VOGAN'S ALGORITHM FOR COMPUTING COMPOSITION SERIES

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Building on joint work [8] with B. Speh, D. A. Vogan [9,10] has obtained an algorithm for computing composition series of the standard induced representations of a semisimple Lie group $G$. The algorithm takes a particularly simple form in the case that the representations are induced from a maximal parabolic subgroup.

On the one hand, this algorithm does not seem to be widely known, possibly because the emphasis in the Vogan papers is somewhat different. And on the other hand, the algorithm is particularly well suited to deciding some irreducibility questions that arise in our paper [1], which settles the contribution to the unitary dual of a linear connected $G$ by Langlands quotients obtained from maximal parabolic subgroups. Thus it seems expedient to provide an exposition of Vogan's algorithm in the context of the examples needed for [1].

This approach to our irreducibility questions was suggested to us by Vogan, and it overlaps with the joint work of Barbasch and Vogan [3], which uses the algorithm to decide irreducibility in "critical cases" for classical groups. (In fact, the specific results that we give here for classical groups are some of the critical cases for Barbasch and Vogan, although our approach may be more direct.) Calculations with the algorithm appear in other papers as well, for example in [2].

The present paper is organized as follows. Section 1 collects some results from [8] and [9] and sketches the steps of the algorithm. Section 2 establishes a theorem that gives an efficient starting point

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for use of the algorithm. And Sections 3-7 show how the algorithm works in $\text{SO}(2n-1,2)$, $\text{Sp}(3,\mathbb{H})$, $\text{SO}^\ast(10)$, $\text{SO}^\ast(2n)$, and some exceptional groups of type $E$, respectively. The worked examples are exactly the ones needed for [1]. We proceed only in the generality needed for our examples—induction from a maximal parabolic subgroup, $G$ linear and connected, rank $G = \text{rank } K$, integral infinitesimal character. Readers interested in relaxing any of these assumptions should consult [8,9,10], particularly pp. 293-294 of [8], §6 of [9], and all of [10].

We are indebted to Vogan for several valuable conversations on this topic and to Barbasch and Vogan for sharing with us the details of their work announced as [3].

1. Notation and algorithm

Let $G$ be a linear connected semisimple group with a simply-connected complexification, let $\Theta$ be a Cartan involution, let $K$ be the corresponding maximal compact subgroup, and assume that rank $G = \text{rank } K$. Let $TA$ be a $\Theta$-stable Cartan subgroup (with $T \subseteq K$ and $A$ equal to a vector group), and let $MA$ be the corresponding Levi factor of the associated cuspidal parabolic subgroups.

Let $P = MAN$ be one of these parabolic subgroups. If $\sigma$ is a discrete series representation of $M$ or nondegenerate limit of discrete series and if $\nu$ is a real-valued linear functional on the Lie algebra $a$ of $A$, we let $U(P,\sigma,\nu)$ be the standard induced representation given by unitary induction as

$$U(P,\sigma,\nu) = \text{ind}^G_P(\sigma \otimes e^\nu \otimes 1).$$

It is known that $U(P,\sigma,\nu)$ has a finite Jordan-Holder series, and Vogan's algorithm addresses the problem of finding the irreducible subquotients and their multiplicities. (We shall be addressing this
problem only when \( \dim A = 1 \) and \( U(P, \sigma, \nu) \) has integral infinitesimal character.

The algorithm takes place in three stages. In the first two stages one considers the same problem for regular integral infinitesimal character. In the first stage we identify some \( U(P, \sigma_k^e, \nu_k) \) with the same infinitesimal character for which the composition series is known. In the second stage we go through a succession of wall-crossing operations, following what happens to the decomposition. At the third stage we pass to our original parameters by means of Zuckerman's \( \psi \) functor [11].

Let us introduce more convenient notation. Let \( \Delta(m^e, \iota^e) \) be the set of roots of \( M \), and let \( M_0 \) be the identity component of \( M \). Pick any irreducible constituent of \( \sigma \big|_{M_0} \), and let \( (\lambda, \Delta^+(m^e, \iota^e)) \) be its Harish-Chandra parameter. Corresponding to each real root \( \beta \) in \( \Delta^+(m^e, \iota^e) \), \( \lambda \) is an element \( \gamma_\beta \) of \( G \) that is the image of \( \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \) in \( SL(2, \mathbb{R}) \) under the homomorphism of \( SL(2, \mathbb{R}) \) into \( G \) built from \( \beta \). Let \( F(T) \) be the finite abelian group generated by the elements \( \gamma_\beta \). The restriction of \( \sigma \) to the \( M \)-central subgroup \( F(T) \) determines a character \( \chi \), and \( (\lambda, \Delta^+(m^e, \iota^e)) \) and \( \chi \) together determine \( \sigma \). Introduce a positive system \( \Delta^+(\delta^e, (i\Theta_\delta^e)^e) \) for the roots of \( G \) containing \( \Delta^+(m^e, \iota^e) \) such that \( \lambda + \nu \) is dominant. (Such a system always exists; if \( \lambda + \nu \) is regular, it is unique.)

Adding to \( \lambda + \nu \) a suitable parameter \( \mu \) that is dominant integral for \( \Delta^+(\delta^e, (i\Theta_\delta^e)^e) \) and adjusting \( \chi \) compatibly, we are led to a standard induced representation \( U(P, \sigma_0^e, \nu_0) \) whose infinitesimal character \( \gamma_0 = \lambda + \nu + \mu \) is regular ([5], Appendix B). Moreover, the Zuckerman \( \psi \) functor [11] satisfies

\[
U(P, \sigma, \nu) = \psi_{\lambda + \nu + \mu} \left( U(P, \sigma_0^e, \nu_0) \right).
\]

We shall be working with several Cartan subgroups, and we need to match carefully the corresponding root systems. Thus let us fix
a compact Cartan subgroup $B$ of $G$ contained in $K$, and let $b$ be its Lie algebra. We may assume that all other Cartan subgroups of interest are obtained by Cayley transform relative to a succession of roots. In particular, for $a$ as above, we can write $a \leftrightarrow \{\ldots\}$, where $\{\ldots\}$ is an ordered set of roots of $(\mathfrak{a}, \mathfrak{b})$ with respect to which we have formed Cayley transforms. Then there will be no confusion if we refer parameters like $\gamma = \lambda + \nu + \mu$ to our compact Cartan subalgebra without introducing new notation for them.

It is important to note that $\gamma, a \leftrightarrow \{\ldots\}$, and the adjusted $\chi$ determine the global character of $U(P, \sigma, \nu_0)$ completely. In fact, we build $i \otimes a$, transform $\gamma_0$ to it, decompose $\gamma_0 = \lambda_0 + \nu_0$, form $\sigma_0$, introduce any $N'$ making $\nu_0$ dominant, and induce $\sigma_0 \otimes \nu_0 \otimes 1$ to obtain a representation $U(P', \sigma_0, \nu_0)$. This representation will have the same global character as $U(P, \sigma_0, \nu_0)$. We write $\pi(\gamma_0, a \leftrightarrow \{\ldots\})$ for this global character, dropping the adjusted $\chi$ from the notation. (Vogan's notation differs slightly from this: He uses $\psi(\cdot)$ for the character, suppressing the explicit mention of $a$ and instead carrying it in the domain of $\gamma_0$.) In our notation, note that $\pi(\gamma_0, a \leftrightarrow \emptyset)$ is a discrete series character of $G$.

Under our assumption that $\gamma_0$ is regular, it follows essentially from [4] that the representation $U(P', \sigma_0, \nu_0)$ has a unique irreducible quotient $J(P', \sigma_0, \nu_0)$. We write $\overline{\pi}(\gamma_0, a \leftrightarrow \{\ldots\})$ for the global character of $J(P', \sigma_0, \nu_0)$. If $N''$ is a second choice of nilpotent subgroup such that $\nu_0$ is dominant, then $J(P'', \sigma_0, \nu_0) \cong J(P', \sigma_0, \nu_0)$ and consequently the global character $\overline{\pi}(\gamma_0, a \leftrightarrow \{\ldots\})$ is independent of the choice of $N'$.

Now let us work with a character $\pi(\gamma, a \leftrightarrow \{\ldots\})$ or $\overline{\pi}(\gamma, a \leftrightarrow \{\ldots\})$ such that $\gamma$ is regular and integral. Let $a$ be a real root (relative to the specified choice of $a$), and let $\chi$ be
the suppressed character of $F(T)$ for this representation. We say that $\alpha$ is a cotangent case for this representation if

$$\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2}$$

or is a tangent case if

$$\chi(\gamma_\alpha) = (-1)^{2\langle \rho_\alpha, \alpha \rangle / |\alpha|^2}.$$

Here $\rho_\alpha$ is half the sum of the roots having positive inner product with $\alpha$. In either event, $n = 2\langle \gamma, \alpha \rangle / |\alpha|^2$ is an integer. We say that $\alpha$ satisfies the parity condition for the representation if either

- $\alpha$ is a cotangent case and $n$ is even

or

- $\alpha$ is a tangent case and $n$ is odd.

If $g$ is simple, it can be shown that the parity condition can fail for integral $\gamma$ only if $\alpha$ is long and $\gamma = \varepsilon(n, \mathbb{R})$ for some $n$.

One starting place for the algorithm is the following character identity due to Schmid [6,7]. See p. 271 of Speh-Vogan [8]; recall we are assuming $G$ is connected.

**Theorem 1.1** (Schmid's identity). Let $\gamma$ be regular integral, and let $\alpha$ be a simple noncompact root (for the system $\Delta^T(g^G, b^G)$ that makes $\gamma$ dominant). If $\alpha$ satisfies the parity condition for $\pi(\gamma, \varepsilon[a])$, then

$$\pi(\gamma, \varepsilon[a]) = \pi(\gamma, \varepsilon[a]) + \pi(\gamma, \varepsilon[a]) + \pi(\rho_\alpha \gamma, \varepsilon[a]),$$

where $\varepsilon_\alpha$ denotes reflection in $\alpha$.

When the parity condition is not satisfied, the corresponding identity is as follows. (See Proposition 6.1 of Speh-Vogan [8].)

**Theorem 1.2.** Let $\gamma$ be regular integral, and let $\alpha$ be a simple noncompact root. If $\alpha$ does not satisfy the parity condition for $\pi(\gamma, \varepsilon[a])$, then
\[ \pi(\gamma, o \leftrightarrow [a]) = \pi' \gamma, o \leftrightarrow [a] \],

i.e., \( \pi(\gamma, o \leftrightarrow [a]) \) is irreducible.

Next we recall the wall-crossing functors and the \( \tau \)-invariant.

Fix \( \gamma \) regular integral, let \( \alpha \) be a simple root in the system that makes \( \gamma \) dominant, let \( \Lambda_\alpha \) be the fundamental weight corresponding to \( \alpha \), and put \( n = 2\langle \gamma, \alpha \rangle / |\alpha|^2 \). In terms of Tuckerman’s \( \psi \) and \( \phi \) functors, define

\[ \psi_\alpha = \psi_{\gamma-n\alpha} \quad \text{and} \quad \phi_\alpha = \phi_{\gamma-n\alpha}. \]

The wall-crossing functor is given by

\[ s_\alpha = \phi_\alpha \psi_\alpha - \psi \]

for any virtual character \( \psi \) whose infinitesimal character is \( \gamma \); \( s_\alpha \) acts on the local expression for a global character by reflection in \( \alpha \) (see Appendix C of [5]). We say that \( \alpha \) is in the \( \tau \)-invariant of \( \pi(\gamma, o \leftrightarrow [\cdots]) \) if \( \psi_\alpha \pi(\gamma, o \leftrightarrow [\cdots]) = 0 \).

For purposes of calculation, we shall want to regard the \( \tau \)-invariant as a subset of integers, say of \( \{1, \ldots, \ell\} \), where \( \ell \) is the rank of \( \gamma \). To do so, we note that the only parameters of interest will be the Cartan subgroups and the various \( w\gamma \) with \( \gamma \) fixed and \( w \) in the Weyl group of \( \gamma \). The positive root systems for \( w\gamma \) and \( \gamma \) are canonically identified via \( w \), and the root systems for the different Cartan subgroups are all identified by our system of Cayley transforms. Thus we can number the roots in a single Dynkin diagram for \( \gamma \) and obtain consistent numberings of the Dynkin diagrams of all the positive root systems we shall consider.

In this way the \( \tau \)-invariant of a character \( \pi \) can be regarded as a subset of \( \{1, \ldots, \ell\} \). (This point will be clearer in the examples.)

If we replace \( \gamma \) by \( w\gamma \) and if \( \alpha \) is a simple root for the system that makes \( \gamma \) dominant, then \( w\alpha \) is simple for the system making \( w\gamma \) dominant. Moreover, as observed in [11], the functor

\[ \psi_\alpha = \psi_{\gamma-n\alpha} \]

is the same as the functor \( \psi_{w\alpha} = \psi_{w\gamma-nw\alpha} \). Similar
remarks apply to $\varphi$, and thus $s_\alpha$ is the same functor as $s_{\psi\alpha}$. In keeping with the notation of the previous paragraph, we can thus denote our wall-crossing functors unambiguously by $s_1, \ldots, s_L$.

The observations about $\psi_\alpha$ in the previous paragraph make it clear that the $\tau$-invariant is an invariant of the Langlands quotient in question. In particular, we can detect inequivalence of two Langlands quotients by seeing that their $\tau$-invariants are different.

The $\tau$-invariant controls what happens at the third stage of the algorithm when we pass back to our original parameters by means of the $\psi$ functor $\psi_{\lambda+\nu+\mu}^{\lambda+\nu+\mu}$. Recall that $\lambda+\nu+\mu$ is dominant for the system $\Delta^+(\mathfrak{g}^E, (i\mathfrak{g})^E)$. From this positive system we obtain a set of singular roots, namely those simple roots orthogonal to $\lambda+\nu$. As usual, we may canonically identify the singular roots with a subset of $\{1, \ldots, l\}$.

**Theorem 1.3** ([8], Theorems 5.15, 6.16, 6.18). Let $\theta$ be an irreducible character with regular integral infinitesimal character $\gamma$ conjugate to $\lambda+\nu+\mu$. Then $\psi_{\lambda+\nu+\mu}^{\lambda+\nu+\mu} \theta$ is irreducible or $0$. It is $0$ if and only if the set of singular roots for $\lambda+\nu$ has nonempty intersection with the $\tau$-invariant of $\theta$.

Computation of $\tau$-invariants is a routine matter because of the next theorem.

**Theorem 1.4** ([8], Theorem 6.16). Let $\gamma$ be regular integral, let $\overline{\mathcal{F}(\gamma, \alpha \leftrightarrow \ldots)}$ be given, and let $\Delta^+(\mathfrak{g}^E, (i\mathfrak{g})^E)$ be the corresponding positive system. Then a simple root $\alpha$ for this system is in the $\tau$-invariant of $\overline{\mathcal{F}(\gamma, \alpha \leftrightarrow \ldots)}$ if and only if $\alpha$ satisfies one of the following:

(a) $\alpha$ is imaginary and $m$-compact

(b) $\alpha$ is complex and $\theta \alpha$ is negative

(c) $\alpha$ is real and satisfies the parity condition for $\overline{\mathcal{F}(\gamma, \alpha \leftrightarrow \ldots)}$.  

Now we turn to computation of the effect of the wall-crossing functors. Theorem 1.5 will show the effect on a full induced character. Then we consider the constituents. If $s_\alpha \emptyset = \emptyset$, then it follows from the definition that $s_\alpha \emptyset = - \emptyset$. Thus we have only to know the effect of $s_\alpha$ on $\emptyset$ in the case that $\alpha$ is not in the $\tau$-invariant; this we write down in Theorem 1.6.

**Theorem 1.5 ([8], Corollary 5.12).** Let $\gamma$ be regular integral, let $\pi(\gamma, \alpha \leftrightarrow \ldots)$ be given, and let $\Delta^+(\alpha, (1, \alpha)' \alpha)$ be the corresponding positive system. If $\alpha$ is a complex simple root for this system, then

$$s_\alpha \pi(\gamma, \alpha \leftrightarrow \ldots) = \pi(s_\alpha \gamma, \alpha \leftrightarrow \ldots).$$

**Remark.** Here $\alpha$ complex and simple makes the positive roots vanishing on $\alpha$ be the same for $\gamma$ and $s_\alpha \gamma$. Hence this theorem is indeed implied by Corollary 5.12 and the sentence before Lemma 5.8 in [8].

**Theorem 1.6 ([8], Theorem 5.16, and [9], Theorem 4.12).** Let $\gamma$ be regular integral, let $\pi(\gamma, \alpha \leftrightarrow \ldots)$ be given, and let $\Delta^+(\alpha, (1, \alpha)' \alpha)$ be the corresponding positive system. Suppose that $\alpha$ is a simple root for this system that is not in the $\tau$-invariant of $\pi(\gamma, \alpha \leftrightarrow \ldots)$. Then the wall-crossing functor $s_\alpha$ satisfies

$$s_\alpha \pi(\gamma, \alpha \leftrightarrow \ldots) = \pi(\gamma, \alpha \leftrightarrow \ldots) + U_{\alpha}(\pi(\gamma, \alpha \leftrightarrow \ldots)),$$

where $U_{\alpha}(\pi(\gamma, \alpha \leftrightarrow \ldots))$ is a sum of true characters as follows:

(a) If $\alpha$ is imaginary and $m$-noncompact, then

$$U_{\alpha}(\pi(\gamma, \alpha \leftrightarrow \ldots)) = \begin{cases} \pi(\gamma, \alpha \leftrightarrow \ldots) + \emptyset & \text{if } s_\alpha \not\in \mathcal{W}(M; \tau) \\ \overline{\pi_1}(\gamma, \alpha \leftrightarrow \ldots) + \pi_2(\gamma, \alpha \leftrightarrow \ldots) + \emptyset & \text{if } s_\alpha \in \mathcal{W}(M; \tau) \end{cases}$$

with $\overline{\pi_1}$ and $\overline{\pi_2}$ differing in how $\chi$ is defined (see p. 264 of [8]).
(b) If \( \alpha \) is complex and \( \Theta \alpha \) is positive, then
\[
U_\alpha(\varpi(\gamma, \alpha \leftrightarrow \{ \ldots \})) = \varpi(s_\alpha \gamma, \alpha \leftrightarrow \{ \ldots \}) + \Theta_0
\]
with \( s_\alpha \) unchanged.

(c) If \( \alpha \) is real and does not satisfy a parity condition for \( \varpi(\gamma, \alpha \leftrightarrow \{ \ldots \}) \), then
\[
U_\alpha(\varpi(\gamma, \alpha \leftrightarrow \{ \ldots \})) = \Theta_0.
\]

Moreover, in all cases \( \Theta_0 \) is a finite sum of irreducible characters, each of which has (the index corresponding to) \( \alpha \) in its \( \tau \)-invariant and each of which occurs in \( \varpi(\gamma, \alpha \leftrightarrow \{ \ldots \}) \).

**Remark.** The main terms of \( U_\alpha \), as well, have \( \alpha \) in their \( \tau \)-invariant, by Lemma 3.11b of [9].

Finally there is a reciprocity theorem that is helpful in computing \( \Theta_0 \).

**Theorem 1.7** ([9], Theorem 4.14). Let \( \gamma \) be regular integral, and suppose \( \Theta \) and \( \Theta' \) are irreducible characters with infinitesimal character \( \gamma \). If \( i \) is in \( \tau(\Theta) \) but \( j \) is not in \( \tau(\Theta) \), and if \( j \) is in \( \tau(\Theta') \) but \( i \) is not in \( \tau(\Theta') \) (and if indices \( i \) and \( j \) do not span a group \( G_2 \)), then the multiplicity of \( \Theta' \) in \( U_j(\Theta) \) equals the multiplicity of \( \Theta \) in \( U_i(\Theta') \), and this common multiplicity is at most one.

Now we can state the algorithm roughly. We begin with the regular integral parameter \( \gamma_0 \) constructed earlier. Since \( \dim \alpha = 1 \), we can write the character of \( U(P', \sigma_0, \nu_0) \) as \( \pi(\gamma_0, \alpha \leftrightarrow \{ \alpha \}) \) for some \( \alpha \). By a succession of reflections in complex roots, we pass from \( \gamma_0 \) to \( \gamma_1, \gamma_2, \ldots, \gamma_k \) to a point where we know how \( \pi(\gamma_k, \alpha \leftrightarrow \{ \alpha \}) \) decomposes. (For example, if \( \alpha \) is simple in the system for \( \gamma_k \), then either a Schmid identity (Theorem 1.1) or Theorem 1.2 will be available. In Section 2 we shall establish a
more efficient starting point.) We write the decomposition of \( \pi(\varphi_K, a \leftrightarrow \{a\}) \) and apply to the whole identity the reflection that takes us to \( \pi(\varphi_K, 1, a \leftrightarrow \{a\}) \), computing the individual terms by Theorems 1.6 and 1.7. (It is not clear whether these tools will be sufficient in general. However, they will suffice in our examples, and the additional tools in [10] will suffice in general.) Then we reflect again to pass to \( \pi(\varphi_{K-2}, a \leftrightarrow \{a\}) \), and so on, until we have a decomposition of \( \pi(\varphi_0, a \leftrightarrow \{a\}) \). Finally we use Theorems 1.3 and 1.4 to pass to our original parameters. If only one nonzero term survives, our original representation \( \Gamma(F, \sigma, \nu) \) was irreducible.

Setting matters up with the initial reflections requires some further explanation. We illustrate matters for an example with \( \text{SO}^*(10) \). The numbering of the Dynkin diagram will be

```
  ______5______
 /            \
1 - 2 - 3 - 4 -
```

Our given data will be a positive system for \( g^\mathcal{E} \) containing \( \Delta^+(m\mathcal{E}, t\mathcal{E}) \) such that the root defining \( a \) is simple. This root \( a \) will be root 2. The white dots are the compact roots (when referred to \( b\mathcal{E} \)), while the black dots are noncompact. We specify \( \lambda \) by attaching \( 2\langle \lambda, \beta \rangle/|\beta|^2 \) to each simple root \( \beta \). Then the diagram is

\[
\lambda:
\begin{array}{cccc}
1 & 0 & 0 & 1 \\
\end{array}
\]  
\[ a = 2 \]

and we investigate reducibility/irreducibility at \( \nu = a \). The diagram for \( \lambda + \nu \) is

\[
\lambda + \nu:
\begin{array}{cccc}
0 & 2 & -1 & 1 \\
\end{array}
\]  
\[ a = 2 \]  
\[ \Delta_2^+ \]
We apply a succession of reflections in roots nonorthogonal to \( \alpha \) in an effort to make \( \lambda + \nu \) dominant. If \( \Delta_1^+ = s_3 \Delta_2^+ \), we obtain

\[ \alpha = 2 + 3 \quad \Delta_1^+ \]

Finally \( \Delta_0^+ = s_2 \Delta_1^+ \) gives us

\[ \alpha = 2 + 3 + 5 \quad \Delta_0^+ \]

This \( \Delta_0^+ \) is a system compatible with \( \Delta^+(\mathfrak{e}_r, \mathfrak{e}_l) \) that makes \( \lambda + \nu \) dominant. We can take it as the system in which \( \gamma_0 \) is to be dominant. The set of singular roots is \( \{1, 3, 4\} \).

There is no need to carry along an explicit value of \( \gamma_0 \); having \( \Delta_0^+ \) and the expression for \( \alpha \) will be enough. Then we can define \( \gamma_1 = s_3 \gamma_0 \), which will be dominant for \( \Delta_1^+ \), and \( \gamma_2 = s_5 \gamma_1 \), which will be dominant for \( \Delta_2^+ \). Since \( \alpha \) is simple for \( \Delta_2^+ \) and satisfies the parity condition (our group is not \( \text{Sp}(n, \mathbb{R}) \)), \( \pi(\gamma_2, \mathfrak{a} \leftrightarrow [\alpha]) \) is given by a Schmid identity. So we have a starting place for the algorithm.

2. An inductive application

When we set up matters as at the end of Section 1 and then proceed with the wall crossings, we typically find that the first few wall-crossing steps are independent of our example. What is happening is that the first few steps take place in a common real rank one example. The theorem below formalizes this process and its result. Because of this theorem, we shall find that the set-up at the end of Section 1 should be done in such a way as to minimize the number of steps that are outside a real rank one subgroup.
Theorem 2.1. Let $\gamma$ be a regular integral infinitesimal character dominant for $\Delta^+$. Suppose that $\alpha$ is a $g$-noncompact root that is the sum of all the simple roots in a single-line real rank one subgroup with positive system $\Delta_0' \subseteq \Delta^+$, and suppose $\alpha$ satisfies the parity condition for $\pi(\gamma, \alpha \leftrightarrow [\alpha])$. Let $n$ be the number of simple roots in $\Delta_0'$, and suppose $n \geq 2$. Let $\epsilon_1$ and $\epsilon_2$ be the nodes among these simple roots, say with $\epsilon_1$ compact and $\epsilon_2$ noncompact. If $n = 2$ and if $[\alpha]$ is abbreviated as $\alpha$, then

$$\pi(\gamma, \alpha \leftrightarrow \alpha) = \pi(\gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha),$$

(2.1)

while if $n \geq 3$, then

$$\pi(\gamma, \alpha \leftrightarrow \alpha) = \pi(\gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha).$$

(2.2)

Proof. We proceed by induction on $n$, treating $n = 2$ and $n = 3$ separately. First let $n = 2$. The relevant part of the Dynkin diagram is

$$\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\quad \alpha = 2
\quad \gamma \ \text{dominant}

\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\quad \alpha = 1 + 2
\quad \gamma \ \text{dominant}

By the Schmid identity (Theorem 1.1) applied to $s_{\epsilon_1} \Delta^+$,

$$\pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1} s_{\epsilon_2} \gamma, \alpha \leftrightarrow \alpha).$$

(2.3)

We shall apply the functor $s_\lambda$. To compute the $\tau$-invariants of the right side of (2.3), we need one more diagram:
The $\tau$-invariants within the set \{1,2\} are

$$\tau(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha)) = \{2\}$$

$$\tau(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)) = \{1\}$$

$$\tau(\overline{\tau}(s_{a x_1} s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)) = \emptyset$$

by Theorem 1.4. Then Theorem 1.6 gives

$$s_1(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha)) = \overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha) + \overline{\tau}(\gamma, a \leftrightarrow \alpha) + \Theta_1$$

$$s_1(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)) = -\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)$$

$$s_1(\overline{\tau}(s_{a x_1} s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)) = \overline{\tau}(s_{a x_1} s_{\epsilon x_1} \gamma, a \leftrightarrow \beta) + \overline{\tau}(s_{a x_1} s_{\epsilon x_1} \gamma, a \leftrightarrow \beta) + \Theta_2.$$

Here $\Theta_2$ is the sum of constituents of $\overline{\tau}(s_{a x_1} s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)$ having 1 in the $\tau$-invariant, and so $\Theta_2 = 0$. Also $\Theta_1$ is the sum of constituents of $\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha)$ having 1 in the $\tau$-invariant, and so (2.1) shows $\Theta_1 = c\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)$. Now Theorem 1.7 gives

$$c = \text{mult } \overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta) \text{ in } U_1(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha)), \quad 1 \notin \tau = \{2\},$$

$$= \text{mult } \overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha) \text{ in } U_2(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha)), \quad 2 \notin \tau = \{1\}.$$

For the latter we write

$$U_2(\overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \beta)) = \overline{\tau}(s_{\epsilon x_1} \gamma, a \leftrightarrow \alpha) + \Theta_3.$$

This shows $c \geq 1$, and Theorem 1.7 says $c = 1$. Applying $s_1$ to (2.3) and using Theorem 1.5, we obtain
\[ \pi(\gamma, \alpha \leftrightarrow \alpha) = s_1 \pi(s_{e_1} \gamma, \alpha \leftrightarrow \alpha) \]
\[ = s_1 \pi(s_{e_1} \gamma, \alpha \leftrightarrow \alpha) + s_1 \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \beta) + s_1 \pi(s_{\epsilon_1} s_{e_1} \gamma, \alpha \leftrightarrow \beta) \]
\[ = \pi(s_{e_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(\gamma, \alpha \leftrightarrow \beta) + \pi(s_{e_1} \gamma, \alpha \leftrightarrow \beta) \]
\[ - \pi(s_{e_1} \gamma, \alpha \leftrightarrow \beta) \]
\[ + \pi(s_{\epsilon_1} s_{e_1} \gamma, \alpha \leftrightarrow \beta) + \pi(s_{\epsilon_1} s_{e_1} \gamma, \alpha \leftrightarrow \epsilon_2) . \]  

(2.4)

Since \( \epsilon_1 \) is compact and \( s_{\epsilon_1} s_{\epsilon_1} = s_{\epsilon_2} \), we have
\[ \pi(s_{\epsilon_1} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \beta) = \pi(s_{\epsilon_1} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \beta) = \pi(s_{\epsilon_2} \gamma, \alpha \leftrightarrow \beta) \]
and
\[ \pi(s_{\epsilon_1} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \epsilon_2) = \pi(s_{\epsilon_1} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \epsilon_1 \epsilon_2) = \pi(s_{\epsilon_2} \gamma, \alpha \leftrightarrow \epsilon_2) . \]

Substitution into (2.4) gives the desired result (2.1).

Next let \( n = 3 \). The corresponding diagrams are

\[ \Delta^+ = s_{\epsilon_1} \Delta^+ \]

\[ \Delta_0^+ \subset \Delta^+ \]

\[ \alpha = 1 + 2 + 3 \]

\[ s_{\epsilon_1} \gamma \text{ dominant} \]

\[ \gamma \text{ dominant} \]

By (2.1) for \( s_{\epsilon_1} \gamma \),
\[ \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_1 + \eta_2} s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) \]
\[ + \pi(s_{\epsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \pi(s_{\epsilon_2 \epsilon_1} \gamma, \alpha \leftrightarrow \beta) . \]  

(2.5)
We shall apply the functor $s_1$. To compute the $\tau$-invariants of the right side of (2.5), we need the diagrams

\[ s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \alpha \quad \eta_2 \varepsilon_1 \eta_2 \quad \alpha = 3 \quad s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma \quad \text{dominant} \]

\[ s_{\varepsilon_2} s_{\varepsilon_1} \Delta \quad \alpha = 2 \quad s_{\varepsilon_2} s_{\varepsilon_1} \gamma \quad \text{dominant} \]

The $\tau$-invariants within $\{1,2,3\}$ are

\[
\tau(\overline{s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha}) = \{2,3\} \\
\tau(\overline{s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha}) = \{1,3\} \\
\tau(\overline{s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha}) = \{2\} \\
\tau(\overline{s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \beta}) = \{1\}. \tag{2.6} \]

Then

\[
s_{\varepsilon_1} \overline{\pi}(s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(\gamma, \alpha \leftrightarrow \alpha) + \Theta_1 \\

s_{\varepsilon_1} \overline{\pi}(s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) = - \overline{\pi}(s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) \\

s_{\varepsilon_1} \overline{\pi}(s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) = \overline{\pi}(s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) + \overline{\pi}(s_{\varepsilon_2} \gamma, \alpha \leftrightarrow \alpha) + \Theta_2 \\

s_{\varepsilon_1} \overline{\pi}(s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \beta) = - \overline{\pi}(s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \beta). \tag{2.7} \]

From (2.5) and (2.6)

\[
\Theta_1 = c_1 \overline{\pi}(s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) + c_2 \overline{\pi}(s_{\varepsilon_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \beta).
\]

Now Theorem 1.7 gives

\[
c_1 = \text{mult} \overline{\pi}(s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) \quad \text{in U}_1(\overline{\pi}(s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha)), \quad 1 \not\in \tau = \{2,3\}, \\
= \text{mult} \overline{\pi}(s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha) \quad \text{in U}_2(\overline{\pi}(s_{\varepsilon_1 \eta_2} s_{\varepsilon_1} \gamma, \alpha \leftrightarrow \alpha)), \quad 2 \not\in \tau = \{1,3\}.\]
For the latter, we write
\[ U_2 \overline{\pi}(s_{e_1}^s_1 \eta_2 \xi, \alpha \leftrightarrow \beta) = \overline{\pi}(s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) + \Theta_3. \]

This shows \( c_1 \geq 1 \), and hence \( c_1 = 1 \). Next,
\[ c_2 = \text{mult} \overline{\pi}(s_{e_2}^s_1 \xi, \alpha \leftrightarrow \beta) \text{ in } U_1 \overline{\pi}(s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha), \quad 1 \not\in \tau = \{2, 3\}, \]
\[ = \text{mult} \overline{\pi}(s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) \text{ in } U_2 \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta), \quad 2 \not\in \tau = \{1\}. \]

Now
\[ U_2 \overline{\pi}(s_{e_2}^s_1 \xi, \alpha \leftrightarrow \beta) = \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) + \Theta_4. \]

Here \( \Theta_4 \) is the sum of constituents of \( \pi(s_{e_1}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta) \)
\[ = \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta) \text{ having } 2 \text{ in the } \tau \text{-invariant, and so } \Theta_4 = 0. \]

Since \( \overline{\pi}(s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) \) and \( \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) \) have respective
\( \tau \)-invariants \([2, 3]\) and \([2]\), they are unequal characters, and thus
\( c_2 = 0 \). Hence
\[ \Theta_1 = \overline{\pi}(s_{e_1}^s_1 \eta_2 \xi, \alpha \leftrightarrow \alpha). \quad (2.3) \]

To compute \( \Theta_2 \), we use the Schmid identity
\[ \pi(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta) = \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) \]
\[ + \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta) \]
and compute that the \( \tau \)-invariants for the terms on the right are
\( [2], [1], \) and \([3]\). Then it follows that
\[ \Theta_2 = c \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta). \]

By Theorem 1.7,
\[ c = \text{mult} \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta) \text{ in } U_1(\overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha)), \quad 1 \not\in \tau = \{2\}, \]
\[ = \text{mult} \overline{\pi}(s_{e_1}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) \text{ in } U_2(\overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta)), \quad 2 \not\in \tau = \{1\}. \]

Since
\[ U_2(\overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \beta)) = \overline{\pi}(s_{e_2}^s_1 s_{e_1}^s_1 \gamma, \alpha \leftrightarrow \alpha) + \Theta_5, \]
we have $c \geq 1$. Thus $c = 1$ and

$$S_2 = \pi(\xi_\varepsilon \nu \cdot \chi, \alpha \leftrightarrow \beta).$$

(2.9)

Finally we apply $s_1$ to both sides of (2.5), use the identity $s_1 \pi(\xi_\varepsilon \nu \cdot \alpha \leftrightarrow \alpha) = \pi(\gamma \cdot \alpha \leftrightarrow \alpha)$ given in Theorem 1.5, and substitute from (2.7), (2.8), and (2.9) to obtain (2.2) for $n = 3$.

Now let $n \geq 4$, and assume inductively that (2.1) and (2.2) have been proved for all cases $\leq n - 1$. The starting diagrams are

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We shall use also the diagrams

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By inductive hypothesis, (2.2) for \( n-1 \) and \( s_{e_1} \gamma \) gives us

\[
\pi(s_{e_1} \gamma, a \leftrightarrow a) = \pi(s_{e_1} \gamma, s \leftrightarrow s) + \pi(s_{e_2} s_{e_1} \gamma, a \leftrightarrow a) + \pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a) + \pi(s_{e_1} s_{e_2} \gamma, s \leftrightarrow s) + \pi(s_{e_1} s_{e_2} \gamma, s \leftrightarrow s) + \pi(s_{e_1} s_{e_2} \gamma, s \leftrightarrow s), \tag{2.13}
\]

and the respective \( \tau \)-invariants for the terms on the right are

\[\{2, ..., n\}, \{2, ..., n-1\}, \{1,3,4, ..., n\}, \text{ and } \{1,3,4, ..., n-1\}.\]

Then

\[
s_{e_1} \pi(s_{e_1} \gamma, a \leftrightarrow a) = \pi(s_{e_1} \gamma, s \leftrightarrow s) + \pi(\gamma, a \leftrightarrow a) + \Theta_1
\]

\[
s_{e_1} \pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a) = \pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a) + \pi(s_{e_2} \gamma, a \leftrightarrow a) + \Theta_2
\]

\[
s_{e_1} \pi(s_{e_1} \gamma, s \leftrightarrow s) = -\pi(s_{e_1} \gamma, a \leftrightarrow a)
\]

\[
s_{e_1} \pi(s_{e_1} s_{e_2} \gamma, s \leftrightarrow s) = -\pi(s_{e_2} \gamma, s \leftrightarrow s)
\]

From (2.13),

\[
\Theta_1 = c_1 \pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a) + c_2 \pi(s_{e_1} s_{e_2} \gamma, s \leftrightarrow s).
\]

By the same argument as when \( n = 3 \), we find \( c_1 = 1 \). Also

\[
c_2 = \text{mult of } \pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a) \text{ in } U_1(\pi(s_{e_1} \gamma, a \leftrightarrow a)),
\]

\[1 \notin \tau = \{2, ..., n-1\},\]

\[
= \text{mult of } \pi(s_{e_1} \gamma, a \leftrightarrow a) \text{ in } U_2(\pi(s_{e_1} s_{e_2} \gamma, a \leftrightarrow a)),
\]

\[2 \notin \tau = \{1,3,4, ..., n-1\}.
\]

Now

\[
U_2(\pi(s_{e_1} s_{e_2} s_{e_1} \gamma, a \leftrightarrow a)) = \pi(s_{e_2} s_{e_1} \gamma, a \leftrightarrow a) + \Theta_3,
\]

with each constituent of \( \Theta_3 \) in \( \pi(s_{e_1} s_{e_2} s_{e_1} \gamma, a \leftrightarrow a) \). By inductive assumption for \( n-3 \) (if \( n > 4 \)) or by a Schmid identity if \( n = 4 \), this \( \pi \) has three or four terms in its expansion, all computed within the diagram (2.12), and \( 1 \) is in the \( \tau \)-invariant of each. Since \( 1 \) is not in the \( \tau \)-invariant of \( \pi(s_{e_1} \gamma, a \leftrightarrow a) \),
\( \varpi(s^1 \epsilon_1, s \leftrightarrow \alpha) \) does not occur in \( \Theta_3 \). But nor do we have

\[ \varpi(s^1 \epsilon_1 \gamma, s \leftrightarrow \alpha) = \varpi(s^2 \epsilon_2 \epsilon_1 \gamma, s \leftrightarrow \alpha), \]

since \( n \) is in the \( \tau \)-invariant of the left side but not the right side. Thus \( c_2 = 0 \), and

\[ \Theta_1 = \varpi(s^1 \epsilon_1 \mu_2 \epsilon_2 \gamma, s \leftrightarrow \alpha). \]  \( (2.15) \)

To compute \( \Theta_2 \), we note that each constituent of \( \Theta_2 \) must occur in \( \pi(s^1 \epsilon_1 \gamma, s \leftrightarrow \alpha) \) and have \( 1 \) in its \( \tau \)-invariant. By inductive assumption for \( n-2 \), this \( \pi \) has four terms in its expansion, all computed within the diagram \( (2.10) \). A term in which \( \alpha \) includes root 2 will not have \( 1 \) in its \( \tau \)-invariant, while a term with \( \alpha \) not including root 2 (or in the case \( n=4 \) -- in which \( \alpha \leftrightarrow \beta \)) will have \( 1 \) in its \( \tau \)-invariant. There are two terms of the latter kind,

\[ \varpi(s^1 \epsilon_1 \eta_2 s^2 \epsilon_2 \gamma, s \leftrightarrow \alpha) \]

and

\[ \begin{cases} 
\varpi(s^1 \epsilon_1 \eta_2 s^2 \epsilon_2 \gamma, s \leftrightarrow \alpha) & \text{if } n = 4 \\
\varpi(s^1 \epsilon_1 \eta_2 s^2 \epsilon_2 \gamma, s \leftrightarrow \alpha) & \text{if } n > 4.
\end{cases} \]

Let \( c_3 \) and \( c_4 \) be the respective coefficients of these terms in \( \Theta_2 \). A familiar argument shows \( c_3 = 1 \). For the computation of \( c_4 \), let us treat \( n = 4 \) and \( n > 4 \) separately.

First suppose \( n = 4 \). Then

\[ c_4 = \text{mult} \varpi(s^1 \epsilon_1 \eta_2 s^2 \epsilon_2 \gamma, s \leftrightarrow \alpha) \text{ in } U_1(\varpi(s^1 \epsilon_1 \gamma, s \leftrightarrow \alpha)), 1 \not\tau = \{2,3\}, \]

and we find that

\[ U_3(\varpi(s^1 \epsilon_1 \eta_2 s^2 \epsilon_2 \gamma, s \leftrightarrow \alpha)) = \varpi(s^1 \epsilon_1 \eta_2 s^2 \gamma, s \leftrightarrow \alpha), \]

The \( \tau \)-invariant of the term on the right turns out to be \( \{1,3\} \), while \( \tau(\varpi(s^1 \epsilon_1 \gamma, s \leftrightarrow \alpha)) \) is \( \{2,3\} \). Thus \( c_4 = 0 \) when \( n = 4 \).
Now suppose $n > 4$. Then
\[ c_n = \text{mult } \overline{\pi}(s_{\eta_{n-1}^{-1}} s_{\varepsilon_2} s_{\eta_1} s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a) \text{ in } U_1(\overline{\pi}(s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)), \]

\[ 1 \notin \tau = \{2, 3, \ldots, n-1\}, \]

\[ = \text{mult } \overline{\pi}(s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a) \text{ in } U_2(\overline{\pi}(s_{\eta_{n-1}^{-1}} s_{\varepsilon_2} s_{\eta_1} s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)), \]

\[ 2 \notin \tau = \{1, 3, \ldots, n-2, n\}, \]

the diagram for the latter $\tau$-invariant being

\[ \eta_2 - e_1^\gamma \eta_2 \quad \ldots \quad \eta_{n-1}^{-e_2} \eta_{n-1}^\gamma \quad \eta_{n-2}^\gamma \eta_{n-1}^{-e_2} \quad \alpha = 3 + \ldots + (n-2) \]

Here

\[ U_2(\overline{\pi}(s_{\eta_{n-1}^{-1}} s_{\varepsilon_2} s_{\eta_1} s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a) = \overline{\pi}(s_{\eta_{n-1}^{-1}} s_{\varepsilon_2} s_{\eta_1} s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)^{+} \Theta_3 \]

(2.16)

with each term of $\Theta_3$ contained in

\[ \pi(s_{\eta_{n-1}^{-1}} s_{\varepsilon_2} s_{\eta_1} s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a). \]

(2.17)

Our inductive assumption for $n=4$ (or a Schmid identity if $n=5$) shows that $1$ is in the $\tau$-invariant of each term of (2.17), while $1$ is not in the $\tau$-invariant of $\overline{\pi}(s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)$. So $\overline{\pi}(s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)$ does not occur in $\Theta_3$. Finally $n$ is in the $\tau$-invariant of the first term on the right of (2.16) but is not in the $\tau$-invariant of $\overline{\pi}(s_{\varepsilon_1} s_{\varepsilon_2}^\gamma, a \leftrightarrow a)$. Thus $c_n = 0$ when $n > 4$.

So all cases $n \geq 4$ have

\[ \Theta_2 = \overline{\pi}(s_{\varepsilon_1} s_{\eta_2} s_{\varepsilon_2}^\gamma, a \leftrightarrow a). \]

(2.18)

Finally we apply $s_1$ to both sides of (2.13), use the identity

\[ s_1 \pi(s_{\varepsilon_1}^\gamma, a \leftrightarrow a) = \pi(s_{\varepsilon_1}^\gamma, a \leftrightarrow a) \]

given in Theorem 1.5, and substitute from (2.14), (2.15), and (2.18)
to obtain (2.2) for \( n \). This completes the induction and the proof of the theorem.

3. \( SO(2n-1,2) \)

For \( SO(2n-1,2) \) we shall consider two sets of examples. For each we specify \( \lambda \) by attaching \( 2\langle \lambda, \beta \rangle / |\beta|^2 \) to each simple root \( \beta \) in the Dynkin diagram of type \( B_n \). The black simple roots are the noncompact ones, and \( P \) is built from \( \alpha \). We number the simple roots from 1 to \( n \), with \( n \) denoting the short simple root. The theorem in each case is that \( U(P, \sigma, \nu) \) is irreducible for the indicated value of \( \nu \).

a. First set of examples

\( n = 3 \)

\[ \lambda: \begin{array}{c}
\circ & 0 & 1 \\
\circ & \circ & \circ \\
\alpha 
\end{array} \]

tangent case

at \( \nu = \frac{t}{2} \alpha \)

\( n \) odd \( \geq 5 \), \( t = \frac{5}{2}(n-1) \)

\[ \lambda: \begin{array}{c}
1 & 1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 2 \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\epsilon_t & \epsilon_{t-1} & \epsilon_1 & \alpha & \gamma_1 & \gamma_2 & \gamma_t 
\end{array} \]

tangent case

at \( \nu = \frac{t}{2} \alpha \)

\( n \) even \( \geq 4 \), \( t = \frac{1}{2}n \)

\[ \lambda: \begin{array}{c}
1 & 1 & 0 & 0 & 1 & 2 \\
\circ & \circ & \circ & \circ & \circ & \circ \\
\epsilon_t & \epsilon_{t-1} & \epsilon_1 & \alpha & \gamma_1 & \gamma_2 & \gamma_t 
\end{array} \]

cotangent case

at \( \nu = (t-1) \alpha \)

The diagrams for \( \lambda + \nu \) are

\[ \lambda + \nu: \begin{array}{c}
\circ & 0 & 1 & 0 \\
\circ & \circ & \circ & \circ \\
\alpha 
\end{array} \] (\( n = 3 \))
For \( n = 3 \), we let \( t = 1 \). In the same way as at the end of Section 1, we apply a sequence of complex root reflections to each of these diagrams. When \( n \) is odd, the sequence is \( s_{2t} \cdots s_{t+2} s_1 \cdots s_{t-1} s_t \). When \( n \) is even, the sequence is \( s_{2t-1} \cdots s_{t+1} s_1 \cdots s_{t-1} \). In all cases the resulting diagram is

\[
\begin{array}{cccccc}
\lambda + \nu: & 1 & 1-t & 2t-1 & l-t & 1 & 2 \\
& \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
& a & & & & & \\
\end{array}
\]

\( (n \text{ odd } \geq 5) \)

\[
\begin{array}{cccccc}
\lambda + \nu: & 1 & 1 & 2-t & 2t-2 & l-t & 1 & 2 \\
& \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

\( (n \text{ even } \geq 4) \)

This \( \Delta_0^+ \) is a system compatible with \( \Delta^+(m^e, e) \) that makes \( \lambda + \nu \) dominant. We take it as the system in which \( \gamma_0 \) is to be dominant. The set of singular roots is \{1, n\}.

The nature of \( \alpha \) allows us to apply Theorem 2.1 immediately to obtain

\[
\pi(\gamma_0, \alpha + \alpha) = \overline{\pi}(\gamma_0, \alpha + \alpha) + \overline{\pi}(s_{\beta_1} \gamma_0, \alpha + \alpha) + \overline{\pi}(s_{\beta_{n-1}} \gamma_0, \alpha + \alpha)
\]

\[
+ \overline{\pi}(s_{\beta_n} \gamma_0, \alpha + \alpha) + \overline{\pi}(s_{\beta_{n-1}} \gamma_0, \alpha + \alpha), \quad (3.1)
\]

except that the last term is replaced by \( \overline{\pi}(s_{\beta_n} \gamma_0, \alpha + \alpha) \) when \( n = 3 \).

The respective \( \tau \)-invariants for the representations on the right side are \{1, \ldots, n-1\}, \{2, \ldots, n-1\}, \{1, \ldots, n-2, n\}, and \{2, \ldots, n-2, n\}.

The only one of these that is disjoint from the set \{l, n\} of singular roots is the second one. By Theorem 1.3, the only term on the right side of (3.1) with nonzero image under the \( \Phi \) functor is the second one, and its image is irreducible. Therefore \( \Phi(P, \sigma, \nu) \) is irreducible.


**b. Second set of examples**

\( n = 3 \)

\[ \lambda: \begin{array}{c}
\begin{array}{cccc}
\cdots & 1 & 0 & 0 \\
\varepsilon_t & \varepsilon_2 & \varepsilon_1 & \alpha \\
& \gamma_1 & \gamma_2 & \gamma_t \\
\end{array}
\end{array} \]

tangent case
at \( \nu = \frac{3}{2} \alpha \)

\( n \) odd \( \geq 5, \ t = \frac{1}{2} (n-1) \)

\[ \lambda: \begin{array}{c}
\begin{array}{cccc}
\cdots & 1 & 1 & 0 & 1 & 1 & 1 \\
\varepsilon_t & \varepsilon_2 & \varepsilon_1 & \alpha & \gamma_1 & \gamma_2 & \gamma_t \\
\end{array}
\end{array} \]

tangent case
at \( \nu = (t - \frac{1}{2}) \alpha \)

\( n \) even \( \geq 4, \ t = \frac{1}{2} n \)

\[ \lambda: \begin{array}{c}
\begin{array}{cccc}
\cdots & 1 & 1 & 0 & 0 & 1 \\
\varepsilon_{t-1} & \varepsilon_2 & \varepsilon_1 & \alpha & \gamma_1 & \gamma_2 & \gamma_t \\
\end{array}
\end{array} \]

cotangent case
at \( \nu = (t - 1) \alpha \)

The diagrams for \( \lambda + \nu \) are

\[ \lambda + \nu: \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
0 & 1 & -1 \\
\alpha \\
\end{array}
\end{array}
\end{array} \]
\( (n = 3) \)

\[ \lambda + \nu: \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 & 1 & t & 2t-1 & 1-t & 1 \\
\alpha \\
\end{array}
\end{array}
\end{array} \]
\( (n \) odd \( \geq 5) \)

\[ \lambda + \nu: \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1 & 2-t & 2t-2 & 1-t & 1 \\
\alpha \\
\end{array}
\end{array}
\end{array} \]
\( (n \) even \( \geq 4) \)

For \( n = 3 \), we let \( t = 1 \). We apply the same sequence of complex root reflections as in Section 3a, obtaining a system \( \Delta_1^+ \), and then we apply the reflection \( s_n \) to \( \Delta_1^+ \) to obtain \( \Delta_0^+ \). The diagrams in all cases are
This $\Delta_0^+$ is a system compatible with $\Delta^+(m^c, n^c)$ that makes $\lambda + \nu$ dominant. We take it as the system in which $\gamma_0$ is to be dominant. The set of singular roots is $\{1, n-1\}$.

We can apply Theorem 2.1 in the system $\Delta_1^+$ to obtain

$$\tau(s_n \gamma_0, \sigma \leftrightarrow \alpha) = \overline{\tau(s_n \gamma_0, \sigma \leftrightarrow \alpha)} + \overline{\tau(s_1 s_n \gamma_0, \sigma \leftrightarrow \alpha)}$$

$$+ \overline{\tau(s_n^{-1} s_n \gamma_0, \sigma \leftrightarrow \alpha)} + \overline{\tau(s_1 s_n^{-1} s_n \gamma_0, \sigma \leftrightarrow \alpha)}, \quad (3.2)$$

except that the last term is replaced by $\overline{\tau(s_n^{-1} s_n \gamma_0, \sigma \leftrightarrow \alpha)}$ when $n = 3$. The respective $\tau$-invariants for the representations on the right side are $\{1, \ldots, n-1\}$, $\{2, \ldots, n-1\}$, $\{1, \ldots, n-2, n\}$, and $\{2, \ldots, n-2, n\}$. We apply $s_n$ to both sides of (3.2). We could go through the step-by-step analysis, but it is simpler to observe that everything that happens in the computation is oblivious to the presence of the double line in the diagram. Therefore the answer has to be of the form given in Theorem 2.1:

$$\tau(\gamma_0, \sigma \leftrightarrow \alpha) = \overline{\tau(\gamma_0, \sigma \leftrightarrow \alpha)} + \overline{\tau(s_1 \gamma_0, \sigma \leftrightarrow \alpha)}$$

$$+ \overline{\tau(s_n \gamma_0, \sigma \leftrightarrow \alpha)} + \overline{\tau(s_1 s_n \gamma_0, \sigma \leftrightarrow \alpha)}.$$

The respective $\tau$-invariants for the representations on the right side are $\{1, \ldots, n-2, n\}$, $\{2, \ldots, n-2, n\}$, $\{1, \ldots, n-1\}$, and $\{2, \ldots, n-2, n\}$. The only one of these that is disjoint from the set $\{1, n-1\}$ of singular roots is the second one. Thus only the image of
$\varphi(s_1\gamma_0, a \leftrightarrow a)$ is nonzero when we apply the $\psi$ functor, and its image is irreducible. Therefore $U(P, \sigma, \nu)$ is irreducible.

4. $Sp(3, \mathbb{R})$

For $Sp(3, \mathbb{R})$ we shall specify $\lambda$ by attaching $2(\lambda, \beta)/|\beta|^2$ to each simple root $\beta$ in the Dynkin diagram of type $C_3$. The black simple roots are the noncompact ones, and $P$ is built from $a$.

$$\lambda: \quad \begin{array}{ccc} & 1 & 0 \\ \alpha & \text{cotangent case} & \end{array} \quad \text{at } \nu = \frac{3}{4}a$$

We shall prove that $U(P, \sigma, \nu)$ is irreducible for the indicated value of $\nu$.

We number the roots as $\{1, 2, 3\}$ from left to right. The diagram of $\lambda + \nu$ is

$$\lambda + \nu: \quad l \quad -1 \quad 1 \quad \alpha = \begin{array}{c} 3 \end{array} \quad \Delta_1^+$$

Put $\Delta_0^+ = s_3\Delta_1^+$. The picture is

$$\begin{array}{ccc} 0 & 1 & 0 \\ \alpha = 2 \begin{array}{c} 2 \end{array} + \begin{array}{c} 3 \end{array} \quad \Delta_0^+ = s_3\Delta_1^+ \end{array}$$

We take $\Delta_0^+$ as the system in which $\gamma_0$ is dominant. The set of singular roots is $\{l, 3\}$. We do not have an immediately available character identity in $\Delta_0^+$ but have one in $\Delta_1^+$. Here $a$ does not satisfy the parity condition, and Theorem 1.2 says that

$$\varphi(s_2\gamma_0, a \leftrightarrow a) = \varphi(s_2\gamma_0, a \leftrightarrow a).$$

The $\tau$-invariant for the right side is $\{l\}$, and we find
$$s_2 \bar{\pi}(s_2 \gamma_0, a \leftrightarrow a) = \bar{\pi}(s_2 \gamma_0, a \leftrightarrow a) + \bar{\pi}(\gamma_0, a \leftrightarrow a) + \theta.$$  

Here the constituents of $\theta$ must occur on the right side of (4.1) and must have 2 in their $\tau$-invariants. So $\theta = 0$. Therefore

$$\bar{\pi}(\gamma_0, a \leftrightarrow a) = s_2 \bar{\pi}(s_2 \gamma_0, a \leftrightarrow a) = s_2 \bar{\pi}(s_2 \gamma_0, a \leftrightarrow a)$$

$$= \bar{\pi}(s_2 \gamma_0, a \leftrightarrow a) + \bar{\pi}(\gamma_0, a \leftrightarrow a).$$

The $\tau$-invariants for the two terms on the right side are $[1]$ and $[2]$. Only the second of these is disjoint from the set $[1, 3]$ of singular roots, and it follows just as in Section 3 that $U(P, \sigma, \nu)$ is irreducible.

5. $SO^*(10)$

For $SO^*(10)$, we specify $\lambda$ as at the end of Section 1. This is a cotangent case and we treat $\nu = \alpha$. We prove that $U(P, \sigma, \nu)$ is irreducible for this value of $\nu$.

We number the roots as at the end of Section 1, and the diagram for $\lambda + \nu$ is what is called $\Delta_2^+$ there. It is a little more convenient to define $\Delta_1^+$ as $s_3 s_1 \Delta_2^+$. We continue with $\Delta_0^+ = s_5 \Delta_1^+$. Then our diagrams are

$$\Delta_1^+$$

```
+---+---+---+---+
|   |   |   |   |
| -1|
+---+---+---+---+
   0  1  1  0
```

$$\alpha = [1] + [2] + [3]$$

$$\Delta_0^+$$

```
+---+---+
|   |   |
| 1 |
+---+---+
   0  1  0  0
```

$$\alpha = [1] + [2] + [3] + [5]$$

We use $\Delta_0^+$ to define $\gamma_0$. The set of singular roots is $[1, 3, 4]$.

We can apply Theorem 2.1 in the system $\Delta_1^+$ to obtain
\[ \pi(s_2\gamma_0, a \leftrightarrow \alpha) = \pi(s_2\gamma_0, a \leftrightarrow \alpha) + \pi(s_1s_2\gamma_0, a \leftrightarrow \alpha) + \pi(s_3s_2\gamma_0, a \leftrightarrow \alpha) + \pi(s_1s_2s_3\gamma_0, a \leftrightarrow \alpha). \] (5.1)

The respective \( \tau \)-invariants of the terms on the right are \{1,2,3\}, \{2,3\}, \{1,2,4\}, and \{2,4\}, and 5 is not in any of these. But even more, we see from Theorem 2.1 that 5 is not in the \( \tau \)-invariant of any constituent of the corresponding representations \( \pi(\ldots, a \leftrightarrow \alpha) \) for the members of the right side of (5.1). This says that all the extra \( \boxtimes \) terms are 0 when we apply \( s_5 \) to the terms on the right side of (5.1). Consequently application of \( s_5 \) to (5.1) gives exactly

\[ \pi(s_5\gamma_0, a \leftrightarrow \alpha) = (\text{same 4 terms as in (5.1)}) \]

\[ + \pi(s_5\gamma_0, a \leftrightarrow \alpha) + \pi(s_1s_5\gamma_0, a \leftrightarrow \alpha) \]

\[ + \pi(s_3s_2s_3\gamma_0, a \leftrightarrow \alpha, \beta) + \pi(s_5s_1s_3s_5\gamma_0, a \leftrightarrow \alpha, \beta). \]

Here \( \beta \) denotes the noncompact root in position 5 for the positive system in question. We readily compute that the respective \( \tau \)-invariants of our four new terms are \{1,2,5\}, \{2,5\}, \{1,2,4,5\}, and \{2,4,5\}. Of our eight \( \tau \)-invariants, the only one that fails to meet the set \{1,3,4\} of singular roots is \{2,5\}. Thus only the image of \( \pi(s_1\gamma_0, a \leftrightarrow \alpha) \) is nonzero when we apply the \( \pi \) functor, and it follows as in Sections 3 and 4 that \( U(P,\sigma,\nu) \) is irreducible.

6. \( S_0^*(2n) \)

For \( S_0^*(2n) \) with \( n \geq 6 \), we shall specify \( \lambda \) by attaching \( 2(\lambda,\beta)/|\beta|^2 \) to each simple root \( \beta \) in the Dynkin diagram of type \( D_n \). The black simple roots are the noncompact ones, and \( P \) is built from \( \alpha \).
n odd, $t = \frac{1}{2}(n-1)$

\[ \lambda: \begin{array}{ccc}
\varepsilon_t & \cdots & 1 & 0 & 0 & 1 & \cdots & 1 & 1 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \]

cotangent case at $\nu = (t-1)\alpha$

n even, $t = \frac{1}{2}(n-2)$

\[ \lambda: \begin{array}{ccc}
\varepsilon_t & \cdots & 1 & \frac{1}{2} & 0 & \frac{1}{2} & \cdots & 1 & 1 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \]

tangent case at $\nu = (t-\frac{1}{2})\alpha$

We shall prove that $U(P, \sigma, \nu)$ is irreducible for the indicated values of $\nu$.

We number the roots on the horizontal as \{1, \ldots, n-1\} and denote by $\alpha$ the root extending upward. The diagrams for $\lambda + \nu$ are

\[ \lambda + \nu: \begin{array}{ccc}
1 & \cdots & 1 & 1-t & 2t-2 & 2-t & \cdots & 1 & 1 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \] (n odd)

\[ \lambda + \nu: \begin{array}{ccc}
1 & \cdots & 1 & 1-t & 2t-1 & 1-t & \cdots & 1 & 1 \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array} \] (n even)

We apply a sequence of complex root reflections to one or the other of these diagrams. When $n$ is odd, the sequence is

$s_{2t-l} \cdots s_{t+2} s_{t-1} s_t$. When $n$ is even the sequence is

$s_{2t-1} \cdots s_{t+2} s_{t-1} s_t$. In both cases the resulting diagram is

\[ \Delta^l_1 \begin{array}{ccc}
0 & 1 & \cdots & 1 & 1 & 0 \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array} \] \quad \alpha = 1 + \ldots + (n-2)
We let $\Delta_0^+ = s_n \Delta_1^+$ and use $\Delta_0^+$ to define $\nu_0$. The diagram is

\[ \begin{array}{cccccccc}
& 0 & 1 & \cdots & 1 & 0 & 0 & \alpha = 1 + \cdots + (n-2) + (n) \\
\end{array} \]

and the set of singular roots is $\{1, n-2, n-1\}$. The argument now proceeds just as with $SO^*(10)$ in Section 5, and the result is that $U(P, \sigma, \nu)$ is irreducible.

7. Groups of type $E$

For groups of type $E$, we shall consider 13 specific examples, of which 2 are in $E_6$, 5 are in $E_7$, and 6 are in $E_8$. The theorem is that $U(P, \sigma, \nu)$ is irreducible in all 13 cases. Our numbering of the roots in $E_8$ is

\[ \begin{array}{cccccccc}
0 & 1 & 3 & 4 & 5 & 6 & 7 \\
\end{array} \]

and we drop root 7 or roots 6 and 7 in $E_7$ or $E_6$.

For each example, we state what $\nu$ is, and we give two diagrams, the left one for the $m$ parameter $\lambda$ (with $2(\lambda, \beta)/|\beta|^2$ attached to the simple root $\beta$) and the right one for the infinitesimal character $\lambda + \nu$. The black simple roots are the noncompact ones, $\alpha$ is circled, and $P$ is built from $\alpha$.

In addition, we give the sequence of reflections used to pass to $\Delta_0^+$. A vertical line in the middle indicates the stage at which we apply Theorem 2.1. (Thus effectively we have only to implement wall crossings for reflections to the left of this line.) With each example we list the set of singular roots.

The 13 examples are listed below. After giving the list, we shall discuss the proof of irreducibility.
(a) $E_6$ with $\nu = \frac{3}{2}\alpha$.

$\lambda$:

\[
\begin{array}{c}
1 \\
\frac{1}{2} \\
0 \\
\frac{1}{2} \\
1
\end{array}
\]

$\lambda + \nu$:

\[
\begin{array}{c}
1 \\
-1 \\
3 \\
-1 \\
1
\end{array}
\]

Reflections: $s_2|s_0s_1s_5s_4$.

Singular set: $\{1,3,5\}$.

(b) $E_6$ with $\nu = 2\alpha$.

\[
\begin{array}{c}
1 \\
0 \\
0 \\
1 \\
1
\end{array}
\]

Reflections: $s_3s_2|s_0s_1s_5s_4$.

Singular set: $\{1,4,5\}$.

(c) $E_7$ with $\nu = 2\alpha$.

\[
\begin{array}{c}
1 \\
1 \\
0 \\
0 \\
1 \\
1
\end{array}
\]

Reflections: $s_2|s_0s_1s_3s_5$.

Singular set: $\{0,3,6\}$.

(d) $E_7$ with $\nu = 3\alpha$.

\[
\begin{array}{c}
1 \\
1 \\
0 \\
0 \\
1 \\
1
\end{array}
\]

Reflections: $s_2|s_0s_1s_5s_4$.

Singular set: $\{0,3,6\}$.
(e) $E_7$ with $v = \frac{5}{2} \alpha$.

Reflections: $s_3s_2s_5s_4$.  

Singular set: $\{1, 4, 6\}$.

(f) $E_7$ with $v = 2 \alpha$.

Reflections: $s_6s_3s_5s_4$.  

Singular set: $\{0, 1, 4, 5\}$.

(g) $E_7$ with $v = 3 \alpha$.

Reflections: $s_8s_3s_5s_4$.  

Singular set: $\{3, 5, 6\}$.

(h) $E_8$ with $v = \frac{5}{2} \alpha$.

Reflections: $s_2s_3s_4s_5$.  

Singular set: $\{0, 3, 7\}$.
(1) $E_8$ with $\nu = 3\alpha$.

Reflections: $s_3^a s_2^a | s_0 s_1 s_3 s_5^a$.
Singular set: $[1,4,7]$.

(2) $E_8$ with $\nu = \frac{5}{2}\alpha$.

Reflections: $s_2^a | s_1 s_3 s_5^a$.
Singular set: $[0,3,7]$.

(3) $E_8$ with $\nu = \frac{5}{4}\alpha$.

Reflections: $s_3^a s_2^a | s_0 s_1 s_3 s_5^a$.
Singular set: $[0,1,4,7]$.

(4) $E_8$ with $\nu = \frac{7}{2}\alpha$.

Reflections: $s_4 s_1 s_3 s_2^a | s_0 s_1 s_5 s_5^a$.
(m) $E_8$ with $\nu = 4\alpha$.

\[
\begin{array}{cccccccc}
0 & 1 & 0 & 1 & 1 & 1 & 1 & -4 \\
1 & 0 & 0 & 1 & 1 & 1 & 1 & 8 & -4 & 8 & -3 & 1 & 1 & 1
\end{array}
\]

Reflections: $s_3 s_5 s_6 s_1 s_2 s_3 s_4 s_5 s_6 s_4$. Singular set: \{2,4,6,7\}.

As we said, $U(\mathbf{P},\mathbf{e},\nu)$ is irreducible in all 13 cases. We turn to the proof. Let $\Delta^+$ be the positive system indicated above, let $\nu$ be the product of the reflections to the right of the vertical line, let $n$ be the number of reflections to the left of the vertical line, and put $\Delta^+_n = \nu \Delta^+$. Then define $\Delta^+_n, \ldots, \Delta^+_0$ by applying one at a time the reflections that are to the left of the vertical line. Then we can define $\gamma_0 = \lambda + \nu + \mu$ as in Section 1, and we can reflect it to obtain $\gamma_j$ dominant for $\Delta^+_n$, $0 \leq j \leq n$. Theorem 2.1 enables us to decompose $\pi(\gamma_n, \nu \rightarrow \alpha)$ as the sum of four irreducible characters.

In every case, the first reflection to the left of the vertical line is $s_2$. Moreover, the root $2$ is not in the $\tau$-invariant of any of the four irreducible characters, nor is it in the $\tau$-invariant of any constituent of any of the corresponding $\tau$'s for these four characters. Applying $s_2$, we then obtain a decomposition of $\pi(\gamma_{n-1}, \nu \rightarrow \alpha)$ as the sum of eight irreducible characters. (This is all very similar to what happened in Section 5.)

When $n=1$, we have only to check that the singular set meets the $\tau$-invariant of 7 of these 8 characters, and then we have irreducibility for $U(\mathbf{P},\mathbf{e},\nu)$. Thus we are done in cases (a), (c), (d), (h), and (j).

When $n=2$, the next (and last) reflection is $s_3$, and we prepare to apply $s_3$ to our expansion of $\pi(\gamma_1, \nu \rightarrow \alpha)$ into the sum of eight irreducible characters. The four characters that occurred also in $\pi(\gamma_2, \nu \rightarrow \alpha)$ have 3 in their $\tau$-invariants and may be
disregarded. In order to handle the 3 terms, it is necessary to
decompose the \( \tau \)'s that correspond to the four additional characters
that first appeared in \( \tau(\gamma_1, \sigma \leftrightarrow \alpha) \). Each of these \( \tau \)'s is seen to
be the sum of eight irreducible characters, all of whose \( \tau \)-invariants
meet the singular set. Therefore all of the 3 terms may be
disregarded. We thus need to consider only our four new characters
and their main new terms under \( U_3 \). Of these 3 characters, 7 have
\( \tau \)-invariants that meet the singular set. Thus \( U(P, \sigma, \nu) \) is
irreducible in cases (h), (e), (i), and (k).

In cases (g) and (l), we have \( n = 4 \). The reflections to the
left of the vertical line are \( s_4 s_1 s_3 s_2 \), and it is necessary to
calculate the decomposition of \( \tau(\gamma_2, \sigma \leftrightarrow \alpha) = s_3 s_2 \tau(\gamma_4, \sigma \leftrightarrow \alpha) \) exactly.
This calculation is more complicated in case (l) than in case (g) but
can be done without tools more advanced than Theorem 1.7. One handles
the last two wall crossings in the spirit of the previous paragraph,
discarding as early as possible any terms that will not affect the
final irreducibility. The details are fairly long and will be omitted,
but the result is that \( U(P, \sigma, \nu) \) is irreducible in cases (g) and (l).

Case (f) has \( n = 3 \). The reflections to the left of the vertical
line are \( s_5 s_3 s_2 \), and we calculate \( \tau(\gamma_1, \sigma \leftrightarrow \alpha) = s_3 s_2 \tau(\gamma_3, \sigma \leftrightarrow \alpha) \) as
in case (g) above. The resulting character identity involves ten
irreducible characters. The root 6 is not in the \( \tau \)-invariant of any
of these ten characters, nor is it in the \( \tau \)-invariant of any
constituent of the corresponding \( \tau \)'s for these ten characters.
Applying \( s_6 \), we obtain a decomposition of \( \tau(\gamma_0, \sigma \leftrightarrow \alpha) \) as the sum
of 20 irreducible characters. The singular set meets the \( \tau \)-invariant
of 19 of these 20 characters, and thus \( U(P, \sigma, \nu) \) is irreducible in
case (f).

Finally we consider case (m), in which \( n = 6 \) and the reflections
are \( s_5 s_4 s_3 s_2 s_1 s_2 \). Here we calculate \( \tau(\gamma_2, \sigma \leftrightarrow \alpha) = s_1 s_3 s_2 \tau(\gamma_6, \sigma \leftrightarrow \alpha) \)
exactly, using Theorem 1.7, and we calculate \( \tau(\gamma_2, \sigma \leftrightarrow \alpha) \)
= \pi(y_3, o \leftrightarrow o) \text{ except for two } \Theta \text{ terms. One handles the final two wall crossings in the spirit of the cases } n=2 \text{ and } n=4. \text{ The details, however, are much more complicated in this situation. But at any rate the result is that } U(P, \sigma, \nu) \text{ is irreducible in case (m).}
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