MAT 342 Applied Complex Analysis  
Spring 2016 Midterm Exam Example  
Solutions

1. a) 1)

\[ 2\sqrt{3} - 1 + i(2 + i^3) = 2\sqrt{3} - 1 + 2i + i^4 = 2\sqrt{3} + 2i = \sqrt{12 + 4e^{i\arctan\left(\frac{2}{\sqrt{3}}\right)}} \]

\[ = 4e^{i\arctan\left(\frac{2}{\sqrt{3}}\right)} = 4e^{i\pi}\]

2)

\[ 1 + \cos\left(\frac{2\pi}{3}\right) + \cos\left(\frac{4\pi}{3}\right) + i\left(\sin\left(\frac{2\pi}{3}\right) + \sin\left(\frac{10\pi}{3}\right)\right) \]

\[ = 1 + \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) + \left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} + 2\pi\right)\right) \]

\[ = 1 + e^{i\frac{2\pi}{3}} + e^{i\frac{4\pi}{3}} = 1 + e^{i\frac{2\pi}{3}} + \left(e^{i\frac{2\pi}{3}}\right)^2 \]

\[ = \frac{\left(e^{i\frac{2\pi}{3}}\right)^3 - 1}{e^{i\frac{2\pi}{3}} - 1} = 0 \]

b) 1)

\[ \frac{3e^{-i\pi}}{\sqrt{(1 - i)(1 + i) + 2 e^{i2\pi}}} = \frac{3e^{-i\pi}}{\sqrt{1 + 1 + 2}} = \frac{3}{2}e^{-i\pi} = -\frac{3}{2} \]

2)

\[ \frac{-1 + i}{\sqrt{3} - i} = \frac{(-1 + i)(\sqrt{3} + i)}{(\sqrt{3} - i)(\sqrt{3} + i)} = \frac{-\sqrt{3} - i + i\sqrt{3} - 1}{4} = \frac{-1 - \sqrt{3} + i\sqrt{3} - 1}{4} \]

2. a) Let \(z = x + iy\) and \(w = a + ib\) be complex numbers. We then have

\[ |z + w|^2 + |z - w|^2 = |(x + a) + i(y + b)|^2 + |(x - a) + i(y - b)|^2 \]

\[ = (x + a)^2 + (y + b)^2 + (x - a)^2 + (y - b)^2 \]

\[ = x^2 + 2xa + a^2 + y^2 + 2yb + b^2 \]

\[ + x^2 - 2xa + a^2 + y^2 - 2yb + b^2 \]

\[ = 2(x^2 + y^2 + a^2 + b^2) = 2(|z|^2 + |w|^2) \]

b) Geometrically, the vectors \(z + w\) and \(z - w\) are the diagonals of a parallelogram spanned by the vectors \(z\) and \(w\). The equation above now states the well-known fact that the sum of the squares of the lengths of the two diagonals equals the sum of the squares of the lengths of the four sides of the parallelogram.
3. a) A set $S$ of complex numbers is called open if it does not contain any of its boundary points. (Equivalently, every point of $S$ is an interior point of $S$.) A set $S$ of complex numbers is called closed if it contains all of its boundary points. (Equivalently, the complement $\mathbb{C} \setminus S$ of $S$ in $\mathbb{C}$ is open.)

b) 1) The set $M_1 = \{ z = 1 + re^{i\theta} \mid r > 0, 0 \leq \theta \leq \frac{\pi}{2} \}$ is neither open nor closed: The point $z = 2 = 1 + e^{i0}$ lies in $M_1$. For any $\varepsilon > 0$, the point $z - i\frac{\varepsilon}{2}$ lies not in $M_1$ since $\operatorname{Arg}(z - i\frac{\varepsilon}{2}) < 0$ but it lies in the $\varepsilon$-neighbourhood of $z$. Hence, $z$ is a boundary point of $M_1$ which lies in $M_1$. Thus, $M_1$ is not open.

We have $1 \notin M_1$, but for any $\varepsilon > 0$ the point $1 + \frac{\varepsilon}{2} = 1 + \frac{\varepsilon}{2} e^{i0}$ lies in $M_1$. Hence, $1$ is a boundary point of $M_1$ which does not lie in $M_1$. This implies that $M_1$ is not closed.

2) The set $M_2 = \{ z \in \mathbb{C} \mid (\operatorname{Re}(z))^2 = \operatorname{Im}(z) \}$ is not open but closed. Let $z \in M_2$, i.e. $z = x + iy$ for some $x \in \mathbb{R}$. Let $\varepsilon > 0$ and let $w = z + \frac{\varepsilon}{2} = x + i(x^2 + \frac{\varepsilon^2}{4})$. Then, $w \notin M_2$ but $w \in B_\varepsilon(z)$. Hence, $z$ is a boundary point of $M_2$ which implies that $M_2$ is not open.

To show that $M_2$ is closed, we will show that any sequence in $M_2$ which converges in $\mathbb{C}$ in fact converges in $M_2$. It is easy to see that this is equivalent to the statement that $M_2$ is closed. Let $(z_n)_{n \in \mathbb{N}}$ be a sequence in $M_2$. Assume that $z_n \to z = x + iy \in \mathbb{C}$ as $n \to \infty$. Let $z_n = x_n + iy_n$. Then, both $x_n \to x$ and $y_n \to y$ as $n \to \infty$. We have to show that $y = x^2$.

Since $z_n \in M_2$, we have $y_n = (x_n)^2$ for all $n \in \mathbb{N}$. Because the function $f : \mathbb{R} \to \mathbb{R}$, $f(t) = t^2$, is continuous, we have $f(x_n) \to f(x) = x^2$ as $n \to \infty$ but also $f(x_n) = (x_n)^2 = y_n \to y$ as $n \to \infty$. Hence, $y = x^2$ which implies that $z \in M_2$. Thus, $M_2$ is closed.

3) The set $M_3 = \left\{ z \in \mathbb{C} \mid \left(\frac{\operatorname{Re}(z)}{4}\right)^2 + \left(\frac{\operatorname{Im}(z)}{4}\right)^2 < 1 \right\}$ is open but not closed.

We first prove, that $M_3$ is open. Let $z = x + iy \in M_3$. Then $\frac{x^2}{16} + \frac{y^2}{4} < 1$.

Let $t = 1 - \frac{x^2}{16} - \frac{y^2}{4}$. Then $t > 0$. Let $\varepsilon = \min\{1, \frac{8t}{2|x|+8|y|+8}\}$. We then have $1 \geq \varepsilon > 0$.

Let $w = a + ib \in B_\varepsilon(z)$. Thus, $|x-a|, |b-y| \leq |z-w| < \varepsilon$.

$$\frac{a^2}{16} + \frac{b^2}{4} = \frac{(a-x+x)^2}{16} + \frac{(b-y+y)^2}{4} = \frac{x^2}{16} + \frac{y^2}{4} + \frac{(a-x)^2}{16} + \frac{2x(a-x)}{4} + \frac{(b-y)^2}{4} + \frac{2y(b-y)}{4} \leq 1 - t + \frac{|a-x|^2}{16} + \frac{2|x||a-x|}{4} + \frac{|b-y|^2}{4} + \frac{2|y||b-y|}{4} \leq 1 - t + \frac{\varepsilon^2}{16} + \frac{2|x|\varepsilon + 4\varepsilon^2 + 8|y|\varepsilon}{4} \leq 1 - t + \frac{\varepsilon^2}{16} + \frac{5 + 2|x| + 8|y|}{16} \leq 1 - t + \frac{t}{2} = 1 - \frac{t}{2} < 1$$
Hence, $w \in M_3$ which implies that $M_3$ is open.

To see that $M_3$ is not closed, consider the point $z = x + iy = 4$. Then, $z \not\in M_3$ since $(\frac{4}{7})^2 + (\frac{0}{7})^2 = 1$. Let $\delta > 0$. If $\delta > 4$, then $0 \in B_\delta(z) \cap M_3$. If $\delta < 4$, then $z - \frac{\delta}{2} \in B_\delta(z) \cap M_3$. Hence, $z$ is a boundary point of $M_3$ which does not lie in $M_3$. Thus, $M_3$ is not closed.

4. a) Let $z_0, w_0 \in \mathbb{C}$ and let $f$ be a function. The notion $\lim_{z \to z_0} f(z) = w_0$ means that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(z) - w_0| < \varepsilon$ whenever $|z - z_0| < \delta$.

   b) 1) By a theorem from the lecture, we need to check whether $\lim_{z \to 0} \frac{\frac{1}{z^2} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + \frac{1}{3z^2}}$ exists. For $z \neq 0$, we have
   \[ \frac{\frac{1}{z^2} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + \frac{1}{3z^2}} = \frac{1 - 2z^2 + z^3}{\pi + i3z}. \]
   Since $\lim_{z \to 0} (1 - 2z^2 + z^3) = 1$ and $\lim_{z \to 0} (\pi + i3z) = \pi \neq 0$, we know that the limit exists with $\lim_{z \to 0} \frac{\frac{1}{z^2} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + \frac{1}{3z^2}} = \frac{1}{\pi}$. Hence, $\lim_{z \to \infty} \frac{\frac{1}{z^2} - \frac{2}{z} + 1}{\frac{\pi}{z^3} + \frac{1}{3z^2}} = \frac{1}{\pi}$.

   2) We have
   \[ \frac{z^2 + 2z - 3}{z^2 - 3z + 2} = \frac{(z - 1)(z + 3)}{(z - 1)(z - 2)} = \frac{z + 3}{z - 2}. \]
   Hence,
   \[ \lim_{z \to 1} \frac{z^2 + 2z - 3}{z^2 - 3z + 2} = \lim_{z \to 2} \frac{z + 3}{z - 2} = \frac{4}{-1} = -4. \]

5. a) Let $f(z) = u(r, \theta) + iv(r, \theta)$. The Cauchy-Riemann equations are fulfilled in $z_0 = r_0 e^{i\theta_0}$ if
   \[ ru_r(r_0, \theta_0) = v_\theta(r_0, \theta_0) \quad \text{and} \quad u_\theta(r_0, \theta_0) = -rv_r(r_0, \theta_0). \]

   b) Since $\sqrt{\cdot}$ is the principle branch of the squareroot function, we have for every $z = re^{i\theta} \in D$
   \[ f(z) = \sqrt{r} e^{i\frac{\theta}{2}} = \sqrt{r} \cos \left(\frac{\theta}{2}\right) + i\sqrt{r} \sin \left(\frac{\theta}{2}\right) = u(r, \theta) + iv(r, \theta). \]
   Hence, the first order partial derivatives of $u$ and $v$ with respect to $r$ and $\theta$ exist and we have
   \[ u_r(r) = \frac{1}{2\sqrt{r}} \cos \left(\frac{\theta}{2}\right) \quad u_\theta(r, \theta) = -\frac{\sqrt{r}}{2} \sin \left(\frac{\theta}{2}\right) \]
   \[ v_r(r) = \frac{1}{2\sqrt{r}} \sin \left(\frac{\theta}{2}\right) \quad v_\theta(r, \theta) = \frac{\sqrt{r}}{2} \cos \left(\frac{\theta}{2}\right). \]
Hence,

\[ ru_r(r, \theta) = \frac{\sqrt{r}}{2} \cos \left( \frac{\theta}{2} \right) = v_\theta(r, \theta) \]

\[ rv_r(r, \theta) = \frac{\sqrt{r}}{2} \sin \left( \frac{\theta}{2} \right) = -u_\theta(r, \theta). \]

Thus, the Cauchy-Riemann equations are fulfilled throughout \( D \). Since the first order partial derivatives of \( u \) and \( v \) are continuous throughout \( D \), \( f' \) exists everywhere in \( D \).

6. a) A function \( f \) defined on an open set \( S \) is called analytic, if it is differentiable at any point \( z \in S \).

b) For \( z = x + iy \in \mathbb{D} \), we have

\[ u(x, y) = e^{\pi x} \cos(\pi y) + 2xy \quad \text{and} \quad v(x, y) = e^{\pi x} \sin(\pi y) - x^2 + y^2. \]

Thus, the first order partial derivates of \( u \) and \( v \) exist and

\[ u_x(x, y) = \pi e^{\pi x} \cos(\pi y) + 2y = v_y(x, y) \]

\[ u_y(x, y) = -\pi e^{\pi x} \sin(\pi y) + 2x = -v_x(x, y). \]

Hence, the Cauchy-Riemann equations are fulfilled throughout \( \mathbb{D} \). Since the first order partial derivatives of \( u \) and \( v \) are continuous throughout \( \mathbb{D} \), the function \( f \) is differentiable throughout \( \mathbb{D} \), hence analytic.