Proceedings of the International Congress of Mathematicians
Vancouver, 1974

A Szegö Kernel for Discrete Series

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The Szegö kernel for the unit ball in C^m is a reproducing kernel that gives a formula for holomorphic functions in the ball in terms of their boundary values, namely

$$F(z) = \frac{(m-1)!}{2\pi^m} \int_{|\zeta|=1} \frac{f(\zeta)\, d\sigma(\zeta)}{(1 - \langle z, \zeta \rangle)^m}, \tag{1}$$

where f is the boundary function for F and $d\sigma$ is Lebesgue measure on the sphere. When $m = 1$, (1) easily transforms into the Cauchy integral formula. In dimension m, the formula extends to be defined on all f in L^2, always yielding holomorphic functions. If we identify holomorphic functions with their boundary values, the extended operator can be regarded as the orthogonal projection from L^2 to the holomorphic functions in L^2. This projection property characterizes the kernel.

In terms of semisimple Lie groups, functions on the sphere suggest nonunitary principal series representations and holomorphic functions on the ball suggest discrete series representations, and the Szegö kernel should suggest a map from the one to the other. Actually formula (1) will not arise with discrete series but with so-called limits of discrete series. For ordinary discrete series representations, we shall use operators that are more analogous to the formula for the $(n-1)$st complex derivative in the disc,

$$F(z) = \frac{(n-1)!}{2\pi i} \oint \frac{f(\zeta)\, d\zeta}{(\zeta - z)^n}, \qquad n > 1. \tag{2}$$

*Supported by NSF grant GP-42459. The author is pleased to acknowledge the extensive influence of conversations with K. Okamoto and early collaboration with E. M. Stein upon the developments in this paper.

Such operators are not projections but do carry general boundary functions into good functions with a known relationship to the original function.

We examine first the role of this mapping for the holomorphic discrete series of $G = \mathrm{SU}(1, 1) = \{(\begin{smallmatrix}\alpha & \beta \\ \bar{\beta} & \bar{\alpha}\end{smallmatrix}), |\alpha|^2 - |\beta|^2 = 1\}$. The holomorphic discrete series is a sequence of square-integrable unitary representations parametrized by an integer $n \geqq 2$. The space is

(3) $\{F(z)$ analytic in $\boldsymbol{C}^1$ for $|z| < 1 \mid \int_{|z|<1} |F(z)|^2(1 - |z|^2)^{n-2}\,dx\,dy < \infty\}$

with group action

$$U(g)F(z) = (\bar{\beta}z + \alpha)^{-n} F\left(\frac{\beta + z\bar{\alpha}}{\alpha + z\bar{\beta}}\right).$$

For the nonunitary principal series, let

$$A = \begin{pmatrix}\cosh r & \sinh r \\ \sinh r & \cosh r\end{pmatrix}, \quad N = \begin{pmatrix}1 + ix & -ix \\ ix & 1 - ix\end{pmatrix}, \quad K = \begin{pmatrix}e^{i\theta} & 0 \\ 0 & e^{-i\theta}\end{pmatrix}.$$

For a the indicated matrix in A, let $\lambda_s(a) = e^{r(s+1)}$ for complex s. Pick some complex s and one of "even" or "odd", say "even", for example. The space of a representation is the space of f defined on the circle K with just "even"-numbered Fourier coefficients. These functions extend uniquely to G by $f(ank) = \lambda_s(a)f(k)$, and the group action $U(g)f(x) = f(xg)$ preserves the space.

Fix $n \geqq 2$, use "even" or "odd" according to what n is, and put $s = \pm(n - 1)$. To fix the ideas, let us take n even. Then the nonunitary principal series representation is reducible. In terms of Fourier coefficients, the situation is as follows: At $s = -(n - 1)$, holomorphic discrete series number n arises as a quotient from coefficients $\{\cdots, -(n + 2), -n\}$, a finite-dimensional representation appears as an invariant subspace from coefficients $\{-(n - 2), -(n - 4), \cdots, (n - 2)\}$, and an antiholomorphic discrete series appears from coefficients $\{n, (n + 2), \cdots\}$. At $s = +(n - 1)$, the numbers are the same, but the roles of quotient and subspace are reversed.

The intertwining operators first constructed by Kunze and Stein [4] are equivariant maps of the representation at s to the one at $-s$, but at $s = -(n - 1)$ the operator has a simple pole. Work by Sally [8] yields an explicit formula for the residue operator in terms of Fourier coefficients, showing that the residue operator is equivariant and maps the nonunitary principal series representation to the sum of the two discrete series. If we compose with the projection to the holomorphic discrete series and reinterpret the principal series as suitable functions on $K/\boldsymbol{Z}_2$ and the discrete series in its form (3), we obtain (2) as the formula for the composition intertwining operator.

We shall attempt to extend these matters to arbitrary (not necessarily holomorphic) discrete series representations of the automorphism groups of noncompact hermitian symmetric spaces. Let G be a connected semisimple Lie group with finite center, let K be a maximal compact subgroup, and assume that G/K is hermitian. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra of

G, and let $\mathfrak{h} \subseteq \mathfrak{k}$ be a maximal abelian subspace. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and we let $\mathfrak{g}^C = \mathfrak{h}^C + \sum \mathfrak{g}_\alpha$ be the root-space decomposition. Here $\mathfrak{g}_\alpha = \boldsymbol{C}E_\alpha$, and we take the E_α to be normalized as in [1]. Introduce an ordering on the roots such that no sum of two noncompact positive roots is a root.

A parametrization of the discrete series was supplied by Harish-Chandra [3]. If we let ρ be half the sum of the positive roots and if, as we may, we assume that ρ is integral, then the parameter space is the set of integral forms $\Lambda + \rho$ on $\mathfrak{h}$ such that $\langle \Lambda + \rho, \alpha \rangle > 0$ for $\alpha > 0$ compact and $\langle \Lambda + \rho, \alpha \rangle \neq 0$ for $\alpha > 0$ noncompact. Since G/K is hermitian, it follows that every such Λ satisfies $\langle \Lambda, \alpha \rangle \geqq 0$ for $\alpha > 0$ compact.

The Langlands conjecture says that the above representation is realized in a certain $\bar{\partial}$-cohomology space, described as follows. Let $Q = \{\alpha > 0 \text{ noncompact} \mid \langle \Lambda + \rho, \alpha \rangle > 0\}$ and $q = |Q|$. To avoid bundles, introduce a symbol $\omega^{-\alpha}$ for each $\alpha > 0$, to be thought of as a $d\bar{z}$-type differential form. If $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_q)$ is any ordered q-tuple of arbitrary positive roots, let $\omega^{-\boldsymbol{\alpha}} = \omega^{-\alpha_1} \wedge \cdots \wedge \omega^{-\alpha_q}$. Consider expressions

$$F = \sum_{|\boldsymbol{\alpha}|=q} F_{\boldsymbol{\alpha}} \omega^{-\boldsymbol{\alpha}},$$

where each $F_{\boldsymbol{\alpha}}$ is a smooth scalar-valued function on G satisfying

$$F_{\boldsymbol{\alpha}}(hx) = \xi_{\Lambda+\alpha_1+\cdots+\alpha_q}(h) F_{\boldsymbol{\alpha}}(x) \quad \text{for } h \in \exp \mathfrak{h},\ x \in G.$$

(Here ξ denotes a character.) The $\bar{\partial}$-operator is given on functions (0-forms) by

$$\bar{\partial} F(x) = \sum_{\alpha > 0} E_{-\alpha} F(x) \omega^{-\alpha},$$

where $E_{-\alpha}$ denotes the right-invariant differentiation computed as the real part plus i times the imaginary part. Extend $\bar{\partial}$ by making $\bar{\partial}^2 = 0$, and let $\bar{\partial}^*$ be the formal adjoint relative to $L^2(G)$. Take ker $\bar{\partial} \cap$ ker $\bar{\partial}^*$ in dimension q as a dense subspace of a representation, and use the norm $(\sum \int_G |F_{\boldsymbol{\alpha}}|^2 \, dx)^{1/2}$. The group operation is a right translation of the coefficient functions. The Langlands conjecture is the statement that the completion of this constructed representation is the discrete series representation with parameter $\Lambda + \rho$. The conjecture is known for $G = \mathrm{SU}(1, 1)$ by easy computation, for general G and $q = 0$ by work of Harish-Chandra [2], for $|\langle \Lambda + \rho, \alpha \rangle| > c$ whenever α is noncompact by Narasimhan and Okamoto [5], and for $\langle \Lambda + \rho, \alpha \rangle > 0$ whenever $\alpha > 0$ is noncompact by Parthasarathy [7]. The proofs by Narasimhan and Okamoto and Parthasarathy are not constructive.

For the nonunitary principal series of G, let $G = ANK$ be an Iwasawa decomposition, let $M = Z_K(A)$, let σ be an irreducible representation of the compact group M, let λ be a linear functional on the Lie algebra of A, and let ρ^+ be half the sum of the positive restricted roots counted with multiplicities. The space consists of vector-valued functions f on K with $f(mk) = \sigma(m) f(k)$, which are then extended to G by the definition $f(ank) = \exp((\lambda + \rho^+) \log a) f(k)$. The group G acts by right translation.

Problem. Give an integral formula for passing from appropriate nonunitary principal series, realized as spaces of functions on K, to discrete series realized in the format of the Langlands conjecture.

Solving the problem would consist of four steps: (1) existence—producing an integral formula so that the members of the image satisfy $\bar{\partial} F = 0$ and $\bar{\partial}^* F = 0$, (2) finiteness—establishing the square-integrability of the K-finite members of the image, (3) uniqueness—proving irreducibility of the image, and (4) identification—computing the character of the image representation. We shall do (1) in general and (2) in some special cases. Steps (3) and (4) follow whenever (1) and (2) and the Langlands conjecture are known.

To define the Szegö kernel, let $\Lambda + \rho$ and Q and q be as before. The linear functional λ is yet to be specified, but we extend arbitrary *scalar-valued* functions f on K to G by $f(ank) = \exp((\lambda + \rho^+) \log a) f(k)$. We map a smooth f into $F = \sum_{|\alpha|=q} F_\alpha \omega^{-\alpha}$ with

$$(4) \quad F_\alpha(x) = \int_K \overline{S_\alpha(k)} f(kx)\, dk = \int_K \overline{S_\alpha(\kappa(kx^{-1}))} \exp((\rho^+ - \lambda) H(kx^{-1})) f(k)\, dk$$

and $S_\alpha(k) = (\tau_\Lambda(k)\phi_\Lambda, \phi_\Lambda)(\mathrm{Ad}(k)E_\alpha, E_Q)$. Here ϕ_Λ is a highest weight vector, and E_α and E_Q denote alternating tensors. This is a group-theoretic generalization of the formula in SU(1, 1) except that the Ad factor was not present in dealing with 0-forms. The first part of (4) shows the map is equivariant, but the second is more useful in computations. The formula for F_Q has been considered by Okamoto [**6**].

Two comments are in order before we state precise theorems. First, no special M-dependence of f is assumed in (4). However expansion of f in Fourier series on M and a change of variables show that only one particular representation of M plays a role. Thus the domain can be regarded as a single nonunitary principal series, but with a finite multiplicity. Second, the choice of A in the Iwasawa decomposition is not arbitrary. [In fact, assume the Langlands conjecture. Then the infinitesimal character of the nonunitary principal series is determined in one way by that of the discrete series and in another way by the character of A and the placement of M in K. These cannot match unless M is placed in K properly.]

Thus we must define A. Any two A's are conjugate by a member of K. Define a standard A_0 by $\mathfrak{a}_0 = \sum \boldsymbol{R}(E_{\alpha_i} + E_{-\alpha_i})$, where α_i is defined inductively as the largest noncompact positive root orthogonal to $\alpha_1, \cdots, \alpha_{i-1}$, and use the basis $\{E_{\alpha_i} + E_{-\alpha_i}\}$ to define an ordering. Let $A = pA_0p^{-1}$, where p is a member of the Weyl group of K yet to be specified.

THEOREM 1. *If p and Q satisfy a compatibility condition* $(*)$, *then to each Λ corresponds a unique $\lambda = \lambda(\Lambda)$ such that the image $F = \sum F_\alpha \omega^{-\alpha}$ satisfies $\bar{\partial} F = 0$ and $\bar{\partial}^* F = 0$ for each f in $C^\infty(K)$.*

We shall state $(*)$ shortly. Our finiteness theorem is as follows.

THEOREM 2. *Let $G = \mathrm{SU}(1, 1)$, or let G be general but $q = 0$. If $(*)$ holds and if $\lambda = \lambda(\Lambda)$, then the image of the trigonometric polynomials under* (4) *is exactly the space of K-finite elements in the cohomology space, and the cohomology space is the discrete series representation with parameter $\Lambda + \rho$.*

Also, by a lengthy computation, we have established square-integrability of the image of $f = S_Q$ for SU(2, 1) when $q = 1, \lambda = \lambda(\Lambda)$, and p is trivial.

To state condition $(*)$, let $\alpha(\Upsilon)$ be the function from positive noncompact roots to the α_i's defined above, given as the first α_i such that Υ is not orthogonal to α_i. Then $\alpha(\Upsilon) - \Upsilon$ is always a positive compact root or 0. Hence $p\alpha(p^{-1}\Upsilon) - \Upsilon$ is always a compact root or 0.

Condition $(*)$. Every positive noncompact root Υ not in Q satisfies $p\alpha(p^{-1}\Upsilon) - \Upsilon \geqq 0$. Also every positive noncompact root Υ in Q satisfies $p\alpha(p^{-1}\Upsilon) - \Upsilon \leqq 0$.

One can show case-by-case that to each Q corresponds some p such that p and Q satisfy $(*)$.

EXAMPLE. $G = \mathrm{SU}(m, 1)$. One can arrange that the positive compact roots are $e_i - e_j$, $i < j \leqq m$, and the positive noncompact roots are $e_i - e_{m+1}$, $i \leqq m$. The element p is a permutation of $\{1, \cdots, m\}$. Then A_0 is built out of $E_{e_1-e_{m+1}} + E_{-(e_1-e_{m+1})}$, and pA_0p^{-1} necessarily is built out of $E_{e_l-e_{m+1}} + E_{-(e_l-e_{m+1})}$ for some l. The set Q has the form $\{e_i - e_{m+1}, i \leqq q\}$. If $0 < q < m$, two choices of l (namely $l = q$ and $q + 1$) are such that p and Q satisfy $(*)$, and generally these lead to really distinct nonunitary principal series. If $q = 0$ or m, there is only one such choice (namely $l = 0$ or m, respectively). However, when $m = 2$ and $q = 1$, it is known that there are three nonunitary principal series, not two, with a given discrete series as quotient.

We conclude with the formula for $\lambda(\Lambda)$. Let

$$n_i^+ = \left|\{\Upsilon > 0 \text{ noncompact} \mid \Upsilon \in Q, \Upsilon \neq p\alpha_i, p\alpha(p^{-1}\Upsilon) = p\alpha_i\}\right|,$$

$$n_i^- = \left|\{\Upsilon > 0 \text{ noncompact} \mid \Upsilon \notin Q, \Upsilon \neq p\alpha_i, p\alpha(p^{-1}\Upsilon) = p\alpha_i\}\right|.$$

Then λ is determined by the values of $\lambda' = \rho^+ - \lambda$ on all $\mathrm{Ad}(p)(E_{\alpha_i} + E_{-\alpha_i})$. If $p\alpha_i$ is not in Q,

$$\lambda'(\mathrm{Ad}(p)(E_{\alpha_i} + E_{-\alpha_i})) = -\left|\alpha_i\right|^{-1}\sqrt{2}\,\langle \Lambda + Q - n_i^+ p\alpha_i, p\alpha_i\rangle.$$

If $p\alpha_i$ is in Q,

$$\lambda'(\mathrm{Ad}(p)(E_{\alpha_i} + E_{-\alpha_i})) = \left|\alpha_i\right|^{-1}\sqrt{2}\,\langle \Lambda + Q + n_i^- p\alpha_i, p\alpha_i\rangle.$$

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