# FINAL EXAM, MAT 362 SPRING 05

This is an open book exam, based on the honor system. You can use any books, lecture notes, etc. to assist you in solving the problems. However, you cannot talk or discuss any issues related to the exam with someone else. The exam should reflect completely your own understanding.

All problems are worth 20pts, even though some are harder or longer than others.

The exam is due:

Tuesday, May 9, 3pm.

Please put your exam in the box by my office door: Math Tower 4-100A, (on the bridge to the Physics Tower - toward the Math-Physics Library).

If you have any questions, you can e-mail me at: andersonmath.sunysb.edu or call at 689-3406.

1. Let $(\phi, \theta)$ be spherical coordinates for the unit sphere $S^2(1)$ in $\mathbb{R}^3$. So one has the usual spherical coordinate or latitude/longitude chart:

$$\sigma(\phi, \theta) = (\cos\theta \cos\phi, \cos\theta \sin\phi, \sin\theta).$$

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, given by

$$f(x, y, z) = (x - y)^2 + z^2,$$

restricted to the surface $S^2(1)$. Find the local form of the function $f$ in the spherical coordinate chart.

2. Let $S$ be a surface of revolution, so that in the natural coordinates, the first fundamental form has the form

$$dr^2 + f^2(r)d\theta^2,$$

where $\theta$ runs over the unit circle, ($\theta \in [0, 2\pi]$, with 0 identified with $2\pi$).

Show that rotation by any angle $\theta_0$ is an isometry of $S$. Thus, define the map

$$F(r, \theta) = (r, \theta + \theta_0).$$

Show that for any $\theta_0 \in [0, 2\pi]$, the map $F$ is an isometry of $S$.

3. Recall that the coefficients of the first and second fundamental forms of a surface $S$ in a given chart are called $E, F, G$ and $L, M, N$ respectively.

Prove that there is no surface $S$ in $\mathbb{R}^3$ which has a chart for which these coefficients have the form:

$$E = G = 1, F = 0 \text{ and } L = 1, M = 0, N = -1.$$

You may *not* use the Codazzi-Mainardi equations for this, which we did not discuss in the course (although they are discussed in the text).

4. Let $S$ be a smooth, compact and oriented surface in $\mathbb{R}^3$. Then $S$ divides $\mathbb{R}^3$ into an inside bounded solid region, and an outside, unbounded solid region. The surface $S$ is called *convex* if, for any two points $p$ and $q$ on $S$, the line segment $\overline{pq}$ joining them lies inside $S$, i.e. in the bounded region. In particular $\overline{pq}$ intersects $S$ only at the points $p$ and $q$; this for any two points on the surface.

Prove that any compact convex surface satisfies

$$K(x) > 0,$$

for all $x \in S$.

5. Suppose a part of a surface $S$ is described the graph of a function $f$, $f : \mathbb{R}^2 \to \mathbb{R}$. So one has a chart for $S$ of the form

$$\sigma(u, v) = (u, v, f(u, v)).$$

Find the following quantities in terms of $f$ and its derivatives.

(a). The vectors $\sigma_u$, $\sigma_v$ forming a basis for the tangent spaces to $S$.
(b). A unit normal vector $N$ to $S$.
(c). Now find the formulas for the 1st and 2nd fundamental forms of $S$ in this chart.
(d). Finally, compute the Gauss curvature of the surface $S$ in the region covered by the chart.

6. Find the total Gauss curvature

$$\int_S K dA,$$

where $S$ is the surface in $\mathbb{R}^3$ given by the equation

$$x^4 + y^6 + z^2 = 1.$$

(Hint: what "standard" surface is $S$ homeomorphic to?).

7. The surface obtained by rotating the curve

$$y = \cosh x,$$

about the $x$-axis is called the catenoid.

(a). Find the Gauss curvature of the catenoid.
(b). Now find the area element $dA$ and compute, from (a), the total Gauss curvature

$$\int K dA.$$

(You may use the fact that the antiderivative (integral) of $1/\cosh^2$ is tanh.

(c). Describe the image of the Gauss map of the catenoid on the sphere $S^2(1)$, and explain how you can recover your result from (b) almost directly, without computation.

(d). Prove (or at least show why) the catenoid is diffeomorphic to the cylinder, obtained by rotating the line $y = 1$ about the $x$-axis.

(e). Determine, by any method, the total Gauss curvature, i.e.

$$\int K dA$$

for the cylinder.

(f). Show that the Euler characteristic of the cylinder is 0. You can use the fact that a finite interval $I$ is diffeomorphic to the whole line $\mathbb{R}$, so the cylinder is diffeomorphic to $I \times S^1$.

After all this, you've shown, by examples, that the Gauss-Bonnet theorem does not hold for non-compact surfaces in the same way that it holds for compact surfaces.