A QUICK PROOF OF THE CLASSIFICATION
OF SIMPLE REAL LIE ALGEBRAS

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(Communicated by Roe W. Goodman)

Abstract. Élie Cartan’s classification of the simple Lie algebras over \( \mathbb{R} \) is derived quickly from some structure theory over \( \mathbb{R} \) and the classification over \( \mathbb{C} \).

Élie Cartan classified the simple Lie algebras over \( \mathbb{R} \) for the first time in 1914. There have been a number of simplifications in the proof since then, and these are described in [3, p. 537]. All proofs assume the classification over \( \mathbb{C} \) and a certain amount of structure theory over \( \mathbb{R} \). Recent proofs tend to run to 25 pages. Here is a shorter argument.

**Theorem.** Up to isomorphism, the only simple Lie algebras over \( \mathbb{R} \) that are neither complex nor compact are those in Cartan’s list as organized in [3, p. 518].

We use terminology as in [3]. Let \( g_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0 \) be a Cartan decomposition of a noncomplex simple Lie algebra over \( \mathbb{R} \), and let \( \theta \) be the Cartan involution. Choose a maximal abelian subspace \( t_0 \) of \( \mathfrak{k}_0 \) and extend to a maximally compact Cartan subalgebra \( h_0 = t_0 \oplus \mathfrak{a}_0 \) of \( g_0 \). Removal of subscripts 0 will indicate complexifications. Let \( \Delta = \Delta(g, h) \) be the root system. Roots are imaginary on \( t_0 \) and real on \( \mathfrak{a}_0 \). All roots are imaginary-valued or complex on \( h_0 \); there are no real-valued roots. Introduce a positive system \( \Delta^+ \) that takes \( it_0 \) before \( \mathfrak{a}_0 \). The map \( \theta \) carries roots to roots and permutes the simple roots. The complex simple roots move in two-element orbits, while the imaginary simple roots are fixed. By the Diagram of \( (g_0, h_0, \Delta^+) \), we mean the Dynkin diagram of \( \Delta \) with the two-element orbits under \( \theta \) so labeled and with the imaginary roots shaded or not, according as the simple root is noncompact (root vector in \( \mathfrak{p} \)) or compact (root vector in \( \mathfrak{k} \)).

**Lemma 1.** If \( (g_0, h_0, \Delta^+) \) and \( (g'_0, h'_0, (\Delta')^+) \) have the same Diagram, then \( g_0 \) and \( g'_0 \) are isomorphic.

**Proof.** We may assume that the complexifications \( (g, h, \Delta^+) \) are the same and that the associated compact forms are the same: \( u_0 = t_0 \oplus i\mathfrak{p}_0 = t'_0 \oplus i\mathfrak{p}'_0 \). Using the conjugacy of compact forms, the conjugacy of maximal abelian subspaces within them, and the standard construction of a compact form from \( h \), we see that we can normalize root vectors \( X_\alpha, \alpha \in \Delta \), as in Theorem 5.5 of [3, p. 176] and obtain \( u_0 \) from \( \{X_\alpha\} \) as in Theorem 6.3 of [3, p. 181].

Received by the editors April 12, 1995.

1991 Mathematics Subject Classification. Primary 17B20, 22E15.

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First suppose $a_0 = 0$, so that all roots are imaginary. For $\alpha$ simple we have $\theta X_\alpha = \pm X_\alpha$, the sign being $+$ if $\alpha$ is compact and $-$ is noncompact. The same formula holds for $\theta'$. Since $\theta$ and the $X_\alpha$'s for $\alpha$ simple generate $g$, it follows that $\theta = \theta'$, hence that $t = t'$ and $p = p'$. Then $g_0 = g_0'$ is recovered as $(u_0 \cap t) \oplus (u_0 \cap p)$.

If $a_0 \neq 0$, we may not have $\theta = \theta'$. For $\alpha \in \Delta$, write $\theta X_\alpha = a_0 X_{\theta \alpha}$. Then $a_0 \theta - \alpha = 1$ and $a_0 a_\theta \alpha = 1$. Since $\theta$ maps $u_0 \cap \text{span}\{X_\alpha, X_{-\alpha}\}$ to $u_0 \cap \text{span}\{X_{\theta \alpha}, X_{-\theta \alpha}\}$, we see that $a_\alpha = a_\alpha$. Therefore $|a_\alpha| = 1$. For each pair of complex simple roots $\alpha$ and $\theta \alpha$, choose square roots $a_\alpha^{1/2}$ and $a_{\theta \alpha}^{1/2}$ whose product is 1. Similarly write $\theta' X_\alpha = b_\alpha X_{\theta \alpha}$ with $|b_\alpha| = 1$, and define $b_{\alpha}^{1/2}$ and $b_{\theta \alpha}^{1/2}$ for $\alpha$ and $\theta \alpha$ simple. Define $H$ and $H'$ in $\mathfrak{h}$ by $u_0$ by the conditions that $\alpha(H) = \alpha(H') = 0$ for $\alpha$ simple imaginary and that $\exp(\frac{1}{2}\alpha(H)) = a_\alpha^{1/2}$, $\exp(\frac{1}{2}\theta \alpha(H)) = a_{\theta \alpha}^{1/2}$, $\exp\left(\frac{1}{2}\alpha(H')\right) = b_\alpha^{1/2}$, and $\exp\left(\frac{1}{2}\theta \alpha(H')\right) = b_{\theta \alpha}^{1/2}$ if $\alpha$ and $\theta \alpha$ are complex simple. A little computation shows that $\theta' \circ \text{Ad}(\exp \frac{1}{2}(H - H')) = \text{Ad}(\exp \frac{1}{2}(H - H')) \circ \theta$, from which it follows that $t' = \text{Ad}(\exp \frac{1}{2}(H - H'))t$, $p' = \text{Ad}(\exp \frac{1}{2}(H - H'))p$, and $g_0' = \text{Ad}(\exp \frac{1}{2}(H - H'))g_0$.

The next step is to identify some pairs of distinct Diagrams that correspond merely to changes of $\Delta^+$. The argument is inspired by [2]. First let us assume that $a_0 = 0$, i.e., that the automorphism of $\Delta$ given by $\theta$ is the identity. Let $\Lambda$ be the subset of $i u_0$ where all roots take integer values and where all noncompact roots take odd-integer values. If $\{\omega_j\}$ is the basis dual to the simple roots, then the sum of those $\omega_j$ corresponding to the noncompact simple roots is a member of $\Lambda$. The set $\Lambda$ is discrete, and we let $H_0$ be a member of $\Lambda$ as close to 0 as possible.

**Lemma 2.** If $(\Delta^+)'$ is a positive system that makes $H_0$ dominant, then there is at most one noncompact simple root, say $\alpha_i$. If the basis dual to the simple roots of $(\Delta^+)'$ is $\{\omega_j\}$, then there cannot exist $i'$ such that $\langle \omega_i - \omega_i', \omega_{i'} \rangle > 0$.

**Proof.** Since $H_0$ is in $\Lambda$ and is dominant, $H_0 = \sum n_j \omega_j$ with all $n_j$ integers $\geq 0$. If $n_i > 0$, then $H_0 - \omega_i$ is dominant and thus has $\langle H_0 - \omega_i, \omega_i \rangle \geq 0$ with equality if and only if $H_0 = \omega_i$. Then $|H_0 - 2\omega_j|^2 \leq |H_0|^2$ with equality only if $H_0 = \omega_j$, and minimality forces $H_0 = \omega_i$. Now let $H_0 = \omega_i$. If $\langle \omega_i - \omega_i', \omega_{i'} \rangle > 0$, then $|H_0 - 2\omega_{i'}|^2 < |H_0|^2$, in contradiction to minimality.

When $a_0 \neq 0$, Lemma 2 is to be applied to the part of $i u_0$ corresponding to the span of the imaginary simple roots. The result is that we can associate to any $g_0$ at least one Diagram in which at most one imaginary root is shaded.

Now we can read off the possibilities. First suppose that the automorphism of $\Delta$ is the identity. If all roots are unshaded, then $g_0$ is the compact form. Otherwise exactly one simple root is shaded. For the classical Dynkin diagrams, let the double line or triple point be at the right end, and let the $i$th root be shaded. In $A_n$, we are led to $\mathfrak{su}(i, n + 1 - i)$. In $B_n$, we are led to $\mathfrak{so}(2i, 2n + 1 - 2i)$. In $C_n$, we are led to $\mathfrak{sp}(i, n - i)$ if $i < n$ and to $\mathfrak{sp}(n, \mathbb{R})$ if $i = n$. In $D_n$, we are led to $\mathfrak{so}(2i, 2n - 2i)$ if $i \leq n - 2$ and to $\mathfrak{so}^*(2n)$ otherwise.

For the exceptional Dynkin diagrams, a little checking that compares the second conclusion of Lemma 2 with the fundamental weights (see [1, pp. 260-275]) shows that $\alpha_i$ in Lemma 2 has to be a node (endpoint vertex) of the Dynkin diagram. Moreover, in $G_2$, $\alpha_1$ has to be the long simple root, while in $E_8$, it cannot be the node on the short branch. In $E_6$ two nodes are equivalent by outer automorphism. Thus we obtain at most three Lie algebras for $E_7$; at most two for $E_6$, $E_8$, $F_4$; and
at most one for $G_2$. These are $E_{II}$, $E_{III}$ for $E_6$, $E_{V}$, $E_{VI}$, $E_{VII}$ for $E_7$; $E_{VIII}$, $E_{IX}$ for $E_8$; $F_1$, $F_{II}$ for $F_4$; and $G$ for $G_2$.

When the automorphism of $\Delta$ is not the identity, the Dynkin diagram is $A_n$, $D_n$, or $E_6$. For $A_n$, there is no imaginary simple root if $n$ is even, and there is one if $n$ is odd. For $n$ even we are led to $\mathfrak{sl}(n+1, \mathbb{R})$, while for $n$ odd we are led to $\mathfrak{sl}(n+1, \mathbb{R})$ if the root is shaded and to $\mathfrak{su}^*(n+1)$ if the root is unshaded. For $D_n$, the first $n-2$ simple roots are imaginary. If all are unshaded, we are led to $\mathfrak{so}(1, 2n-1)$. If the $i^{th}$ simple root is shaded, $i \leq n-2$, we are led to $\mathfrak{so}(2i+1, 2n-2i-1)$. For $E_6$, the triple point and the node on the short branch are imaginary. If neither is shaded, we are led to $E_{IV}$, while if either one is shaded, we are led to $E_{I}$.

Note added in proof. David Vogan has pointed out that any Dynkin diagram marked with an involution and having a subset of its one-element orbits shaded is a Diagram for some $\mathfrak{g}_0$. The proof is in the spirit of Lemma 1. Existence of the exceptional simple real Lie algebras follows.

REFERENCES


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