1. Some answers to problems from §3.1

3) The equilibrium position of a string is clearly that when the string is at rest and its vertical displacement is trivial, that is \( u(x, t) = 0 \). This is just common sense. That is reflected in the answer to this problem.

For if \( \nu(x) \) is a solution of the wave equation with homogeneous boundary conditions, then we must have

\[
\begin{align*}
\ddot{\nu}(x) & = 0, \quad 0 < x < a, \\
\nu(0) & = 0, \\
\nu(a) & = 0.
\end{align*}
\]

The general solution to the ode is given by \( \nu(x) = c_1 x + c_2 \), for arbitrary constants \( c_1 \), \( c_2 \). If we search for the solution that vanishes at 0 and \( a \), we must have \( 0 = c_1 \cdot 0 + c_2 \) and \( 0 = c_1 a + c_2 \). Thus, \( c_1 = c_2 = 0 \), and \( \nu(x) \equiv 0 \) is the equilibrium solution (as common sense indicates).

2. Some answers to problems from §3.2

2) The general solution with arbitrary initial conditions \( f \) and \( g \) is given in the textbook as

\[
u(x, t) = \sum_{n=1}^{\infty} \sin \left( \lambda_n x \right) \left( a_n \cos \left( \lambda_n ct \right) + b_n \sin \left( \lambda_n ct \right) \right)
\]

where

\[
a_n = \frac{2}{a} \int_0^a f(x) \sin \left( \lambda_n x \right) \, dx,
\]

and

\[
b_n \lambda_n = \frac{2}{a} \int_0^a g(x) \sin \left( \lambda_n x \right) \, dx.
\]

Here \( \lambda_n = n \pi / a \).

In the specific case we are asked to analyze, \( g(x) = 0 \). Therefore, the integral above for \( b_n \lambda_n \) is zero for all \( n \), and so \( b_n = 0 \). On the other hand, \( f(x) = \sin \left( \pi x / a \right) = \sin \left( \lambda_1 x \right) \) and we have

\[
a_n = \frac{2}{a} \int_0^a \sin \left( \lambda_1 x \right) \sin \left( \lambda_n x \right) \, dx.
\]

We have seen that \( \sin \left( \lambda_1 x \right) \) and \( \sin \left( \lambda_n x \right) \) are orthogonal to each other for \( n \neq 1 \). Therefore, for all \( n > 1 \) we have \( a_n = 0 \), and so, only the \( a_1 \) survives in the series above. Since

\[
a_1 = \frac{2}{a} \int_0^a \sin \left( \lambda_1 x \right) \sin \left( \lambda_1 x \right) = \frac{2}{a} \int_0^a \frac{1 - \cos \left( 2 \lambda_1 x \right)}{2} \, dx = 1,
\]

we obtain that

\[
u(x, t) = \sin \left( \frac{\pi x}{a} \right) \cos \left( \frac{\pi ct}{a} \right) + \frac{1}{2} \sin \left( \frac{\pi (x - ct)}{a} \right) + \frac{1}{2} \sin \left( \frac{\pi (x + ct)}{a} \right),
\]

the sum of two waves, one moving to the right and the other moving to the left.

5a) We begin b writing the solution as

\[
p(x, t) = p_0(x) + \bar{p}(x, t),
\]
where \( p_0(x) \) also satisfies the wave equation and has boundary conditions \( p_0 \) at both, \( x = 0 \) and \( x = a \). This implies that \( p_0(x) = p_0 \), a constant, and we have

\[ p(x, t) = p_0 + \tilde{p}(x, t). \]

Since \( p(x, t) \) is assumed to be a solution to the wave equation with boundary condition \( p_0 \) at both ends, the function \( \tilde{p}(x, t) \) must satisfy the equation

\[ \frac{\partial^2 \tilde{p}}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \tilde{p}}{\partial t^2}, \]

and its boundary conditions are

\[ \tilde{p}(0, t) = 0, \quad \tilde{p}(a, t) = 0. \]

We apply the method of separation of variables in order to find the associated eigenvalue problem, and the corresponding eigenvalues and eigenfunctions, that is what the question is asking for.

Hence, suppose \( \tilde{p}(x, t) = \varphi(x)T(t) \) is a solution of the problem above. Therefore,

\[ \ddot{\varphi}(x)T(t) = \frac{1}{c^2} \varphi(x)\dddot{T}(t), \]

and separating the variables \( x \) and \( t \), we obtain that

\[ \frac{\ddot{\varphi}(x)}{\varphi(x)} = -\lambda^2 = \frac{\dddot{T}(t)}{c^2T(t)}, \]

for some constant \( \lambda \). Since we want \( \varphi(x)T(t) \) to vanish at \( x = 0 \) and \( x = a \), this leads to the eigenvalue problem

\[ \begin{aligned} &\ddot{\varphi} + \lambda^2 \varphi = 0, \\ &\varphi(0) = 0, \\ &\varphi(a) = 0, \end{aligned} \]

(1)

while \( T(t) \) must be a solution of the equation

\[ \dddot{T} + \lambda^2 c^2 T = 0. \]

The most general solution of the differential equation in (1) is

\[ \varphi(x) = c_1 \cos (\lambda x) + c_2 \sin (\lambda x). \]

Since \( \varphi(0) = 0 \), this forces \( c_1 \) to be zero. Hence, we must only consider solutions of the form \( \varphi(x) = c_2 \sin (\lambda x) \). Now, since \( \varphi(a) = c_2 \sin (\lambda a) \) must be zero, we obtain that

\[ \sin (\lambda a) = 0, \]

and therefore, \( \lambda a \) is forced to be a non-trivial multiple of \( \pi \): Thus, the eigenvalues of our problem are

\[ \lambda_n = \frac{n\pi}{a}, \quad n = 1, 2, 3, \ldots \]

The corresponding eigenfunctions, the solutions to (1) associated with \( \lambda_n \), is given by

\[ \varphi(x) = \varphi_n(x) = \sin \left( \frac{n\pi x}{a} \right). \]
12) We have a function $u(x, t)$ defined by

$$u(x, t) = \psi(x + ct) + \phi(x - ct),$$

where $\psi(s)$ and $\phi(s)$ are two functions of one real variable, each one of which has at least two derivatives.

By the chain rule, we have that

$$\frac{\partial u}{\partial x} = \psi(x + ct) \frac{\partial x + ct}{\partial x} + \phi(x - ct) \frac{\partial x - ct}{\partial x} = \dot{\psi}(x + ct) + \dot{\phi}(x - ct),$$

and

$$\frac{\partial u}{\partial t} = \psi(x + ct) \frac{\partial x + ct}{\partial t} + \phi(x - ct) \frac{\partial x - ct}{\partial t} = c \dot{\psi}(x + ct) - c \dot{\phi}(x - ct).$$

We apply the chain rule once again to compute the second partial derivatives of $u$. For example, we have that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t} \left( c \dot{\psi}(x + ct) - c \dot{\phi}(x - ct) \right) = c(\ddot{\psi}(x + ct) - (-c)\ddot{\phi}(x - ct)) = c^2(\ddot{\psi}(x + ct) + \ddot{\phi}(x - ct)).$$

Similarly,

$$\frac{\partial^2 u}{\partial x^2} = \ddot{\psi}(x + ct) + \ddot{\phi}(x - ct).$$

Comparing these last two results, we conclude that

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2},$$

as desired.

Notice that for these computations to make sense, the functions $\psi$ and $\phi$ have to be differentiable twice. That is how the hypothesis on these functions is used.