ON THE BARTNIK CONJECTURE FOR THE
STATIC VACUUM EINSTEIN EQUATIONS

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ABSTRACT. We prove that given any smooth metric γ and smooth positive function H on S^2, there is a constant $\lambda > 0$, depending on (γ, H), and an asymptotically flat solution (M, g, u) of the static vacuum Einstein equations on $M = \mathbb{R}^3 \setminus B^3$, such that the induced metric and mean curvature of (M, g, u) at ∂M are given by $(\gamma, \lambda H)$. This gives a partial resolution of a conjecture of Bartnik.

1. INTRODUCTION

The Bartnik conjecture [5], [7] is a statement concerning the existence and uniqueness of solutions to a geometrically and physically natural boundary value problem for the static Einstein equations in vacuum. In general relativity, it is important in connection with the concepts of mass and quasi-local mass of initial data sets (Cauchy surfaces). Mathematically, the conjecture concerns the unique global solvability of a very interesting but rather complicated elliptic boundary value problem for a system of PDE's on a 3-manifold M.

We first recall the statement of the conjecture. Let $B^3 \subset \mathbb{R}^3$ be a 3-ball (not necessarily round) and let M be a 3-manifold (with boundary) diffeomorphic to $\mathbb{R}^3 \setminus B^3$. Thus ∂M is diffeomorphic to the 2-sphere S^2. The free data for a static vacuum solution on M are a pair (g, u) consisting of a smooth asymptotically flat Riemannian metric g on M and a smooth positive (potential) function $u : M \to \mathbb{R}^+$ asymptotic to 1 at infinity. The static vacuum Einstein equations on (g, u) are given by

$$(1.1) \qquad uRic_g = D^2 u, \quad \Delta u = 0,$$

where the Hessian D^2 and Laplacian $\Delta = trD^2$ are taken with respect to g. From the metric g and potential u, one can construct a 4-dimensional solution to the (usual) vacuum Einstein equations

$$(1.2) \qquad Ric_{g_{\mathcal{M}}} = 0,$$

where $\mathcal{M} = \mathbb{R} \times M$ and the 4-metric is given by $g_{\mathcal{M}} = \pm u^2 dt^2 + g$. The equations (1.1) on M and (1.2) on $\mathcal{M}$ are equivalent.

Given a solution (M, g, u) of the static vacuum Einstein equations, let γ be the Riemannian metric induced on $S^2 \simeq \partial M$ and let H be the mean curvature of $\partial M \subset (M, g)$, (with respect to the inward unit normal into M). View the pair (γ, H) as boundary data of the static vacuum solution (M, g, u) and suppose the data (g, u) are C^∞ smooth up to ∂M. The boundary data thus takes values in the space $Met^\infty(S^2) \times C^\infty(S^2)$ of smooth metrics and smooth functions on S^2. Observe that the potential u on ∂M is not part of the boundary data. Let $C_+^\infty(S^2)$ be the space of positive C^∞ functions $H : S^2 \to \mathbb{R}^+$.

The Bartnik conjecture then states that arbitrary boundary data $(\gamma, H) \in Met^\infty(S^2) \times C_+^\infty(S^2)$ are realized uniquely (up to diffeomorphism) by an asymptotically flat solution (M, g, u) of the static vacuum Einstein equations (1.1) on M, smooth up to ∂M. Thus, given an arbitrary smooth metric

γ on S^2 and an arbitrary positive smooth function H on S^2, there exists a smooth asymptotically flat solution (M, g, u) of the static vacuum Einstein equations for which the induced metric and mean curvature at ∂M are given by (γ, H). Further, the solution is unique, up to a smooth diffeomorphism fixing ∂M.

In this paper, we obtain a partial resolution of the existence part of the Bartnik conjecture. One version of the main result is the following.

Theorem 1.1. *Given any smooth boundary data* $(\gamma, H) \in Met^\infty(S^2) \times C_+^\infty(S^2)$, *there is a constant* $\lambda > 0$, *depending on* (γ, H), *and an asymptotically flat static vacuum solution* (M, g, u), *smooth up to* ∂M, *such that the induced metric and mean curvature of* (M, g, u) *at* ∂M *are given by* $(\gamma, \lambda H)$.

Thus, up to a single (multiplicative) scalar degree of indeterminacy on the mean curvature, arbitrary smooth data (γ, H) are realized as the boundary metric and mean curvature of a smooth asymptotically flat static vacuum solution (M, g, u). In particular, the boundary metric γ on S^2 may be arbitrarily specified. Clearly (by rescaling g) one may instead shift the indeterminacy to a scaling of the boundary metric, (so that given (γ, H), there is a constant $\lambda > 0$ such that $(\lambda\gamma, H)$ are realized as boundary data of an asymptotically flat static vacuum solution).

The idea of considering rescalings λH of the mean curvature appears in earlier work of Jauregui [11], (cf. also [12]), in connection with filling *in* prescribed boundary data $(\gamma, \lambda H)$ on S^2 with a metric of non-negative scalar curvature on a 3-ball B^3 with boundary S^2. Similarly, it appears in the context of filling *in* such boundary data by static vacuum solutions (the "interior Bartnik problem") in [2].

In the interior context, such a rescaling of H is *necessary*; for a typical $\mathbb{R}^+$-orbit $(\gamma, \lambda H)$, $\lambda \in \mathbb{R}^+$, only for at most finitely many λ are the data $(\gamma, \lambda H)$ realized as boundary data of interior static vacuum solutions. In fact, the space of interior (strictly) static vacuum solutions is generically transverse to the $\mathbb{R}^+$-orbits of H, cf. [2].

Such a restriction is not necessary in the current "exterior" context. There are simple examples where for a given data (γ, H), all H-rescalings $(\gamma, \lambda H)$ are also realized as boundary data of static vacuum solutions. The simplest instance of this is the curve of Schwarzschild metrics, given as

(1.3)
$$g_m = (1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{S^2(1)},$$

on the space-like hypersurface $M \simeq \mathbb{R}^3 \setminus B$ where $r \geq 1 > 0$ and $r > 2m$. Assume $m < \frac{1}{2}$. Then the boundary ∂M may be taken at the locus $r = 1$ and the metric on $\{r = 1\}$ is fixed,

$$\gamma = \gamma_{S^2(1)}.$$

A simple computation shows that at the boundary $\partial M = \{r = 1\}$,

$$H = 2\sqrt{1 - 2m},$$

so that H is a constant depending only on m. As m varies over the interval $(-\infty, \frac{1}{2})$, $H \in (0, \infty)$ is monotone decreasing.

It is worthwhile to place the Bartnik conjecture and Theorem 1.1 in a more geometric setting. To do this, let $\mathbb{E}^{m,\alpha}$ be the space of all asymptotically flat (AF) solutions (g, u) of the static vacuum equations (1.1), which are $C^{m,\alpha+}$ on M up to ∂M. Here $C^{m,\alpha+}$ is the usual "little" Hölder space of functions, i.e. the closure of the space of C^∞ smooth functions on M with respect to the $C^{m,\alpha}$ Hölder norm. It is well-known, cf. [15] for instance, that $C^{m,\alpha+}$ is a closed separable subspace of the Hölder space $C^{m,\alpha}$ (which itself is not separable). We assume throughout the paper that $m \geq 3$ and $\alpha \in (0, 1)$. As above, we also assume $u \to 1$ at infinity in M. The topology on $\mathbb{E}^{m,\alpha}$ is the standard weighted $C^{m,\alpha}$ Hölder topology used for AF manifolds, cf. [3] for instance.

Let $\mathrm{Diff}_1^{m+1,\alpha}(M)$ be the group of $C^{m+1,\alpha+}$ diffeomorphisms $\psi : M \to M$ with $\psi = id$ on ∂M. The group $\mathrm{Diff}_1^{m+1,\alpha}(M)$ acts freely on $\mathbb{E}^{m,\alpha}$ fixing the boundary data (γ, H), (since an isometry of (M, g) fixing ∂M pointwise is necessarily the identity). The quotient space $\mathcal{E}^{m,\alpha}$ is the moduli space of static vacuum solutions; an element in $\mathcal{E}^{m,\alpha}$ represents an equivalence class of isometric metrics.

Next, as above abusing notation slightly, let $Met^{m,\alpha}(S^2)$ be the space of $C^{m,\alpha+}$ metrics on $S^2 \simeq \partial M$ and $C^{m-1,\alpha}(S^2)$ be the space of $C^{m-1,\alpha+}$ functions on S^2. One thus has a natural map, mapping a static vacuum solution to its Bartnik boundary data:

(1.4) $$\Pi_B : \mathcal{E}^{m,\alpha} \to Met^{m,\alpha}(S^2) \times C^{m-1,\alpha}(S^2),$$

$$\Pi_B(g) = (\gamma, H).$$

It is proved in [1], [3] that the space $\mathcal{E}^{m,\alpha}$ is a smooth (infinite dimensional) Banach manifold, and the map Π_B is C^∞ smooth and Fredholm, of Fredholm index 0.[1] This means that the derivative or linearization $D\Pi_B$, at any solution $(M, g, u) \in \mathcal{E}^{m,\alpha}$, has finite dimensional kernel $Ker D\Pi_B$ and finite dimensional cokernel $Coker D\Pi_B$,

$$dim Ker D\Pi_B < \infty \;\; \text{and} \;\; dim Coker D\Pi_B < \infty,$$

and the range of $D\Pi_B$ is closed. Further the Fredholm index, given by the difference of these terms, satisfies

(1.5) $$dim Ker D\Pi_B - dim Coker D\Pi_B = 0.$$

This result implies that the boundary map Π_B is at least locally reasonably well-behaved, as a map of Banach manifolds.

Let $\mathcal{E}_+^{m,\alpha}$ be the open submanifold of $\mathcal{E}^{m,\alpha}$ for which the mean curvature H on ∂M is positive, i.e.

$$\mathcal{E}_+^{m,\alpha} = (\Pi_B)^{-1}(Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)).$$

The Bartnik conjecture above may thus be rephrased as stating that the smooth map Π_B, restricted to the open submanifold $\mathcal{E}_+^{m,\alpha}$,

(1.6) $$\Pi_B : \mathcal{E}_+^{m,\alpha} \to Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2),$$

is surjective and injective. (Here we generalize slightly from $m = \infty$ to general $m \geq 3$). If one adds the condition that the linearization $D\Pi_B$ is an isomorphism everywhere (i.e. $Ker D\Pi_B = 0$ everywhere) it follows from the inverse function theorem in Banach manifolds that Π_B is conjecturally a smooth global diffeomorphism.

However, it is proved in [3] that this most optimistic version of the conjecture does not hold; Π_B in (1.6) cannot be a diffeomorphism. For example, consider the "trivial" static solution $\mathbb{R}^3$ with $u = 1$ and $g = g_{Eucl}$. It is easy to construct a curve of embedded positive mean curvature 2-spheres $F_t : S^2 \to \mathbb{R}^3$, $t \in (0, 1]$ such that as $t \to 0$, the boundary data converge to a limit in the target space, i.e. $(\gamma_t, H_t) \to (\gamma, H) \in Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$ with $F_t \to F_0$, where $F_0(S^2)$ is only an immersed but not an embedded 2-sphere. It follows that the exterior or unbounded components of $\mathbb{R}^3 \setminus F_t(S^2)$ define static vacuum solutions $(M_t, (g_{Eucl})_t, 1)$ for which the boundary data $\Pi_B(M_t, (g_{Eucl})_t, 1)$ converge to a limit in $Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$, but although the limit $(M_0, (g_{Eucl})_0, 1)$ makes sense, it is not a static vacuum solution on a smooth manifold with boundary. The structure of

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[1]This is proved in [1], [3] with respect to the usual Hölder $C^{m,\alpha}$ spaces. The proof relies only on the fact that the linearized static vacuum equations, together with the linearization of the boundary conditions (γ, H), form a formally self-adjoint elliptic boundary value problem in a suitable gauge, (cf. Section 3 and in particular Theorem 3.5 of [3]). The same proof, based on elliptic regularity and Fredholm theory, applies also to the little Hölder spaces $C^{m,\alpha+}$ and the Banach manifolds modeled thereon. While such extensions of elliptic regularity and Fredholm theory to the little Hölder spaces are forklore, an explicit and general proof is given in Appendix A of [9] for instance.

a manifold-with-boundary degenerates, with uniform control on the boundary data (γ, H). This means that the inverse map $(\Pi_B)^{-1}$ (assuming it exists) cannot be continuous everywhere.

The same construction can be carried out in any background (exterior) static vacuum solution (M, g, u) (in place of $(\mathbb{R}^3, g_{Eucl}, 1)$) so that this phenomenon is ubiquitous.

To understand Theorem 1.1 in this context, define an equivalence relation on $C_+^{m-1,\alpha}(S^2)$ by $[H_1] = [H_2]$ if and only if

$$(1.7) \qquad\qquad H_2 = \lambda H_1,$$

for some constant $\lambda > 0$. Let $\mathcal{C}_+^{m-1,\alpha}(S^2) = C_+^{m-1,\alpha}(S^2)/\sim$ be the space of equivalence classes; this is the quotient of $C_+^{m-1,\alpha}(S^2)$ by the $\mathbb{R}^+$-action given by multiplication by a positive scalar. Since the action is smooth and free, the space $\mathcal{C}_+^{m-1,\alpha}(S^2)$ is also a smooth Banach manifold, and so one has an induced smooth map

$$(1.8) \qquad\qquad \Pi : \mathcal{E}_+^{m,\alpha} \to Met^{m,\alpha}(S^2) \times \mathcal{C}_+^{m-1,\alpha}(S^2),$$

$$\Pi(M, g, u) = (\gamma, [H]).$$

The map Π is a smooth Fredholm map, of Fredholm index 1, i.e.

$$(1.9) \qquad\qquad dim Ker D\Pi - dim Coker D\Pi = 1.$$

Theorem 1.1 then follows from the following slightly more general statement.

Theorem 1.2. *The map Π in (1.8) is surjective.*

The proof of Theorems 1.1 and 1.2 follow in Section 2 below. The proof builds on the prior work of the author in [3] which will be used quite often here. In particular, we refer to [3] for a more detailed presentation of certain definitions. Section 3 contains a discussion of related work and lists several open problems suggested by Theorem 1.2.

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2. Proofs.

Given the results of [3], the primary difficulty in understanding the validity of the Bartnik conjecture is the degeneration of the manifold-with-boundary structure under control of the boundary data (γ, H) discussed in the Introduction.

Given a static vacuum solution $(M, g, u) \in \mathcal{E}^{m,\alpha}$, let N be the unit inward normal of ∂M in M and let $inj_{\partial M}$ denote the injectivity radius of the normal exponential map from ∂M into (M, g, u). The normal exponential map $exp_N : N(\partial M) \to M$ maps vectors in the normal bundle of ∂M into M and $inj_{\partial M}$ is the largest value such that exp_N is a diffeomorphism when restricted to normal vectors aN with $a < inj_{\partial M}$.

The behavior described in the paragraph following (1.6) is characterized by the fact that there are curves (or sequences) of static vacuum solutions (M, g_t, u_t) (with uniformly bounded curvature and derivatives of curvature for instance) such that

$$inj_{\partial M_t} \to 0, \quad \text{as} \quad t \to 0,$$

but with $(\gamma_t, H_t) \to (\gamma, H) \in Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$. Equivalently, there are a pair of curves x_t^1, x_t^2 in ∂M_t with $dist_{\partial M_t}(x_t^1, x_t^2) \geq c_0 > 0$ but such that

$$dist_{(M, g_t)}(x_t^1, x_t^2) \to 0, \quad \text{as} \quad t \to 0.$$

Thus the distance between x_t^1 and x_t^2 remains bounded away from zero in ∂M, but converges to zero in (M, g_t).

Motivated by the work in [3], we will use a somewhat stronger notion than the injectivity radius. Let $S(t) = \{x \in M : dist(x, \partial M) = t\}$ be the geodesic t-sphere about ∂M and $B(t) = \{x \in M : dist(x, \partial M) < t\}$ the associated open geodesic ball about ∂M. For $t < inj_{\partial M}$, $S(t)$ is a $C^{m+1,\alpha}$ smooth hypersurface, diffeomorphic to ∂M, with $\partial B(t) = S(t) \cup \partial M$. (For $t > inj_{\partial M}$, $S(t)$ is no longer in general smooth).

Define the *outer-minimizing radius* of ∂M in $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$ by

(2.1) $$\mathcal{O}_{\partial M} = \sup\{\rho : area(\Sigma) > area(\partial M)\},$$

where Σ is any smooth surface in $B(\rho)$ homologous to ∂M, $\Sigma \neq \partial M$. Thus $\mathcal{O}_{\partial M}$ is the largest radius such that ∂M is outer-minimizing in $B(\mathcal{O}_{\partial M})$, i.e. ∂M has least area among all surfaces $\Sigma \subset B(\mathcal{O}_{\partial M})$ homologous to ∂M. Recall that the concept of outer-minimizing boundary plays an important role in [3].

By definition, $H > 0$ on ∂M for $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$. Hence, there is an $\varepsilon > 0$ (depending on (M, g, u)) such that $H(s) > 0$ for all $s < \varepsilon$, where $H(s)$ is the mean curvature of $S(s)$. Thus, $B(\varepsilon)$ has a foliation by surfaces of positive mean curvature. Since $H > 0$ corresponds to infinitesimal area expansion, it is easy to see (and well-known) that ∂M is outer-minimizing in $B(\varepsilon)$, so that

$$\mathcal{O}_{\partial M} \geq \varepsilon > 0.$$

Thus $\mathcal{O}_{\partial M}$ is a positive function on $\mathcal{E}_+^{m,\alpha}$.

It is also well-known that if the Riemann curvature Rm of (M, g) is uniformly bounded, $|Rm| \leq \Lambda$, then $\mathcal{O}_{\partial M} \geq d_0 > 0$ implies $inj_{\partial M} \geq d_1 > 0$, where d_1 depends only on d_0 and Λ; (for a proof see the beginning of the proof of [3, Theorem 4.1] for instance). The converse is also true, i.e. $inj_{\partial M} \geq d_1 > 0$ implies $\mathcal{O}_{\partial M} \geq d_0 > 0$, with $d_0 = d_0(d_1, \Lambda)$.

The following result from [3] shows that there is no degeneration in $\mathcal{E}_+^{m,\alpha}$ with uniform control on the boundary data (γ, H) and a uniform lower bound on $\mathcal{O}_{\partial M}$.

Proposition 2.1. *Let S be a set of static vacuum metrics $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$ for which the corresponding set of boundary data (γ, H) is a compact subset of $Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$ and for which there exists a constant $d_0 > 0$ such that for all $(M, g, u) \in S$,*

$$\mathcal{O}_{\partial M} \geq d_0 > 0.$$

Then S is compact in $\mathcal{E}_+^{m,\alpha}$.

Proof: This result is discussed following the statement of [3, Theorem 1.2] and the proof, which follows from the proof of [3, Theorem 4.3], is discussed in [3, Remark 4.4]. ∎

The basic idea in the proof of Theorem 1.2 is to build into the boundary data (γ, H) a term controlling the behavior of $\mathcal{O}_{\partial M}$. We do this by modifying the boundary map Π_B by a suitable term related to $\mathcal{O}_{\partial M}$. Of course this is no longer a "boundary map" in the strict sense, but as will be seen, it is sufficient to prove Theorem 1.2.

We first note the following:

Lemma 2.2. *The function*

$$\mathcal{O}_{\partial M} : \mathcal{E}_+^{m,\alpha} \to \mathbb{R}^+ \cup \{\infty\}$$

is lower semi-continuous.

Proof: Let (M_i, g_i, u_i) be a sequence in $\mathcal{E}_+^{m,\alpha}$ converging to a limit (M, g, u). Assume that

$$\mathcal{O}_0 \equiv \liminf \mathcal{O}_{\partial M}(g_i) < \infty,$$

(since otherwise there is nothing to prove). In each closed ball $\bar{B}_i(\mathcal{O}_{\partial M}(g_i))$ about ∂M in (M, g_i) there is a surface $\Sigma_i \neq \partial M$ such that

$$area_{g_i}(\Sigma_i) = area_{g_i}(\partial M).$$

Since the convergence to the limit (M, g, u) is sufficiently smooth (i.e. $C^{m,\alpha}$), there exists a limit surface $\Sigma \subset \bar{B}(\mathcal{O}_0) \subset (M, g)$ such that

$$area_g(\Sigma) = area_g(\partial M).$$

It follows that

$$\liminf \mathcal{O}_{\partial M}(g_i) \geq \mathcal{O}_{\partial M},$$

which proves lower semi-continuity. ∎

Let $\mathcal{V}^{m,\alpha} \subset \mathcal{E}_+^{m,\alpha}$ be the set of static vacuum solutions (M, g, u) such that ∂M is outer-minimizing in all of M, i.e.

$$\mathcal{V}^{m,\alpha} = \{(M, g, u) \in \mathcal{E}_+^{m,\alpha} : \mathcal{O}_{\partial M} = \infty\}.$$

We claim that $\mathcal{V}^{m,\alpha}$ is open in $\mathcal{E}_+^{m,\alpha}$. This follows from the fact that the solutions (M, g, u) are asymptotically flat, (and the convergence in $\mathcal{E}^{m,\alpha}$ preserves this structure). Namely, surfaces $\Sigma \subset M \setminus B(R)$ homologous to ∂M have very large area (for R sufficiently large) so that for all such surfaces

$$area(\Sigma) > area(\partial M).$$

In particular if $(M, g_i, u_i) \to (M, g, u) \in \mathcal{E}^{m,\alpha}$ and $\mathcal{O}_{\partial M}(M) = \infty$, then $\mathcal{O}_{\partial M}(g_i) = \infty$ for i sufficiently large.

The reciprocal function

(2.2)
$$\mathcal{I}_{\partial M} = \frac{1}{\mathcal{O}_{\partial M}} : \mathcal{E}_+^{m,\alpha} \to \mathbb{R}^+ \cup \{0\} \subset \mathbb{R}$$

is upper semi-continuous. (Recall f is upper semi-continuous if $\limsup_{x \to x_0} f(x) \leq f(x_0)$).

Now the space $\mathcal{E}^{m,\alpha}$ is a smooth separable Banach manifold locally modeled on the little Hölder spaces $C^{m,\alpha+}$ and has smooth partitions of unity, cf. [10], [8] and references therein. In particular $\mathcal{E}^{m,\alpha}$ is a perfectly normal topological space and by a classical theorem of Baire, any upper semi-continuous function on $\mathcal{E}_+^{m,\alpha}$ is then the pointwise limit of a decreasing sequence of continuous functions; see for instance [14] for the definition and discussion of such results. Thus let

$$\widehat{\mathcal{I}}_{\partial M} : \mathcal{E}_+^{m,\alpha} \to \mathbb{R}^+ \cup \{0\}$$

be a continuous approximation to $\frac{1}{\mathcal{O}_{\partial M}}$ with $\widehat{\mathcal{I}}_{\partial M} \geq \frac{1}{\mathcal{O}_{\partial M}}$.

Moreover, via smooth local partitions of unity, one may convolve $\widehat{\mathcal{I}}_{\partial M}$ with a smooth local mollifier to obtain a C^∞ smooth approximation $\widetilde{\mathcal{I}}_{\partial M}$ to $\widehat{\mathcal{I}}_{\partial M}$ which satisfies

(2.3)
$$\widetilde{\mathcal{I}}_{\partial M} \geq \frac{1}{2}\mathcal{I}_{\partial M}.$$

We choose the approximation and smoothing above so that

(2.4)
$$\widetilde{\mathcal{I}}_{\partial M} = 0 \quad \text{on} \quad \widetilde{\mathcal{V}}^{m,\alpha},$$

where $\widetilde{\mathcal{V}}^{m,\alpha}$ is a proper open subset of $\mathcal{V}^{m,\alpha}$ containing the exterior Schwarzschild metrics (1.3). The Schwarzschild metrics have $\mathcal{O}_{\partial M} = \infty$ so $\mathcal{I}_{\partial M} = 0$.

Next the function $H \to \min H$ is a continuous function on $C_+^{m-1,\alpha}(\partial M)$, with values in $\mathbb{R}^+$. The ratio

$$\frac{H}{\min H} : C_+^{m-1,\alpha}(\partial M) \to \mathbb{R}^+,$$

is thus also continuous. Analogous to the above, let $minH$ be a smooth approximation to $\min H$. The ratio

$$\frac{H}{minH} : C_+^{m-1,\alpha}(\partial M) \to \mathbb{R}^+,$$

is thus smooth.

Given this background, consider now the map

(2.5)
$$\widetilde{\Pi} : \mathcal{E}^{m,\alpha} \to Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2),$$

$$\widetilde{\Pi}(g,u) = (\gamma, \widetilde{H}),$$

where

(2.6)
$$\widetilde{H} = H(1 + \frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M}) = H + \frac{H}{minH}\widetilde{\mathcal{I}}_{\partial M}.$$

Thus we are altering the boundary map Π_B by a scaling factor $\lambda = (1 + \frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})$ on H. This choice of scaling factor is not unique or canonical any sense; it is a useful or convenient choice. While (γ, H) depend only on the boundary data, the term $\widetilde{\mathcal{I}}_{\partial M}$ approximating $\frac{1}{\mathcal{O}_{\partial M}}$ depends on the global geometry of $(M, g, u) \in \mathcal{E}^{m,\alpha}$.

Observe that

(2.7)
$$\widetilde{\Pi} = \Pi_B$$

in domains of $\mathcal{E}_+^{m,\alpha}$ where $\widetilde{\mathcal{I}}_{\partial M} = 0$, i.e. on the domain $\widetilde{\mathcal{V}}^{m,\alpha}$ defined in (2.4). As noted above, the standard exterior $\{r \geq R\}$ regions of the Schwarzschild metric are in $\widetilde{\mathcal{V}}^{m,\alpha}$. Note also that the family of maps

$$\widetilde{\Pi}_\delta(g,u) = (\gamma, \widetilde{H}_\delta),$$

where

$$\widetilde{H}_\delta = H(1 + \frac{\delta}{minH}\widetilde{\mathcal{I}}_{\partial M}),$$

interpolates continuously between Π_B and $\widetilde{\Pi}$ as δ ranges over the interval $[0,1]$.

Proposition 2.3. *The map $\widetilde{\Pi}$ is a smooth Fredholm map of Banach spaces, of Fredholm index 0.*

Proof: The map $\widetilde{\Pi}$ is a smooth map between Banach manifolds, by construction. To prove it is Fredholm, recall from the Introduction that the boundary map Π_B is smooth and Fredholm, (as proved in [1], [3]). The derivatives of $\widetilde{\Pi}$ and Π_B are closely related. One has

(2.8)
$$D\widetilde{\Pi}(h,u') = (h^T, H_h'(1 + \frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})) + (0, H(\frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})').$$

Here h^T is the restriction of the metric variation h of g to ∂M, while H_h' is the induced variation of the mean curvature. The first term in (2.8), denoted D_1, differs from $D\Pi_B$ by a single fixed scale factor $(1 + \frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})$ and since $D\Pi_B$ is Fredholm, it is clear that D_1 is also a Fredholm map of Banach spaces. For the second term in (2.8), denoted D_2, the derivative (at any given point $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$) of the smooth function $(\frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})'$ is a bounded linear functional of (h, u') and so $H(\frac{1}{minH}\widetilde{\mathcal{I}}_{\partial M})'$ is a compact operator (in fact an operator of rank at most 1). Since the sum of a Fredholm operator and a compact operator is Fredholm, this proves that $\widetilde{\Pi}$ is Fredholm.

Exactly the same argument shows that the interpolating maps $\widetilde{\Pi}_\delta$ are also smooth and Fredholm. It is well-known that the Fredholm index (cf. (1.5)) is deformation invariant, so that the Fredholm index of $D\widetilde{\Pi}$ is independent of the point $(M, g, u) \in \mathcal{E}^{m,\alpha}$ at which the linearization is taken, in any component of $\mathcal{E}_+^{m,\alpha}$. Similarly, the Fredholm index of $\widetilde{D}\Pi_\delta$ (at any base point) is independent of δ. Since the Fredholm index of $D\Pi_B$ is zero (everywhere) it follows that the index of $D\widetilde{\Pi}$ is also zero everywhere.

The key to obtaining global information from a Fredholm map is to prove that the map is proper. Recall that a continuous map $F : X \to Y$ is called *proper* if, for any compact set $K \subset Y$, the inverse image $F^{-1}(Y)$ is compact in X. In the context of the map $\widetilde{\Pi}$, this amounts to showing that control of the boundary data $(\gamma, \widetilde{H})$ (i.e. data in a compact set of the target space) implies control of corresponding solutions (M, g, u) realizing such boundary data.

The examples of boundary degeneration discussed following (1.6) show that Π_B is not proper.

Theorem 2.4. *The map $\widetilde{\Pi}$ is proper.*

Proof: Let (M_i, g_i, u_i) be a sequence in $\mathcal{E}_+^{m,\alpha}$ such that $\widetilde{\Pi}(M_i, g_i, u_i)$ converges to a limit in $Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$, so that

$$(2.9) \qquad \gamma_i \to \gamma_\infty, \ \ \widetilde{H}_i \to \widetilde{H}_\infty,$$

in the $C^{m,\alpha}$ and $C^{m-1,\alpha}$ topologies respectively. We need to prove that a subsequence of (M_i, g_i, u_i) converges to a limit in $\mathcal{E}_+^{m,\alpha}$. The basic idea is to use Proposition 2.1. For this we show that (2.9) implies that the hypotheses of Proposition 2.1 hold. (Of course S is taken to be the set $\{(M, g_i, u_i)\}$).

We first claim there is a constant $C_0 < \infty$ such that (on the sequence (M, g_i, u_i))

$$(2.10) \qquad \widetilde{\mathcal{I}}_{\partial M} \leq C_0 < \infty.$$

Namely, since $H > 0$ in $\mathcal{E}_+^{m,\alpha}$ one has

$$\widetilde{H} > \frac{H}{\min H} \widetilde{\mathcal{I}}_{\partial M} \geq \widetilde{\mathcal{I}}_{\partial M},$$

since $\frac{H}{\min H} \geq 1$. Thus (2.9) implies (2.10). Via (2.3), the bound (2.10) implies that $\mathcal{O}_{\partial M} \geq c_0 > 0$, for some constant $c_0 > 0$.

Next we show that (2.9) implies

$$(2.11) \qquad H_i \to H_\infty \ \ \text{and} \ \ H_\infty > 0,$$

so that $H_i \to H_\infty \in C_+^{m-1,\alpha}(S^2)$.

To do this, we first prove that $\min H_i$ is bounded away from 0; there exists $h_0 > 0$ such that

$$(2.12) \qquad H_i \geq h_0 > 0.$$

For if not, then

$$(2.13) \qquad \min H_i \to 0$$

(in a subsequence). Choose points x_i such that $H(x_i) = \min_{x \in \partial M} H_i(x)$. Then $\widetilde{H}(x_i) = H(x_i) + \widetilde{\mathcal{I}}_{\partial M}$ and since $\widetilde{H}(x_i)$ is bounded away from 0 by (2.9), it follows that there exists a constant $c_0 > 0$ such that

$$(2.14) \qquad c_0 \leq \widetilde{\mathcal{I}}_{\partial M} \leq C_0.$$

It then follows from (2.14) and (2.9) that the function

$$\frac{H}{\min H} + H$$

is uniformly bounded away from 0 and ∞ on the sequence (M, g_i, u_i). Since by assumption $\min H_i \to 0$, this implies that

$$H_i \to 0$$

uniformly on ∂M. (The geometry of ∂M is uniformly controlled also by (2.9)). By (2.10) (i.e. $\mathcal{O}_{\partial M} \geq c_0 > 0$) and Remark 4.4 of [3], a subsequence of (M, g_i, u_i) then converges in $\mathcal{E}^{m,\alpha}$ to a limit $(M, g, u) \in \mathcal{E}^{m,\alpha}$ with boundary data $(\gamma, H) = (\gamma_\infty, 0)$. (This is the extension of Proposition

8

2.1 to the closure $\bar{\mathcal{E}}_+^{m,\alpha}$). A result of Miao, [13], which is a variant of the black hole uniqueness theorem, implies that (M, g, u) is the Schwarzschild metric (1.3) for some mass m. In particular, if γ_∞ in (2.9) is not a round metric of some fixed radius, then one already has a contradiction to (2.13). The Schwarzschild metric has $\mathcal{O}_{\partial M} = \infty$ and is in the space $\widetilde{\mathcal{V}}^{m,\alpha}$ where $\widetilde{\mathcal{I}}_{\partial M} = 0$. Since $\widetilde{\mathcal{V}}^{m,\alpha}$ is open, $(M, g_i, u_i) \in \widetilde{\mathcal{V}}^{m,\alpha}$ for i sufficiently large. This contradicts (2.14) which then establishes (2.12).

Thus (2.9) implies that $\min H_i$ is uniformly bounded away from 0 and ∞ and $\widetilde{\mathcal{I}}_{\partial M}$ is uniformly bounded above on the sequence (M, g_i, u_i). From this, it follows trivially from (2.9) that $H_i \to H > 0$ in $C_+^{m-1,\alpha}(S^2)$ (in a subsequence). This proves (2.11).

The result then follows from Proposition 2.1.

∎

A proper Fredholm map F of index 0 between connected and separable Banach manifolds has a $\mathbb{Z}_2$-valued degree, the Smale degree $deg_{\mathbb{Z}_2} F$, given by

$$(2.15) \qquad deg_{\mathbb{Z}_2} F = \#\{F^{-1}(y)\} \mod 2,$$

where y is a regular value of F, i.e. for any $x \in F^{-1}(y)$ the linearization $(DF)_x$ is surjective. Here $\#$ denotes the cardinality; the set $\{F^{-1}(y)\}$ is a finite set of points, by the inverse function theorem and properness assumption. Of course the degree is independent of the choice of regular value y. We refer to [17] (or also [3] here) for further detail.

Apriori, the separable Banach manifold $\mathcal{E}_+^{m,\alpha}$ may not be connected. Let then $\mathcal{E}_{+,0}^{m,\alpha}$ be the component of $\mathcal{E}_+^{m,\alpha}$ containing the exterior Schwarzschild metrics as described following (1.3). This is the same as the component containing the flat static vacuum solutions $(M, g_{Eucl}, 1)$ where M the the region exterior to a smooth compact convex subset $B \subset \mathbb{R}^3$.

The map $\widetilde{\Pi}$ restricts to a map

$$(2.16) \qquad \widetilde{\Pi} : \mathcal{E}_{+,0}^{m,\alpha} \to Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2).$$

By Proposition 2.3 and Theorem 2.4, $\widetilde{\Pi}$ is a proper Fredholm map of index 0 between connected and separable Banach manifolds. Hence

$$deg_{\mathbb{Z}_2} \widetilde{\Pi} \in \mathbb{Z}_2,$$

is well-defined.

Proposition 2.5. *For the map $\widetilde{\Pi}$ in (2.16), one has*

$$(2.17) \qquad deg_{\mathbb{Z}_2} \widetilde{\Pi} = 1.$$

In particular $\widetilde{\Pi}$ is surjective onto $Met^{m,\alpha}(S^2) \times C_+^{m,\alpha}(S^2)$.

Proof: The proof is a simple modification of the proof of Theorem 5.1 of [3]. On the curve of Schwarzschild metrics (1.3) with ∂M taken at a locus with r fixed, one has

$$(\gamma, H) = (\gamma_{S^2(r)}, \frac{2}{r}\sqrt{1 - \frac{2m}{r}}).$$

Thus any constant curvature metric $\gamma_{S^2(r)}$, and any constant $H_0 \in (0, \infty)$ are realized in $Im\Pi_B$. Since $\mathcal{I}_{\partial M} = 0$ on the Schwarzschild curve (so that (2.7) holds) the same boundary data satisfy $(\gamma_{S^2(r)}, H_0) \in Im(\widetilde{\Pi})$.

Now fix $r = 1$ and conversely consider boundary data as above $(\gamma_{S^2(1)}, t_i)$ for $\widetilde{\Pi}$, with $\widetilde{H}_{t_i} = t_i \to 0$ as $t_i \to 0$. Let (M, g_i, u_i) be any sequence of points in $(\widetilde{\Pi})^{-1}(\gamma_{S^2(1)}, t_i)$. On such a sequence one has

$$H(1 + \frac{1}{\min H}\widetilde{\mathcal{I}}_{\partial M}) = t_i \to 0,$$

9

which implies that

$$H = const \to 0 \quad \text{and} \quad \mathcal{I}_{\partial M} \to 0.$$

It then follows then as above in the proof of Theorem 2.4, namely from [3, Remark 4.4] (see also the beginning of the proof of [3, Theorem 5.1]) that (M, g_i, u_i) converges again to the Schwarzschild metric of mass $m = \frac{1}{2}$.

Now suppose

$$(2.18) \qquad\qquad\qquad\qquad deg_{\mathbb{Z}_2}\widetilde{\Pi} = 0.$$

By the Sard-Smale theorem [17], regular values of $\widetilde{\Pi}$ are open and dense. Choose a sequence of regular values (γ_i, H_i) in $Met^{m,\alpha}(S^2) \times C^{m-1,\alpha}_+(S^2)$ converging to $(\gamma_{S^2(1)}, 0)$. For any i, the finite set $(\widetilde{\Pi})^{-1}(\gamma_i, H_i)$ (if non-empty) consists of at least two distinct static vacuum solutions (M, g_i^1, u_i^1), (M, g_i^2, u_i^2). As above, it follows that both sequences converge to the Schwarzschild metric (1.3) with $m = \frac{1}{2}$.

Thus, as discussed following (2.4) (and in the proof of Theorem 2.4 above),

$$\widetilde{\mathcal{I}}_{\partial M}(g_i^j) = 0,$$

for $j = 1, 2$ and i sufficiently large, so that $\widetilde{\Pi} = \Pi_B$ at and near each g_i^j. The rest of the proof of [3, Theorem 5.1] now applies verbatim to give a contradiction to (2.18), which establishes (2.17).

It is standard that any map of degree one is necessarily surjective. (Any point not in the image of F is a regular value of F).

$\blacksquare$

Proposition 2.5 leads trivially to the proof of Theorem 1.2.

Proof of Theorem 1.2.
Let $\pi : Met^{m,\alpha}(S^2) \times C^{m-1,\alpha}_+(S^2) \to Met^{m,\alpha}(S^2) \times \mathcal{C}^{m-1,\alpha}_+(S^2)$ be the projection to the orbit space of the $\mathbb{R}^+$ action as following (1.7). The map π is clearly surjective, and hence by Proposition 2.5, $\pi \circ \widetilde{\Pi} : \mathcal{E}^{m,\alpha}_+ \to Met^{m,\alpha}(S^2) \times \mathcal{C}^{m-1,\alpha}_+(S^2)$ is also surjective. Since

$$(2.19) \qquad\qquad\qquad\qquad \Pi = \pi \circ \widetilde{\Pi},$$

this proves the result.

$\blacksquare$

Proof of Theorem 1.1.
Let $(\gamma, H) \in Met^\infty(S^2) \times C^\infty_+(S^2)$ be smooth boundary data. Then of course (γ, H) is in the little Hölder space $Met^{m,\alpha}(S^2) \times C^{m,\alpha}_+(S^2)$, for any m, α. By Theorem 1.2, there exists a static vacuum solution $(M, g, u) \in \mathcal{E}^{m,\alpha}_{+,0} \subset \mathcal{E}^{m,\alpha}_+$ realizing the boundary data $(\gamma, [H])$, so the mean curvature of (M, g, u) at ∂M is given by λH for some $\lambda > 0$. The data (g, u) are $C^{m,\alpha}$ up to ∂M. Recall from the Introduction that the static vacuum Einstein equations with boundary data (γ, H) form an elliptic boundary value problem (in a suitable gauge). By elliptic boundary regularity, it follows that (g, u) are C^∞ up to ∂M, so that $(M, g, u) \in \mathcal{E}^\infty_{+,0}$. This completes the proof.

$\blacksquare$

3. Discussion

Consider the map Π in (1.8). By the Sard-Smale theorem [17], regular values are open and dense in the target space $Met^{m,\alpha}(S^2) \times \mathcal{C}^{m-1,\alpha}_+(S^2)$; as in (2.15), y is a regular value of F if $(DF)_x$ is surjective, for each $x \in F^{-1}(y)$.

Let $(\gamma, [H])$ be any regular value of Π. Since $\widetilde{\Pi}$ is proper and π as in (2.19) has 1-dimensional fibers, it follows that

(3.1) $$\Pi^{-1}(\gamma, [H]),$$

is a finite collection of smooth curves in $\mathcal{E}_+^{m,\alpha}$. This corresponds to (1.9), that the Fredholm index of Π in (1.8) is one. Let (M_t, g_t, u_t) denote any one such curve. The induced boundary data are (γ, H_t) with $H_t = \lambda_t H$, for some fixed function H, so that $\lambda_t \in \mathbb{R}^+$ depends smoothly on t.

Question: Is λ_t surjective onto $\mathbb{R}^+$?

This is the question of whether Π_B in (1.4) is surjective onto the $\mathbb{R}^+$ orbits of H; it is surjective onto the orbit space by Theorem 1.2. If this is the case (which would be surprising in the author's opinion) then of course the Bartnik boundary map Π_B is essentially surjective; it has at least open and dense range. However, as discussed in the Introduction, one does not expect Π_B to be injective or have a well-behaved inverse. Consider for example the curve of metrics $(M_t, (g_{Eucl})_t, 1)$ in $\mathbb{R}^3$ where ∂M_t converges to an immersed sphere of positive mean curvature in $\mathbb{R}^3$ as $t \to 0$. The limit configuration $(M_0, (g_{Eucl})_0, 1)$ makes sense, and has a well-defined boundary value $(\gamma_0, H_0) \in Met^{m,\alpha}(S^2) \times C_+^{m-1,\alpha}(S^2)$, but $(M_0, (g_{Eucl})_0, 1)$ is not a solution in the space $\mathcal{E}_+^{m,\alpha}$. On the other hand, Theorem 1.2 implies there is a $\lambda > 0$ such that $(\gamma_0, \lambda H_0)$ is realized as boundary data of a solution $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$.

Next, an immediate consequence of Theorem 1.2 is that any $C^{m,\alpha+}$ metric γ on S^2 may be extended to a smooth asymptotically flat metric of zero scalar curvature on M, with mean curvature $H > 0$ on ∂M prescribed up to a multiplicative constant. Modulo the scalar indeterminacy on H, this extends previous work of Shi-Tam [16] who showed that boundary data (γ, H) can be extended to scalar-flat asymptotically flat metrics provided the Gauss curvature $K_\gamma > 0$. The work of Shi-Tam is based on the method of Bartnik [6] on constructing asymptotically flat metrics of prescribed (non-negative) scalar curvature with given boundary data (γ, H).

Finally we discuss issues related to the mass of solutions in $\mathcal{E}_+^{m,\alpha}$. Let m denote the ADM mass of a static vacuum solution (M, g, u). As is well-known, m is given as a Komar-type integral (mass),

$$m = \frac{1}{4\pi} \int_{\partial M} N(u) dv_\gamma.$$

In particular, the mass

$$m : \mathcal{E}_+^{m,\alpha} \to \mathbb{R}$$

is a smooth function on $\mathcal{E}_+^{m,\alpha}$. Clearly the space $\mathcal{E}_+^{m,\alpha}$ contains solutions with both positive and negative mass (e.g. the Schwarzschild curve). Thus $\mathcal{H}_0 = \{m = 0\}$ is a hypersurface in $\mathcal{E}_+^{m,\alpha}$; it seems likely that $\mathcal{H}_0$ is a smooth hypersurface, but this remains to be proved.

Let (M_t, g_t, u_t) be a smooth curve component of $\Pi^{-1}(\gamma, [H])$ as in (3.1), so that m_t is a smooth function of t.

Question: Is λ_t a function of m_t or vice versa? Is λ_t monotone in m_t?

Note this is the case for the curve of Schwarzschild metrics g_m described following (1.3); here λ_t takes all values in $(0, \infty)$ and is monotone decreasing as a function of $m_t \in (-\infty, \frac{1}{2})$.

Question: Are there criteria on (γ, H) which guarantee that $m > 0$ or $m \geq 0$?

This question is motivated by the original context of the Bartnik conjecture and the definition of the Bartnik quasi-local mass [4], [5], which we briefly recall. Let $(\widehat{M}, \widehat{g})$ be a complete asymptotically flat manifold of non-negative scalar curvature and Ω a bounded smooth domain in $\widehat{M}$ with $H > 0$ at $\partial\Omega$. Let $M = \widehat{M} \setminus \Omega$, assumed to be diffeomorphic to $\mathbb{R}^3 \setminus B$. Then $\partial M = \partial\Omega \simeq S^2$ with induced Bartnik boundary data (γ, H). A special case of the Bartnik conjecture is that (γ, H) are boundary data of a unique solution $(M, g, u) \in \mathcal{E}_+^{m,\alpha}$. By the results of [16] and the positive mass

theorem, $m \geq 0$ on (M, g, u). In this context, one would like to characterize or understand the space of solutions in $\mathcal{E}_+^{m,\alpha}$ which have $m > 0$ or $m \geq 0$ in terms of the boundary data (γ, H).

A result of Jauregui [11] shows that, given such data (γ, H) on S^2, and assuming $K_\gamma > 0$, there is a maximal value $\lambda_0 < \infty$ (depending on (γ, H)) such that $(\gamma, \lambda H)$ has an in-filling 3-ball B^3 with a metric of non-negative scalar curvature for all $\lambda < \lambda_0$, but no such in-filling exists for $\lambda > \lambda_0$. See also [12] for a refinement of this result.

It would be interesting to know if such a criterion holds for the mass, e.g. whether $m \geq 0$ if any only if $\lambda \leq \lambda_0$, for some λ_0. This is true on the Schwarzschild curve, where $m \geq 0$ if and only if $H \leq 2$ for $\gamma = \gamma_{S^2(1)}$.

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