Imbedding Discrete Series in $L^2(G/H)$

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This report partly discusses joint work with M. W. Baldoni-Silva and partly carries out a suggestion of D. A. Vogan.

Let $G/H$ be a semisimple symmetric space, with $G$ linear connected semisimple and with $H$ the identity component of the group of fixed points of an involution. The problem is: For a generic discrete series representation of $L^2(G/H)$, with given Langlands parameters (in the classification of irreducible representations of $G$), find an explicit nonzero intertwining operator $C$ carrying the Langlands quotient into the realization in $L^2(G/H)$.

**Heuristic Principle.** A natural intertwining operator $T$ from $\text{ind}_H^G \pi_1$ to $\text{ind}_H^G \pi_2$ is given by some interpretation of

$$Tf(x) = \int_{H_2/(H_1 \cap H_2)} \pi_2(\delta) f(xh \delta) \, dh,$$

provided that $\pi_1$ and $\pi_2$ act on the same Hilbert space and are equal on $H_1 \cap H_2$ except for the natural combination of change-of-measure factors. This formula is to be valid, with a suitable interpretation, even when $H_2/(H_1 \cap H_2)$ has no invariant measure.

This formula, together with an analytic continuation, accounts for the standard Knapp-Stein operators.

It accounts also for a simplification in the search for $C$ in the present situation. Namely the Langlands realization is as a quotient of some $\pi = \text{ind}_H^G \text{MAN}(\sigma \otimes \varepsilon^* \otimes 1)$ for a parabolic subgroup $\text{MAN}$. Thus $H_1 = MAN$, $\pi_1 = \sigma \otimes \varepsilon^* \otimes 1$, $H_2 = H$, and $\pi_2 = 1$. For the compatibility property to apply directly, one needs $\sigma$ and $1$ to act on the same space. This situation is quite special. When it occurs, we can interpret $H/(H \cap MAN)$ as $H \cap K$, where $K$ is maximal compact in $G$, and $T$ becomes $Tf(x) = \int_{K} ...$.

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\[ \mathcal{E} \] we can take as the desired \( \mathcal{E} \). In general, with \( \sigma \) and \( 1 \) acting on different spaces, \( \mathcal{E} \) should involve as an additional ingredient a passage from \( \sigma \) to \( 1 \). The Reduction Theorem below will make precise the fact that the passage from \( \sigma \) to \( 1 \) is the only obstacle to finding \( \mathcal{E} \).

If the desired \( \mathcal{E} \) can be found, then the linear functional \( e \) on the analytic vectors of \( \pi = \text{ind}_{\text{MAX}}^G(\sigma \otimes e^\nu \otimes 1) \) given by \( e = (\text{evaluation at } 1) \circ \mathcal{E} \) satisfies

1. \( e \) is \( H \)-invariant \( (e(x) = e \text{ for } h \in H) \) and continuous
2. \( \{ e(\pi(x)f) \mid x \in G \} \) is bounded in \( C \) for each analytic vector \( f \)
3. \( e(\mathcal{E}^{-1}(\psi_\lambda)) \neq 0 \), where \( \psi_\lambda \) is Flensted-Jensen's generating element of the given discrete series of \( L^2(G/H) \).

Conversely any linear functional \( e \) on the analytic vectors of the correct \( \pi \) satisfying (1), (2), and

(3') \( e \neq 0 \)
determines a nonzero intertwining operator into \( L^2(G/H) \).

For a generic Flensted-Jensen parameter \( \lambda \), Schlichtkrull found the Langlands parameters of the discrete series representation of \( L^2(G/H) \) with Flensted-Jensen parameter \( \lambda \). Let \( g = h \oplus q \) be the decomposition of the Lie algebra of \( G \) according to the involution, and let \( g = t \oplus p \) be a compatible Cartan decomposition. Flensted-Jensen's parameter is a linear functional on a maximal abelian subspace \( t \) of \( t \cap q \) (and \( t \) is assumed maximal abelian in \( q \)). Let \( L \) be the centralizer of \( t \) in \( G \). Let \( A \) be the abelian factor of an Iwasawa decomposition of \( L \), and let \( MAN \) be a corresponding parabolic subgroup with \( N \) chosen suitably. Also let \( K \) be the compact subgroup corresponding to \( t \), and let \( \mu_\lambda \) be the highest weight of the \( K \)-type constructed from \( \lambda \) by Flensted-Jensen.

According to Schlichtkrull, the Langlands parameters are \( \nu = \rho_\lambda \) (half the sum of the positive restricted roots of \( L \), with multiplicities) and the discrete series \( \sigma \) of \( M \) constructed by the Vogan algorithm from the \( K \)-type \( \mu_\lambda \). The quotient \( M/(H \cap M) \) is a semisimple symmetric space, and the theorem below reduces the construction of the linear functional \( e \) for \( G \) to the construction of a corresponding linear functional \( l \) for \( M \).

**Reduction Theorem.** Let the discrete series \( \sigma \) act on \( V^\sigma \). Suppose \( l \) is a linear functional on the analytic vectors of \( V^\sigma \) such that

1. \( l \) is \( (H \cap M) \)-invariant \( (l(\sigma(h)) = l \text{ for } h \in H \cap M) \) and continuous
\((2_M)\) \(\{l(\sigma(m)v) \mid m \in M\}\) is bounded in \(C\) for each analytic vector \(v\).

\((3_M)\) \(l(v_{H \cap M \cap K}) \neq 0\) if \(v_{H \cap M \cap K}\) is a nonzero \(H \cap M \cap K\) fixed vector in the minimal \(M \cap K\) type of \(\sigma\).

Define \(\epsilon\) on the analytic vectors \(f\) of \(\text{ind}_{M \cap K}^{G}(\sigma \otimes e^v \otimes 1)\) by

\[\epsilon(f) = \int_{H \cap K} l(f(k)) \, dk.\]

Then \(\epsilon\) satisfies \((1), (2), (3)\) and defines the required nonzero intertwining operator into \(L^2(G/H)\).

One expects \(l\) to be defined by integration:

\[l(v) = \int_{H \cap M} \{\sigma(h)v, v_{H \cap M \cap K}\} \, dh.\]  \((*)\)

The following partial results support this expectation:

(a) The integral \((*)\) always converges for \(\sigma\) generic and \(v\) analytic, and \((1_M)\) holds.

(b) For \(l\) as in \((*)\), \((2_M)\) and \((3_M)\) hold when \(M\) is compact, when \(M/(H \cap M)\) is a group case, and when \(M/(H \cap M) = SO(2,1)/SO(1,1)\).

(c) For \(l\) as in \((*)\), \((3_M)\) holds if \(\sigma\) is a holomorphic discrete series representation of \(M\).