-221.

[11] A. Kasue, Gap phenomena for minimal submanifolds in Euclidean space, J. Math. Soc. Japan

[12] W.Littman, G.Stampacchia, and H.Weinberger, Regular points for elliptic equations with discontinuous coefficients, Ann.Sc.Norm.Sup.Pisa (3), 17 (1963) 43-77.

[13] J.Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961), 577-591.

[14] R. Osserman, On complete minimal surfaces, Arch. Rat. Mech. Anal. 13 (1963), 392-404.

[15] R. Osserman, Global properties of minimal surfaces in $E^3$ and $E^N$, Ann. of Math. 80 (1964), 340-364.

[16] R. Schoen, Uniqueness, symmetry and embeddedness of minimal surfaces,

J. Diff. Geometry 18 (1983), 791-809.

[17] R. Schoen, Analytic aspects of the harmonic map problem, Springer Verlag MSRI Series, vol 2, (1984).

[18] B. White, Complete minimal surfaces of finite total curvature, (preprint).

[19] F. Xavier, The Gauss map of a complete non-flat minimal surface cannot omit 7 points of the sphere, Ann. of Math. 113 (1981), 211-214.