Irreducible Tempered Representations

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The group G ultimately will be a real semisimple Lie group of matrices, with K a maximal compact subgroup, and the actual problem is to classify the irreducible admissible representations of G. Langlands [5] was able to reduce this problem to the classification of those irreducible representations that are tempered. We say that the admissible representation π on a Hilbert space is <u>tempered</u> if $(\pi(g)u, v)$ is in $L^{2+\epsilon}(G)$ for every $\epsilon > 0$ and for all K-finite vectors u and v in the Hilbert space. We do not give the Langlands reduction but will concentrate on the tempered representations.

§1. Special case

We begin by describing tempered representations when G is $SL(2,\mathbb{R})$. The results in this case are due to Bargmann [1]. A typical element of G is $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We recall the definitions of the principal series and discrete series of unitary representations of G.

The representations of the principal series are parametrized by $(\pm,\mathbb{R})$. The Hilbert space for each is $L^2(\mathbb{R})$, and the group action is

$$\mathcal{P}^{\pm,it}(g)f(x) = \begin{cases} |bx+d|^{-1-it} f\left(\frac{ax+c}{bx+d}\right) & \text{if } + \\ \operatorname{sgn}(bx+d)\,|bx+d|^{-1-it} f\left(\frac{ax+c}{bx+d}\right) & \text{if } - . \end{cases}$$

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The representations of the discrete series are parametrized by $(\pm,n)$ with n an integer ≥ 2. "Discrete series" representations are those irreducible representations for which $(\pi(g)u,v)$ is in $L^2(G)$. For the representation with parameters $(+,n)$, the space is a space of functions analytic in the upper half plane, and the group action is

$$\mathcal{D}^{+,n}(g)f(z) = (bz+d)^{-n} f\left(\frac{az+c}{bz+d}\right).$$

$\mathcal{D}^{-,n}$ is given by using complex conjugates everywhere.

These representations are all tempered. The list $\{\mathcal{P}^{\pm,it}, \mathcal{D}^{\pm,n}\}$ fails to be a classification of irreducible tempered representations in three ways:

(1) $\mathcal{P}^{-,0}$ is reducible (the others are irreducible).

(2) $\mathcal{D}^{+,1}$ and $\mathcal{D}^{-,1}$ are missing from the list (limits of discrete series).

(3) There is redundance, $\mathcal{P}^{+,it} \cong \mathcal{P}^{+,-it}$ and $\mathcal{P}^{-,it} \cong \mathcal{P}^{-,-it}$.

Fact (3) has to do with a Weyl group action, is well understood, and will not concern us for now. The point is that facts (1) and (2) are related by the formula

$$\mathcal{P}^{-,0} \cong \mathcal{D}^{+,1} \oplus \mathcal{D}^{-,1}$$

and this formula is the only thing that prevents a naïve classification from working.

Before passing to the general case, we shall recast the results for $SL(2,\mathbb{R})$ in more group-theoretic fashion. This G has the two nonconjugate Cartan subgroups

$$H = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, a \in \mathbb{R}^{\times}\right\}, \quad B = \left\{\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\right\}.$$

From each we obtain a parabolic subgroup

$$H \to MAN \text{ with } M = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix} \text{ and } \epsilon = \pm 1,$$

$$A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \text{ and } a > 0,\ N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

$$B \to G .$$

The principal series representations can be viewed as induced from the subgroup MAN. The correspondence in notation is that

$$\pm \leftrightarrow \sigma = \text{one-dimensional representation of } M,\quad \sigma\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \pm 1,$$

$$t \leftrightarrow \nu,\quad \nu\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a^{it} \quad \text{for } a > 0 .$$

The data (σ, ν) thus provide us with a character of the Cartan subgroup H. Let $\mu\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = a$. The space for the induced representation (apart from technical considerations) is

$$\{f : G \to \mathbb{C} \mid f(manx) = \mu(a)\nu(a)\sigma(m)f(x)\ \}$$

and the G action is

$$\mathcal{P}^{\pm, it}(g)f(x) = f(xg) .$$

The equivalence with the previous definition of $\mathcal{P}^{\pm, it}$ is achieved by restriction of the functions on G to the subgroup $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$.

The discrete series representations are not induced but are intrinsic to G. They can be parametrized by the nonsingular integral linear forms on $i\mathfrak{b}$, where $\mathfrak{b}$ is the Lie algebra of B. Namely, the parametrization is $\begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} \to ik\theta$, $k \neq 0$, the parametrization having to do with the form of the character on B

$$\Theta(k, \operatorname{sgn} k) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \frac{-(\operatorname{sgn} k)e^{ik\theta}}{e^{i\theta} - e^{-i\theta}}$$

If $k > 0$, $\Theta(k, \operatorname{sgn} k)$ is the character of $\mathcal{D}^{+, k+1}$. The characters of

$\mathcal{D}^{\pm,1}$ are $\Theta(0,+)$ and $\Theta(0,-)$, and these add to the character of $p^{-,0}$. We get the two constituent characters of $p^{-,0}$, then, by using the discrete series characters and letting the parameter become singular from both directions. This character formulation is another sense in which $\mathcal{D}^{\pm,1}$ are "limits of discrete series."

§2. General case

For general G, fix a maximal compact subgroup K and let θ be the corresponding Cartan involution. From any θ-stable Cartan subgroup H, we can construct a parabolic subgroup MAN. Some properties of this parabolic subgroup are that

A = vector part of H

MA = centralizer of A in G

N = nilpotent group built from positive roots of (G, A)
("positive" depends on the choice of an ordering)

$H \cap K$ = compact Cartan subgroup of M, with Lie algebra $\mathfrak{h}^-$.

The group M is reductive (not necessarily semisimple) and may be mildly disconnected. Since it has a compact Cartan subgroup, a theorem [2] of Harish-Chandra's says it has discrete series and tells how to construct their characters. If δ denotes half the sum of the positive roots, the parameters are

λ = nonsingular linear form on $i\mathfrak{h}^-$ such that $\lambda+\delta$ is integral

η = character of center of M compatible with $e^{\lambda+\delta}$,

with $(\lambda, \eta) \sim (\lambda', \eta')$ if $\eta = \eta'$ and λ is in the same orbit as λ' under the Weyl group $W(M:H\cap K)$. The character of the discrete series representation with parameters (λ, η) is denoted

$\Theta^M(\lambda, C, \eta)$, where C = unique Weyl chamber of $i\mathfrak{h}^-$ with respect to which λ is dominant.

With limits of discrete series [7], the characters are of the same form, except that λ is allowed to be singular. The nice ones are nonsingular with respect to compact roots; the others contribute only redundance. A basic character is a character of the form

$$\mathrm{ind}_{MAN}^{G}\ \Theta^M(\lambda, C, \eta)\otimes e^{\nu}\otimes 1\ ,$$

where Θ^M is a limit of discrete series character and ν is an imaginary linear functional on the Lie algebra of A. Every basic character is tempered.

Theorem (Trombi [6], Langlands [5], Harish-Chandra). Every irreducible tempered character is a constituent of a basic character with Θ^M a discrete series character on M.

One knows how to check on the irreducibility of basic representations for which the Θ^M is a discrete series character. This check is done by means of a finite subgroup R of the Weyl group $W(G:A)$. The group R is always of the form $\Sigma\mathbb{Z}_2$, and its order tells how many constituents to look for. Each element of R points to a reduction formula like that in $SL(2,\mathbb{R})$. An example here will be more useful than the abstract theorem.

Example. Let G be the split real group whose complexification is simply-connected with Dynkin diagram D_4. We may take G to be a double covering of $SO_e(4,4)$. Start with H a split Cartan subgroup, so that all roots are real. In standard notation, we can denote the simple roots by e_1-e_2, e_2-e_3, e_3-e_4, e_3+e_4 .

We shall start with a representation that is induced from a character on the minimal parabolic subgroup $M_pA_pN_p$ that is trivial on A_p. Here M_p has order 16, and we can describe M_p explicitly as follows. For each root α, we can form a corresponding $SL(2,\mathbb{R})$

subgroup of G in a natural way, and we let γ_α be the element of G that corresponds to $\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$ in $SL(2,\mathbb{R})$. Each γ_α is of order 2, and M_p is the 16-element abelian group generated by the γ_α's with α simple.

Define a character σ of M_p by

$$\sigma(\gamma_{e_1-e_2}) = \sigma(\gamma_{e_3-e_4}) = \sigma(\gamma_{e_3+e_4}) = -1$$
$$\sigma(\gamma_{e_2-e_3}) = +1 \quad .$$

Form the induced representation using σ on M_p and 1 on A_p. Then the group R turns out to be

$$R = \{1,\ p_{e_1-e_2}p_{e_3-e_4},\ p_{e_1-e_2}p_{e_3+e_4},\ p_{e_3-e_4}p_{e_3+e_4}\} ,$$

where p_α denotes the Weyl group reflection corresponding to the root α.

The element $p_{e_1-e_2}p_{e_3-e_4}$ points to a new Cartan subgroup in which e_1-e_2 and e_3-e_4 are imaginary-valued; for the corresponding parabolic MAN, M has two components and its identity component is

$$M_0 = SL(2,\mathbb{R})\otimes SL(2,\mathbb{R}) \quad .$$

The character of our induced representation splits as the sum of basic characters

$$\mathrm{ind}_{MAN}^{G}[\Theta^M(0,C^+)\otimes 1\otimes 1 + \Theta^M(0,C^-)\otimes 1\otimes 1]$$

for suitable chambers C^+ and C^-. In turn,

$$\mathrm{ind}_{MAN}^{G}[\Theta^M(0,C^\pm)\otimes 1\otimes 1] \qquad (*)$$

has R-group $\{1, p_{e_3+e_4}\}$ and we are led to a third Cartan subgroup, in which e_1-e_2, e_3-e_4, and e_3+e_4 are imaginary-valued. In terms of this third Cartan subgroup, both basic characters (*) split again, and the result is that the original character is the sum of four irreducible basic characters.

Returning to the general case, one can formulate an explicit theorem as in [3] and [4] that generalizes the above example. In order to avoid introducing more notation, we shall confine ourselves to a qualitative statement of part of the result:

(1) Every irreducible tempered character is basic and can be written with data in a suitably nondegenerate form.

(2) A basic character with nondegenerate data is irreducible if and only if its R-group is trivial.

(3) Two basic characters with nondegenerate data are the same if and only if their sets of data are conjugate in the obvious sense.

References

[1] Bargmann, V., Irreducible unitary representations of the Lorentz group, Ann. of Math. (2) 48 (1947), 568-640.

[2] Harish-Chandra, Discrete series for semisimple Lie groups II, Acta Math. 116 (1966), 1-111.

[3] Knapp, A.W., and G. Zuckerman, Classification of irreducible tempered representations of semisimple Lie groups, Proc. Nat. Acad. Sci. USA 73 (1976), 2178-2180.

[4] ________, Classification theorems for representations of semisimple Lie groups, Non-Commutative Harmonic Analysis, Springer-Verlag Lecture Notes in Math. 587 (1977), 138-159.

[5] Langlands, R.P., On the classification of irreducible representations of real algebraic groups, mimeographed notes, Institute for Advanced Study, 1973.

[6] Trombi, P., The tempered spectrum of a real semisimple Lie group, Amer. J. Math. 99(1977), 57-75.

[7] Zuckerman, G., Tensor products of infinite-dimensional and finite-dimensional representations of semisimple Lie groups, Ann. of Math. (2), to appear.

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