HOMEWORK 3, MAT 568, FALL 06

Due: Thursday, Dec 7.

0. Think over topics or general areas in geometry you would like to see discussed next Semester in MAT 569. We'll discuss this soon in class.

1. A well-known theorem of Nash states that any say compact Riemannian manifold (M, g) isometrically embeds in Euclidean space, i.e. there is an embedding

$$F : M \to \mathbb{R}^N$$

such that $F^*(g_{Eucl}) = g$. The dimension N depends only on $dim M$.

Let g_0 be the flat product metric on the torus $T^2 = S^1 \times S^1$. Show that there exists an isometric embedding

$$F : T^2 \to \mathbb{R}^4.$$

Prove that there is no isometric embedding $f : T^2 \to \mathbb{R}^3$. Hint: Assuming there is such, look at the behavior of the 2nd fundamental form A of $T^2 \subset \mathbb{R}^3$ at a point farthest away from some given point, say 0 in $\mathbb{R}^3$.

2. Let M be a compact manifold, (for simplicity). A sequence of metrics g_i is said to *collapse* if

$$inj_{g_i}(x) \to 0, \quad \text{as} \quad i \to \infty,$$

for all $x \in M$. The sequence is said to *collapse with bounded curvature* if it collapses and there is a constant $\Lambda < \infty$ such that

$$|R|_{g_i}(x) \leq \Lambda,$$

where $|R|_{g_i}$ is the norm of the curvature tensor of g_i; equivalently, all of the sectional curvatures at all points in (M, g_i) are uniformly bounded.

(a). Show that any compact manifold has a collapsing sequence of metrics.

(b). On $N = S^1 \times M$, consider the family of metrics

$$g_\varepsilon = \varepsilon^2 d\theta^2 + g_M,$$

where g_M is any fixed metric on M.

Prove that g_ε collapses with bounded curvature as $\varepsilon \to 0$.

(c). Replace S^1 in (b) above by S^2 or S^k, for any $k > 1$. Prove that then g_ε collapses, but does not collapse with bounded curvature.

(d). Using (c) as a hint, prove that any manifold of the form $N = S^k \times M$, for M arbitrary, (compact), admits metrics of positive scalar curvature.

3. As above, let M be a compact manifold. A sequence of metrics g_i is said to *volumecollapse* if

$$vol(M, g_i) \to 0, \quad \text{as} \quad i \to \infty,$$

and to volume collapse with bounded curvature if the statement above holds and the curvature of g_i remains uniformly bounded, as in 2. above.

Assume the Gauss-Bonnet theorem for compact oriented surfaces (Σ, g):

$$\int_\Sigma K dA = 2\pi\chi(\Sigma).$$

Prove that the only surface which volume collapses with bounded curvature, (i.e. admits a sequence of metrics which collapse with bounded curvature), is the torus. (We assume also you know the classification of surfaces).

Your proof, if correct, will generalize to higher dimensions, using analogues of Gauss-Bonnet in higher dimensions, (e.g. using curvature expressions for characteristic classes of TM).

In fact, its true that the only compact oriented surface which collapses with bounded curvature is the torus. This is harder to prove.

Double Extra Credit if you can do this, or have some good ideas how to do it!!

4. Let (M, g) be a complete Riemannian manifold and let

$$inj(M) = inf_{x \in M} inj(x).$$

Let $\ell(\gamma)$ denote the length of the shortest closed geodesic in (M, g). Show that

$$inj(M) \le \frac{1}{2}\ell(\gamma).$$

What does this statement say if there are no closed geodesics in (M, g). Can you give an example of (M, g) with no closed geodesics?