Group Representations and Harmonic Analysis from Euler to Langlands, Part II

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The essence of harmonic analysis is to decompose complicated expressions into pieces that reflect the structure of a group action when there is one. The goal is to make some difficult analysis manageable.

In the seventeenth and eighteenth centuries, the groups that arose in this connection were the circle $\mathbb{R}/2\pi\mathbb{Z}$, the line $\mathbb{R}$, and finite abelian groups. Embedded in applications were decompositions of functions in terms of *multiplicative characters*, continuous homomorphisms of the group into the nonzero complex numbers. In the case of the circle, the decomposition is just the expansion of a function on $(-\pi, \pi)$ into its Fourier series

$$(1) \qquad \begin{aligned} f(x) &\sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \\ c_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx. \end{aligned}$$

In the case of the line, the decomposition is given by the Fourier transform and the Fourier inversion formula, which for sufficiently good functions we write as

$$(2) \qquad \begin{aligned} \hat{f}(y) &= \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy}\, dx, \\ f(x) &= \int_{-\infty}^{\infty} \hat{f}(y) e^{2\pi ixy}\, dy. \end{aligned}$$

And in the case of a finite abelian group G, the expansion is simply

$$(3) \qquad f(x) = \frac{1}{|G|} \sum_{\omega} \Big[\sum_{y \in G} f(y)\overline{\omega(y)} \Big] \omega(x),$$

the sum being taken over all multiplicative characters of the group.

Multiplicative characters are less helpful in exploiting a nonabelian group of symmetries because a multiplicative character must send every commutator $xyx^{-1}y^{-1}$ into 1. To be able to do harmonic analysis with nonabelian groups, one introduces a multidimensional generalization

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of multiplicative character, the group representation.

A *representation* of a group G on a complex vector space V is a group action of G on V by linear transformations, i.e., a homomorphism of G into the group of invertible linear transformations on V. Often the group G and the vector space V are topologized, and the group action is then normally assumed to be continuous. A multiplicative character ω gives a representation on the 1-dimensional space $\mathbb{C}$ of complex numbers, the action by $g \in G$ being multiplication by $\omega(g)$.

The end of the nineteenth century was a period when Lie and Klein were leading mathematicians and when group actions were being intensely studied, including group actions by linear fractional transformations. In this atmosphere it is natural to expect that people would have looked at group actions by linear transformations as well, thereby discovering group representations. But this is not at all how group representations were introduced.

Finite Groups

In his work on algebraic number theory, Dedekind noticed a curious thing about finite abelian groups. Let $G = \{g_1 = 1, g_2, \ldots, g_h\}$ be a finite group of order h, and let $x_{g_1}, \ldots, x_{g_h}$ be commuting independent variables parametrized by the elements of G. Dedekind worked with the determinant $\theta(x_{g_1}, \ldots, x_{g_h})$ of the matrix $(x_{g_i g_j^{-1}})$, and in the abelian case he proved that θ admits a factorization

$$\theta(x_{g_1}, \ldots, x_{g_h}) = \prod_{\chi} \Big(\sum_{j=1}^{h} \chi(x_{g_j}) x_{g_j} \Big),$$

the product being taken over all multiplicative characters of G.

Dedekind wondered to Frobenius how this result might generalize to the nonabelian case, and Frobenius ([4], vol. III) began his work in representation theory in 1896 by introducing (irreducible) characters for arbitrary finite groups and solving Dedekind's problem. Today a character is the trace of a representation, but Frobenius did not introduce representations right away. Instead, doing mathematics that looks strange today, he initially worked directly with characters, introducing finite-dimensional representations only in a later paper.

Burnside, starting in 1904, and the young I. Schur, ([13], vol. I), starting in 1905, each redid the theory, the primary objects of each study being matrix representations (homomorphisms into the group of invertible matrices of some size). According to E. Artin ([1], p. 528), "It was Emmy Noether who made the decisive step. It consisted in replacing the notion of a matrix by the notion for which the matrix stood in the first place, namely, a linear transformation of a vector space." Noether's definition was thus essentially the modern general definition of representation given above. For Burnside and Schur the spaces of representations were spaces $V = \mathbb{C}^n$ of column vectors, and the linear transformations were viewed as matrices. Later when representation theory was extended to Lie groups and when quantum mechanics forced infinite-dimensional representations into the study, it would have been awkward to proceed without Noether's viewpoint.

Two finite-dimensional representations of G, π on V and π' on V' are *equivalent* if there is an invertible linear map $E : V \to V'$ such that $\pi'(g)E = E\pi(g)$ for all $g \in G$. An *invariant subspace* U for π is a vector subspace such that $\pi(g)U \subseteq U$ for all $g \in G$. The finite-dimensional representation π is said to be *irreducible* if V has no proper nonzero invariant subspaces.

The outcome of the work of Burnside and Schur, partly reworded in terms of linear transformations, was an abstract theory establishing principles for finite groups that in retrospect one might look for in other settings:

(P1) (Unitarity and complete reducibility). Every finite-dimensional representation is equivalent to a representation by unitary matrices. Then the orthogonal complement of an invariant subspace is invariant, and it follows that every finite-dimensional representation is the direct sum of irreducible representations. (These conclusions were already known; Burnside's contribution was to observe that complete reducibility is a consequence of unitarity.)

(P2) (Schur's Lemma). If π on V and π' on V' are irreducible representations and $E : V \to V'$ is a linear map such that $\pi'(g)E = E\pi(g)$ for all $g \in G$, then $E = 0$ or E is invertible. If $V = V'$, then E is scalar. (The first conclusion is due to Burnside, the second to Schur.)

(P3) (Schur orthogonality). If π and π' are inequivalent irreducible unitary representations, then

$$\sum_{g \in G} \pi_{ij}(g)\overline{\pi'_{kl}(g)} = 0.$$

Also

$$\frac{1}{|G|} \sum_{g \in G} \pi_{ij}(g)\overline{\pi_{kl}(g)} = \begin{cases} 1/\dim \pi & \text{if } (i,j) = (k,l) \\ 0 & \text{otherwise.} \end{cases}$$

(P4) (Fourier inversion). Let π vary through a complete set of inequivalent irreducible unitary representations of G. If f is a complex-valued function on G, define $\pi(f) = \sum_{x\in G} f(x)\pi(x)$. Then

$$f(1) = \frac{1}{|G|}\sum_{\pi}(\dim\pi)\mathrm{Trace}(\pi(f)).$$

(P5) (Completeness). Let π vary through a complete set of inequivalent irreducible unitary representations of G. If f is a complex-valued function on G, define $\pi(f) = \sum_{x\in G} f(x)\pi(x)$. Then

$$\sum_{x\in G}|f(x)|^2 = \frac{1}{|G|}\sum_{\pi}(\dim\pi)\|\pi(f)\|^2,$$

where $\|\cdot\|$ denotes the Hilbert-Schmidt norm (the square root of the sum of the absolute value squared of the entries).

To do ordinary harmonic analysis with a particular finite group is only a little more complicated than in the finite abelian case. We can illustrate some of the five principles above with the symmetric group on three letters. For this group G, there are three inequivalent irreducible representations, of dimensions 1, 1, and 2. They are the trivial representation 1, the sign representation, and the representation π on the plane obtained by placing an equilateral triangle with its center at the origin and considering the effect of permuting the vertices. For the 2-dimensional representation π, suppose that the vertices in terms of polar coordinates are $(1, 0°)$, $(1, 120°)$, $(1, 240°)$, numbered $1, 2, 3$. We convert each linear transformation $\pi(g)$ to a matrix, using the standard basis, and obtain

$$\pi((1\,2)) = \begin{pmatrix}\cos 120° & \cos 30° \\ \sin 120° & \sin 30°\end{pmatrix} \quad\text{and}$$
$$\pi((2\,3)) = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix},$$

with π given by a corresponding product on each of the other permutations. We can view the entries as functions on G as follows:

$g \setminus$ entry	$\pi_{11}(g)$	$\pi_{12}(g)$	$\pi_{21}(g)$	$\pi_{22}(g)$
(1)	1	0	0	1
(1 2 3)	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/2$	$-1/2$
(1 3 2)	$-1/2$	$\sqrt{3}/2$	$-\sqrt{3}/2$	$-1/2$
(1 2)	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/2$	$1/2$
(2 3)	1	0	0	-1
(1 3)	$-1/2$	$-\sqrt{3}/2$	$-\sqrt{3}/2$	$1/2$

For the sign representation the corresponding entries as a function of g are $1, 1, 1, -1, -1, -1$, and for the trivial representation they are all 1. Direct computation shows that the six columns are mutually orthogonal. The displayed columns

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Ferdinand Georg Frobenius

have norm squared equal to 3, and the columns for the sign and trivial representations have norm squared equal to 6. This is (P3). Because of the orthogonality the six columns form a basis for the 6-dimensional space of complex-valued functions on G, and (P5) follows from linear algebra. In a sense (P4) and (P5) are equivalent: Define convolution on G by $f * h(x) = \sum_{y\in G} f(xy^{-1})h(y)$. Then (P5) amounts to (P4) applied to the function $f * f^*$, where $f^*(x) = \overline{f(x^{-1})}$. So (P5) is a special case of (P4). But the functions $f * f^*$ span the space of all functions, and therefore the special case (P5) implies the general case (P4). This example is worked out in more detail in Gross [5].

Another part of the abstract theory is the idea of an induced representation, which is due to Frobenius. Induction is a way of forming a representation of G from a representation of a subgroup H. Let φ be a representation of H on a space V^{φ}. Then the *induced representation* $\pi = \mathrm{ind}_H^G\,\varphi$ acts in the vector space

$$\{f : G \to V^{\varphi} \mid f(xh) = \varphi(h)^{-1}(f(x)), h \in H\}$$

by $(\pi(g)f)(x) = f(g^{-1}x)$. If φ is the trivial representation of H, then π is the *left regular* representation of G on functions on G/H, i.e., the representation l given by $(l(g)f)(x) = f(g^{-1}x)$. Induced representations of finite groups play a role with Artin L functions, which we shall discuss shortly.

Making use of harmonic analysis with a particular finite group does require knowing the irreducible representations of the group, or at least their characters. These were worked out over a period of time for the symmetric and alternating groups by Frobenius and Young independently. "Young diagrams" remain the standard device for manipulating such representations.

One of the first serious applications of the representation theory of finite groups to something other than representation theory was the following theorem of Frobenius (1901): A transitive permutation group on n symbols whose operations other than the identity move all or all but one of the symbols contains a normal subgroup of order n. Another early application was the theorem of Burnside (1904) that any group of order $p^a q^b$ is solvable if p and q are prime. After those early results, representation theory continued to play a key role at various stages in the classification of finite simple groups.

Another application of the representation theory of finite groups occurs with Artin L functions, which Artin introduced in the 1920s. An Artin L function over the rationals $\mathbb{Q}$ encodes in a generating function information about how an irreducible monic polynomial over $\mathbb{Z}$ factors when reduced modulo each prime. For the polynomial x^2+1, the L function is

$$L(s, \mathbb{Q}(i)/\mathbb{Q}, \mathrm{sgn}) = \prod_{p \text{ odd prime}} \frac{1}{1-\left(\frac{-1}{p}\right)p^{-s}}, \tag{4}$$

where $\left(\frac{-1}{p}\right)$ is the Legendre symbol that yields $+1$ if -1 is a square modulo p and yields -1 if not. This L function is subtly different from one introduced by Euler, in which $\left(\frac{-1}{p}\right)$ is replaced by an expression $\chi^-(p)$ that is $+1$ or -1 according as p is congruent to 1 or 3 modulo 4. The fact that $\left(\frac{-1}{p}\right) = \chi^-(p)$ is well known as a preliminary case of quadratic reciprocity, and thus Euler's L function and (4) are equal. This role for reciprocity admits a vast generalization, in which representation theory predominates, and we shall return to this matter a little later. But let us see where representation theory enters the very definition of Artin L functions. A more general Artin L function encodes certain information about prime ideals in the ring of integers of a number field (finite extension of $\mathbb{Q}$). The L function depends on a complex parameter s, a finite Galois extension K/k of number fields, and a representation of the (finite) Galois group of K over k. The exact definition, which generalizes (4), will not concern us. However, when $k = K$ and the representation is trivial, the L function reduces to what is called the ζ function of K. Induced representations play an important role in understanding L functions. The L function does not change when k is replaced by a smaller field k_0 and the representation is replaced by the induced representation from $\mathrm{Gal}(K/k)$ to $\mathrm{Gal}(K/k_0)$. Taking $k = K$, we see that the ζ function of K equals the Artin L function for K/k_0 and the left regular representation of $\mathrm{Gal}(K/k_0)$ on functions on $\mathrm{Gal}(K/k_0)$. Decomposition of this representation into irreducible summands gets reflected in a factorization of the ζ function of K into a product of L functions. Thus Artin L functions are canonical factors of ζ functions of number fields, and they arise naturally by applying the representation theory of the Galois group.

Lebesgue Integration, Fourier Series, and Fourier Transform

At almost the same time as the development of representation theory for finite groups, the theory of Fourier series and the Fourier transform began to expand rapidly. The impetus was the introduction of the Lebesgue integral in Lebesgue's 1902 thesis and his 1904 book. The version of (P4) that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ if $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ and f is continuous was already known, as was the significance of (P5) Parseval's equality $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2$ for the completeness of the system of exponentials. But the Lebesgue integral paved the way for the Riesz-Fischer Theorem in 1907 that any square-summable sequence $\{c_n\}_{n=-\infty}^{\infty}$ is the sequence of Fourier coefficients of an L^2 function on $(-\pi, \pi)$, thus for a full understanding that the Fourier coefficient mapping $f \mapsto \{c_n\}$ is an isometric linear map of one L^2 space onto another.

Plancherel proved a version of (P5) for the Fourier transform in 1910, and all later generalizations of this property have been called the *Plancherel formula*. In the notation of (2), his result was that the Fourier transform mapping $f \mapsto \widehat{f}$ on $L^1 \cap L^2$ satisfies $\|\widehat{f}\|_2^2 = \|f\|_2^2$ and that the Fourier transform therefore extends to an isometric mapping of L^2 into L^2. Because the inversion formula in (2) is of the same type as the transform itself, the Fourier transform was then automatically onto L^2, and the formalism needed for harmonic analysis was all in place.

Historically the first operator using the Lebesgue integral significantly that could be written with Fourier series in the form

$$T\left(\sum c_n e^{inx}\right) = \sum b_n c_n e^{inx} \tag{5}$$

seems to be the Hilbert transform. Here $b_n = -i \,\mathrm{sgn}\, n$. This operator arises by regarding

f as a function on the unit circle, using the Poisson Integral Formula to obtain a harmonic function on the unit disc, passing to the conjugate harmonic function normalized to be 0 at the origin, and finally taking boundary values. This study was carried out independently by Privalov (1918) and Plessner (1923), and a version for the Fourier transform and the half plane may be handled similarly. In 1927 M. Riesz proved that the Fourier-series Hilbert transform is bounded on L^p if $1 < p < \infty$, and it follows easily that the partial sums of the Fourier series of L^p functions converge to the original function in L^p if $1 < p < \infty$. A similar boundedness result is valid for the version of the Hilbert transform appropriate to the Fourier transform and the half plane. More complicated operators commuting with translations were studied beginning in the 1930s, and the subject expanded into several variables. For the Fourier transform in $\mathbb{R}^n$, for example, the inversion formula is as in (2), but with integrations taken over $\mathbb{R}^n$ and with xy replaced by the dot product $x \cdot y$. The books of Zygmund [18] and Stein [14] give expositions of these theories. One result worthy of special note because of the way it was adapted later is Bochner's Theorem of 1932 on positive definite functions. A *positive definite function* f on a group G is one for which the matrix $\{f(x_i x_j^{-1})\}$ is always positive semidefinite Hermitian. The theorem is that among the continuous functions on $\mathbb{R}^n$, the positive definite functions are exactly those functions that are Fourier transforms of finite measures.

Work of Gårding in 1953 combined the Fourier transform on $\mathbb{R}^n$ with an earlier invention, a "freezing principle", to extend the scope of the Fourier transform to situations that do not exhibit symmetry under a group. "Gårding's inequality" gives a lower bound for the inner product (Lu, u), where L is a linear elliptic real differential operator of order m and where u has compact support. Use of the Plancherel formula handles the case that all terms are of order m and have constant coefficients. Behavior of a general operator near a point is approximated by behavior of one of these special operators with coefficients constantly equal to the value of the leading coefficients at that point (thus the "freezing principle"), and such estimates are pieced together with a partition of unity. The book of Bers, John, and Schechter [3] recites the details. The idea of a freezing principle in this context is one motivation for the more modern theory of pseudodifferential operators and its generalizations. The freezing principle will come up again when we consider nilpotent Lie groups.

AMS files

Hermann Weyl

Compact Groups

Early in the twentieth century all that was needed to extend parts of representation theory from finite groups to compact groups was invariant integration, and this was already in place for the rotation groups and the unitary groups in 1897 in a paper of A. Hurwitz. The abstract theory and identification of irreducible representations came side by side. Schur observed in 1924 that (P1), (P2), and (P3) extend as soon as one has invariant integration, the sums over G being replaced by integrals and $|G|$ being replaced by the total volume. Also Schur worked out the irreducible representations of the rotation groups and the unitary groups.

Already in 1913 É. Cartan had proved by algebraic means the Theorem of the Highest Weight, which classifies the irreducible representations of complex semisimple Lie algebras. But it is doubtful that he saw at that time how close this result is to a classification of the irreducible representations of compact connected Lie groups (at least when they are simply connected) or even that he attached special significance to this problem. Weyl, inspired partly by Schur's 1924 paper, developed the theory for compact connected Lie groups analytically in the years 1924–26. He used invariant integration in terms of differential forms, showed that every element of the group is conjugate to an element of a maximal torus, gave an integration formula in terms of integration over conjugacy classes, and used characters and the integration formula to reduce a version of the Theorem of the Highest Weight to the theory of Fourier series on the maximal torus. The well-known Peter-Weyl Theorem, establishing (P5), followed in 1927, and (P4) for smooth functions is a consequence. Unlike the case of finite groups, the Peter-Weyl Theorem has to use some analysis, and the Spectral Theorem for compact self-adjoint operators is invariably the tool. In a 1929 paper Cartan tied the algebraic and analytic theories together by showing the full relationship between complex semisimple Lie algebras and the real Lie algebras of compact Lie groups. In the early 1930s the proofs of existence and uniqueness of Haar measure by Haar and von Neumann allowed the abstract theory (P1) through (P5) to be extended routinely to all compact topological groups.

The first mathematical application of harmonic analysis for compact groups was Cartan's 1929 reinterpretation of a portion of the theory of special functions in terms of compact Riemannian symmetric spaces. In the simplest example this work clarifies the way in which spherical harmonics and Legendre polynomials arise from the action of the rotation group $SO(3)$ on the sphere S^2. Gross [5] explains just what the Cartan theory says about this example.

Rather few applications of harmonic analysis for compact groups are analogous to what happens in (5): that the Fourier coefficients of a function on the group itself get multiplied by something and one examines the resulting operator. One case of such an application is to L^p convergence of the partial sums of the Fourier expansion of a function on a compact group constant on conjugacy classes. Herz and R. Stanton treated this problem for compact semisimple groups, and L^p convergence takes place for some range of p's with $1+\epsilon < p < 1+1/\epsilon$. Unlike the case of classical Fourier series, the best ϵ for a given compact semisimple group is strictly positive.

In most applications of the harmonic analysis of compact groups, a compact group G acts nontransitively on a measure space X, and one analyzes an operator on $L^2(X)$ that commutes with the action of G. A case in point is the Fourier transform on $L^2(\mathbb{R}^n)$, which commutes with rotations. One expects harmonic analysis with the rotation group $SO(n)$ to yield some information. In fact, Bochner carried out such an investigation in 1951, using spherical harmonics and (1-dimensional) Hankel transforms. The book by Stein and Weiss [15] reproduces this investigation with more emphasis on the group theory.

Applications of compact groups to physics are of this sort. The representation theory of the special unitary group $SU(n)$, for specific values of n, has played a role in the study of nuclear interactions in elementary-particle theory. In quantum mechanics the things that one observes from experiments are eigenvalues (or members of the spectrum if there are not discrete eigenvalues) of certain self-adjoint operators on Hilbert spaces. Conservation laws correspond to self-adjoint operators A for which the one-parameter group of unitary operators e^{itA} commutes with the Hamiltonian. Thus a system of conservation laws leads to a group of symmetries, namely, a group of unitary operators commuting with the Hamiltonian. A part of this symmetry group, as the theory goes, is some particular $SU(n)$. Simultaneous observables correspond to commuting one-parameter groups, hence to a commuting subspace of the Lie algebra. For $SU(n)$, a commuting subspace of the Lie algebra has dimension $\leq n-1$ (and may as well be taken to be diagonal), and hence at most $n-1$ simultaneously observable physical quantities can be distinguished by the $SU(n)$ theory. Elementary particles correspond to certain vectors in the spaces of irreducible representations, say, to simultaneous eigenvectors of the diagonal subgroup (identified when they are equal up to scalar multiples). Quarks correspond to the standard basis vectors in the standard representation. The power of the theory comes from the way in which interactions of particles are to be understood: one takes the tensor product of the representations, decomposes it according to (P1), and sees what combinations of particles occur.

Locally Compact Abelian Groups

In 1934–35 Pontrjagin and van Kampen proved a duality theorem for locally compact abelian groups, and later Weil [16] based a theory of harmonic analysis on this duality theorem. For a noncompact locally compact abelian group, such as $\mathbb{R}$, not all the multiplicative characters are of interest. For $\mathbb{R}$, the function $x \mapsto e^{-2\pi ixy}$ is a multiplicative character whenever y is complex, but only those characters with y real are of interest. A multiplicative character is said to be *unitary* if it is everywhere of absolute value 1. (More generally a representation π in a Hilbert space is *unitary* if each operator $\pi(x)$ is unitary.) If G is a locally compact abelian group, then the unitary multiplicative characters form a group $\widehat{G}$ under pointwise multiplication, and $\widehat{G}$ becomes a locally compact abelian group (the *dual group*) when topologized with the topology of uniform convergence on compact sets. For G equal to a torus $\mathbb{T}^n$ or a group $\mathbb{R}^n$, $\widehat{G}$ consists exactly of the unitary multiplicative characters that we have been using, and the topology is the ordinary one. Thus $\widehat{G}$ is isomorphic with $\mathbb{Z}^n$ or $\mathbb{R}^n$ in the two cases. In complete generality the group $\widehat{G}$ is locally compact abelian and has a dual $\widehat{\widehat{G}}$, and there is a canonical continuous homomorphism of G into $\widehat{\widehat{G}}$: If g is in G, then the corresponding member of $\widehat{\widehat{G}}$, evaluated on a character $\omega \in \widehat{G}$, takes the value $\omega(g)$. The duality theorem says that this homomorphism $G \to \widehat{\widehat{G}}$ is a topological isomorphism onto. Pontrjagin and van Kampen proved the duality theorem as a consequence of a structure theory that they developed. Weil went on to define a Fourier transform $f \mapsto \widehat{f}$ carrying functions on G to functions on $\widehat{G}$:

$$\widehat{f}(\omega) = \int_G f(x)\overline{\omega(x)}\,dx,$$

where dx is Haar measure on G. Weil's inversion formula, valid for integrable continuous f's whose Fourier transforms are integrable, says that there exists a normalization of Haar measure $d\omega$ on $\widehat{G}$ so that

$$f(x) = \int_{\widehat{G}} \widehat{f}(\omega)\omega(x)\,d\omega.$$

The Plancherel formula $\|\widehat{f}\|_2^2 = \|f\|_2^2$ is a consequence, and thus versions of (P4) and (P5) are valid for G. Rudin [12] shows how the duality theorem and harmonic analysis could be developed together, bypassing the structure theory.

Adeles and Ideles

Harmonic analysis on locally compact abelian groups has an important application in algebraic number theory, due to Tate. To establish the setting, it is necessary to take another look at L functions as they emerged in the 1920s. Even before Artin introduced his L functions, Hecke was introducing other kinds. One kind that Hecke treated was a generalization of Euler's L function and other Dirichlet L functions; it was an L function associated to something called a *Grossencharacter*, which we discuss shortly. In a complicated way, Hecke proved that these L functions, which are convergent in a right half plane, extend meromorphically to the complex plane, satisfy a functional equation relating the values at s and $1 - s$, and are entire except in the case of $\zeta(s)$ itself. Because of some work of Takagi, Artin realized that his L functions, in the case that the Galois group is abelian, ought always to equal Hecke L functions of Grossencharacters. This expectation led Artin to formulate and then to prove Artin reciprocity, which is a far-reaching generalization of quadratic reciprocity and is a cornerstone of abelian class field

Photograph courtesy of Michael Artin

Emil Artin

theory. In turn Artin reciprocity enabled Artin to prove the expected equality of L functions. Therefore Artin L functions in the case of abelian Galois groups satisfy functional equations, and for nontrivial multiplicative characters of such Galois groups they are entire.

In their original formulation Grossencharacters have something to do with the ideal class group of Dedekind, but Chevalley found another formulation in 1936. A *place* v of a number field k is an isomorphism class of field maps of k onto a dense subfield of a nondiscrete locally compact field k_v. For $k = \mathbb{Q}$, the places are the embeddings of $\mathbb{Q}$ into $\mathbb{R}$ and into the field $\mathbb{Q}_p$ of p-adic numbers for each prime p. Each of the locally compact fields has an absolute value mapping, the elements of $\mathbb{Q}_p$ of absolute value 1 being the closure of those rationals with numerator and denominator prime to p. Chevalley introduced the group of *ideles* of k as the multiplicative subgroup of all elements in the product of the multiplicative groups $k_v^\times$ whose absolute value is 1 at all but finitely many v, and he made this abelian group into a locally compact abelian group by defining a topology suitably. The multiplicative group $k^\times$ turns out to have a diagonal embedding into the group of ideles. In this terminology a Grossencharacter is simply a multiplicative character of the group of ideles that is trivial on the diagonally embedded $k^\times$.

In his 1950 thesis Tate applied harmonic analysis for locally compact abelian groups in this situation. The *adeles* of k are the additive subgroup of all elements in $\prod k_v$ whose absolute value is ≤ 1 at all but finitely many v's. With coordinate-wise multiplication and with a suitable topology, they form a commutative locally compact ring. By using harmonic analysis on the adeles and exploiting the interplay between adeles and ideles, Tate reinterpreted Hecke's L functions and showed that the analytic continuation and functional equation were a consequence of a Poisson Summation Formula for this setting. Shortly we shall return to a nonabelian analog of this result.

Locally Compact Groups

Following the development of quantum mechanics about 1927, representation theory expanded to include locally compact groups that were neither compact nor abelian. The two concrete situations studied before 1940 were what now would be called the Heisenberg group in n complex variables and the inhomogenous Lorentz group. In the language of the times, the representation theory of the Heisenberg group was studied in the form of projective representations of the group $\mathbb{C}^n$, functions π satisfying $\pi(x+y) = c(x,y)\pi(x)\pi(y)$ for nonzero scalars $c(x,y)$. But we shall stick to group representations.

The Heisenberg group H^n in n complex variables is the group of all (z,t) with $z \in \mathbb{C}^n$ and $t \in \mathbb{R}$ having multiplication

$$(w,t)(z,t') = (w+z, t+t' + \operatorname{Im} w^* z),$$

where w^* means w conjugate transpose. The group is isomorphic to the group of matrices given in block form as

$$\begin{pmatrix} 1 & z^* & \frac{1}{2}|z|^2 + it \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

and has $(z,t)^{-1} = (-z,-t)$. The center Z is the set of all $(0,t)$, and $H^n/Z \cong \mathbb{C}^n$. The early representation theory of H^n emerged from research of Heisenberg, Weyl, Stone, and von Neumann.

The group H^n has the property that every finite-dimensional unitary representation is trivial on Z and thus factors through to a representation of the quotient $\mathbb{C}^n$. Consequently such representations do not distinguish all the points of the group. In fact, (P1) is applicable for such representations φ, and we may assume that φ is an irreducible unitary representation of finite dimension $\dim \varphi$. Property (P2) is valid, and thus $\varphi(0,t)$ is scalar, hence of the form $e^{2\pi i t x}$ times the identity. Meanwhile a little calculation shows that

$$(w,t)(z,t')(w,t)^{-1}(z,t')^{-1} = (0, \operatorname{Im}(w^*z - z^*w)).$$

The determinant of φ of the left side is the product of the determinant of φ of each factor and hence is 1. Thus $1 = \exp(2\pi i \operatorname{Im}(w^*z - z^*w) x \dim \varphi)$ for all w and z, and hence $x = 0$.

To do harmonic analysis meaningfully, we are thus led to consider unitary representations that are infinite-dimensional. This adjustment requires some refinement of our definitions: The total space will now be a Hilbert space, the invariant subspaces of interest are the closed ones, and a representation is *irreducible* if it contains no proper nonzero closed invariant subspaces. In this language the Stone-von Neumann Theorem says that the infinite-dimensional irreducible unitary representations of H^n, up to equivalence, are parametrized by a nonzero $\lambda \in \mathbb{R}$. The representation π_λ with parameter λ acts in $L^2(\mathbb{R}^n)$ with

$$\pi_\lambda(x+iy,t)f(x') = e^{2i\lambda y^* x'} e^{i\lambda(t+y^*x)} f(x'+x). \tag{6}$$

An infinitesimal version of π_λ and the connection with quantum mechanics will not concern us here and are discussed at length in Mackey [11].

The other noncompact nonabelian locally compact group that was studied before 1940 was the 10-dimensional inhomogenous Lorentz group. Again the motivation was quantum mechanics. E. Wigner's well-known work on the representation theory of this group appeared in 1938 and is discussed in Wightman [17].

In 1936, before Wigner had studied this second noncompact nonabelian group, Murray and von Neumann made a study of rings of operators that has implications for the abstract theory of group representations. If π is a unitary representation of G on a separable Hilbert space, let $R(\pi)$ be the smallest weakly closed algebra of bounded linear operators containing all $\pi(g)$ for $g \in G$. We say that π is *primary* if the center of $R(\pi)$ consists only of scalar operators. The orthogonal sum of a finite or countable number of copies of a single irreducible representation is always a primary representation. If the primary representation π is of this special form, we say that π is of *type I*. The group G is of type I if all of its primary representations are of type I. One of the discoveries of Murray and von Neumann is that there are locally compact groups that are not of type I; it is known that the (discrete) free group on two generators is such a group. From work of Mackey and Glimm, it is known that a separable group G is of type I if and only if the space of equivalence classes of irreducible unitary representations is a standard Borel space; thus the groups that are not of type I are pathological for our current purposes, and we shall want to discard them.

A second contribution to the abstract theory came from Gelfand and Raikov in 1943. If π is a unitary representation of a locally compact group G and v is a vector, then the function $g \mapsto (\pi(g)v, v)$ is continuous positive definite. They noted the converse (that a continuous positive definite function is always of this form for some π and v) and then proved that π is irreducible if and only if $\pi(G)v$ generates the whole Hilbert space and $(\pi(g)v, v)$ lies on an extreme ray in the cone of all continuous positive definite functions. Applying the Krein-Milman Theorem suitably, they were able to conclude that the irreducible unitary representations of G separate points. Thus in principle there should be enough irreducible unitary representations to do harmonic analysis.

In 1946, inspired by Weil [16], Mackey began a systematic study of the representation theory of locally compact groups. By a circuitous route described in [8], pp. 892–893, Mackey was led to realize that induced representations, introduced by Frobenius for finite groups, were meaningful and important for separable locally compact groups. The only restriction was that the subgroup from which the inducing would take place had to be closed. In fact, the representations (6) of H^n were induced from 1-dimensional representations. Wigner's representations of the inhomogeneous Lorentz group were induced, and so were many others. Mackey went on to develop a theory in the direction of classifying the irreducible representations of semidirect products of groups when the representations of each factor are known and the action is tame enough.

The final piece of the abstract theory is the Plancherel formula. Mautner and Segal proved independently in 1950 that any type I group that is *unimodular* (i.e., has left Haar measure equal to right Haar measure) has a unique measure $d\mu$ on the space of equivalence classes of irreducible unitary representations such that $\|f\|^2 = \int \|\pi(f)\|_{\mathrm{HS}}^2 \, d\mu(\pi)$ for all $f \in L^2(G)$. Further exposition of the abstract theory may be found in Mackey [10].

Thus early in the development of representation theory of noncompact nonabelian groups, three key questions about particular groups emerged: Is G of type I? If so, what are the irreducible unitary representations of G? If so and if G is unimodular, what concretely is the Plancherel measure? These questions have turned out to be much harder for general locally compact groups than for compact or abelian groups.

Among the classes of locally compact groups that have been studied are nilpotent Lie groups, solvable type I Lie groups, semisimple Lie groups with finite center, general connected Lie groups of type I, and certain p-adic groups. We discuss only some of these.

Nilpotent and Solvable Lie Groups

A nilpotent Lie group has a simply connected covering group homeomorphic to $\mathbb{R}^n$, and the prototype is any connected closed subgroup of upper-triangular complex matrices with 1's on the diagonal. The Heisenberg group H^n is an example. A nilpotent Lie group is always unimodular of type I. Kirillov showed that the irreducible unitary representations are always induced from 1-dimensional representations of closed subgroups, he classified these representations, and he found the explicit Plancherel measure. In his work he developed for irreducible representations an "orbit picture" that had been introduced earlier by Harish-Chandra in the compact case. The orbit picture associates representations to data attached to orbits of the action by the group on the vector-space dual of the Lie algebra. It is described at length in Howe [6].

Folland in 1973 used harmonic analysis for the Heisenberg group to give new estimates for a subelliptic differential operator in Euclidean space that amounts to the operator $\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b$ associated with the boundary of the unit ball in $\mathbb{C}^n$. It turns out that this operator can be regarded as a constant-coefficient differential operator on H^{n-1}, and analysis using H^{n-1} gives sharper estimates than the traditional Euclidean approach. In 1974 Folland and Stein combined this idea for H^{n-1} with a freezing principle to investigate $\Box_b$ for strictly pseudoconvex domains. Rothschild and Stein in 1976 combined representation theory for other nilpotent Lie groups with a freezing principle to analyze subelliptic operators $\sum X_i^2$ on a manifold when the X_i are vector fields that generate, with their commutators through a fixed order, the full tangent space at each point of the manifold. No example is known of broken symmetry in mathematics that can be handled by combining a freezing principle with the representation theory of some nonnilpotent group.

A solvable Lie group has a simply connected covering group homeomorphic to $\mathbb{R}^n$, and the prototype is any connected closed subgroup of upper triangular complex matrices. Such a group may fail to be unimodular, and it may fail to be of type I. L. Auslander and Kostant in 1969 extended Kirillov's orbit picture to classify the irreducible unitary representations of type I solvable Lie groups. Pukanszky gave a version of a Plancherel formula that did not require the group to be unimodular or even type I.

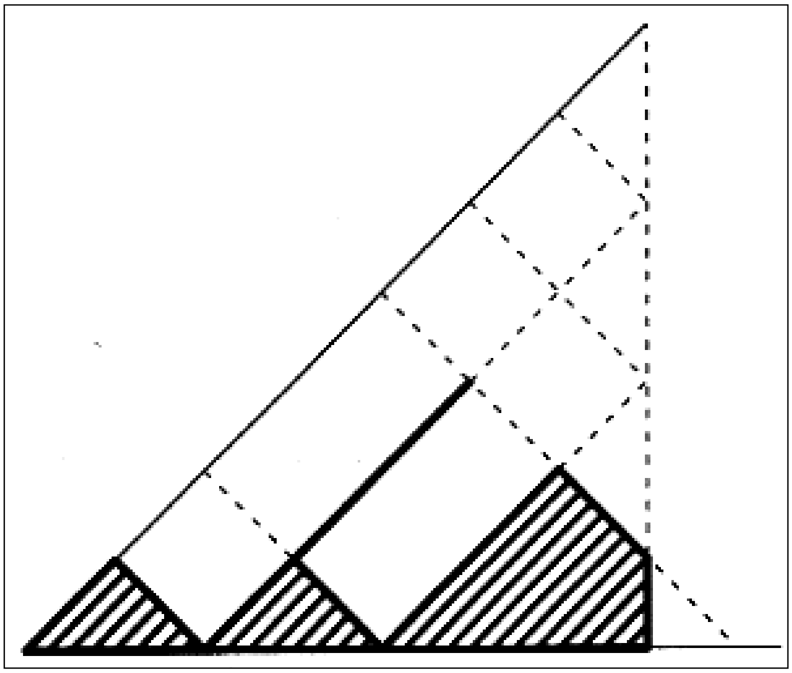

Figure 1. Unitary points in a nonspherical principal series of *Sp*(6, 2) with a real character on *A*.

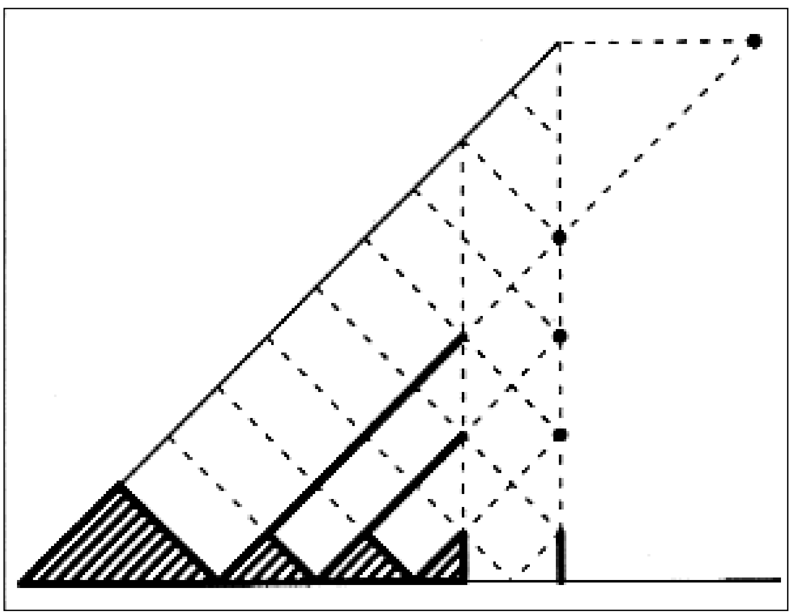

Figure 2. Known unitary points in the spherical principal series of *Sp*(6, 2) with a real character on *A*.

Semisimple Lie Groups

The largest area of research has been on semisimple Lie groups, which are always unimodular and were shown to be of type I by Harish-Chandra in a 1953 paper. We concentrate on semisimple Lie groups that can be realized as groups of real or complex matrices. Apart from isomorphism, the semisimple Lie groups of matrices are exactly the connected closed subgroups of complex matrices that are closed under conjugate transpose and have discrete center (necessarily then finite). Special linear groups, symplectic groups, and various isometry groups of quadratic forms provide examples. For a book-length exposition of the theory, see [7]. The first such group to be considered was the group $SL(2,\mathbb{R})$ of real 2-by-2 matrices of determinant one. Bargmann, presumably motivated by a desire to supplement Wigner's 1938 paper, classified the irreducible unitary representations of this group (or, more precisely, a conjugate of it) in a famous 1947 paper. Bargmann even gave information about decomposing L^2 of the group in a way that anticipated the idea of a Plancherel formula. The irreducible unitary representations occur in series. One of these, now called the spherical *principal series*, contains a representation $P^{+,i\nu}$ for each real ν. The representation acts as in $L^2(\mathbb{R})$ by

$$P^{+,i\nu}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x) = |bx+d|^{-1-i\nu} f((ax+c)(bx+d)^{-1}), \tag{7a}$$

and only the representations with $\nu \geq 0$ are needed in the classification. The representation $P^{-,i\nu}$ of the nonspherical principal series acts in $L^2(\mathbb{R})$ for real ν by

$$P^{-,i\nu}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x) = \operatorname{sgn}(bx+d) P^{+,i\nu}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(x), \tag{7b}$$

and only the representations with $\nu > 0$ are needed in the classification. Bargmann found also two sets of representations now called *discrete series*. One of these contains a representation $D^{+,n}$ for each integer $n \geq 2$. The representation acts in the space of analytic functions on the upper half plane square integrable with respect to $y^{n-2}\,dx\,dy$. The action is

$$D^{+,n}\begin{pmatrix} a & b \\ c & d\end{pmatrix} f(z) = (bz+d)^{-n} f((az+c)(bz+d)^{-1}). \tag{8}$$

There is a corresponding representation $D^{-,n}$ in the space of complex conjugates. Another series, the *complementary series*, acts as in (7a) but with $i\nu$ replaced by a real parameter u between 0 and 1 and with a complicated inner product that is not obviously positive definite. And there were the trivial 1-dimensional representation and two representations now called *limits of discrete series*, $D^{+,1}$ and $D^{-,1}$. That was all. Most striking of all, Bargmann saw that only the principal series and the discrete series contributed to the left regular representation $l(g)F(x) = F(g^{-1}x)$ of the group on L^2 of the group. The complementary series played no role. Moreover, the discrete series contributed discretely; this was a new phenomenon not seen before in noncompact groups.

Also in 1947 Gelfand and Naimark investigated $SL(2, \mathbb{C})$. They found principal series and complementary series, and they found that there were no discrete series. In 1950 Gelfand and Naimark published a book giving extensive information about the representation theory of complex classical semisimple Lie groups. No discrete series appeared. The 1950 book also showed how to use integral geometry to obtain an explicit Plancherel formula (decomposing $L^2(G)$) for $G = SL(n, \mathbb{C})$.

Mackey seems to have been the first to realize that the representations (7) are induced representations, induced from 1-dimensional representations of the upper-triangular subgroup. Harish-Chandra obtained the Plancherel formula for $SL(2, \mathbb{R})$ in 1952, and Gelfand and Graev generalized the argument to $SL(n, \mathbb{R})$ in 1953. From these results a picture emerged of $L^2(G)$ as having finitely many pieces, each one related to the harmonic analysis of a certain kind of maximal abelian subgroup called a *Cartan subgroup*. For $SL(n, \mathbb{C})$ there is only one such group up to conjugacy (the diagonal subgroup), and the Plancherel formula reduces to Fourier analysis of the diagonal subgroup. For $SL(2, \mathbb{R})$, there are two nonconjugate Cartan subgroups (the diagonal subgroup and the rotation subgroup). The analysis leading to the Plancherel formula is subtle, but the heart of it is Fourier analysis on the two Cartan subgroups. For $SL(n, \mathbb{R})$, there are $\left[\frac{n}{2} \right] + 1$ nonconjugate Cartan subgroups.

Harish-Chandra realized that it was important to find representations corresponding to a compact Cartan subgroup and that these should be the discrete series—the representations occurring discretely in the Plancherel formula. He first constructed generalizations of (8) to other groups G that act on bounded symmetric domains, but he knew that was not enough since such groups do not exhaust the groups with compact Cartan subgroups. Finally after a long sequence of deep papers culminating in an epic 1966 paper in *Acta Mathematica*, he completed a classification of the discrete series. Later he proved that the representations contributing to the Plancherel formula are induced from parabolic subgroups written MAN, each MAN constructed from a distinct conjugacy class of Cartan subgroups, with the inducing representation consisting of a discrete series on M, a unitary multiplicative character on the Euclidean group A, and the trivial representation on the simply connected nilpotent group N. Harish-Chandra published the final steps in the proof of the Plancherel formula in 1976.

Photograph courtesy of Springer-Verlag, New York

Harish-Chandra

As was already true in $SL(2, \mathbb{R})$, there are other irreducible unitary representations than those needed for the Plancherel measure. Their classification has been more elusive and remains unsolved. A giant advance was made by Langlands in 1973, who classified all irreducible "admissible" representations, which include all irreducible unitary representations. In the final form of this classification, irreducible admissible representations are seen to be quotients of induced representations from the same subgroups MAN as above (except that it may be necessary to use more than one N for given MA). On M is a discrete series or a limit of discrete series, and on A is a multiplicative character whose modulus has logarithm in a certain cone. Again the trivial representation is used on N. The question about unitarity therefore reduces to deciding which Langlands quotients, as they are called, can be made unitary by introducing a new inner product. For $SL(2, \mathbb{R})$ the complementary series are obtained this way, and one requires an answer in general. Progress has been slow and has been based on finding methods for settling unitarity of classes of Langlands quotients. These methods are now sufficiently powerful that Vogan has completely settled $SL(n, \mathbb{C})$ and essentially $SL(n, \mathbb{R})$, and Barbasch has settled the complex orthogonal and symplectic groups. But there is a possibility that the answer for all G may be too complicated to state reasonably. Figures 1 and 2, taken from [2], illustrate two situations in a group $Sp(6, 2)$; a representation of an M has been fixed in each case, and the picture shows the 2-dimensional picture of logarithms of real multiplicative characters of A in the appropriate cone. The heavily marked regions, lines, and points indicate parameters that correspond to unitary representations. The picture is complete in the case of Figure 1 and is believed to be complete in the case of Figure 2. One can prove for general G that the parameters of the unitary representations form polygonal complexes of this sort, but the description of the parameters of the unitary representations in these examples is already so complicated that most researchers in the field are looking for "important" irreducible unitary representations rather than all irreducible unitary representations.

We mention two applications of harmonic analysis with semisimple groups. One is to the decomposition of spaces of functions on semisimple symmetric spaces G/H, where H is the fixed group of an involution of G. This theory is now fairly complete and may be viewed as a simultaneous and remarkable extension of three theories: Cartan's 1929 theory for the compact case, a theory for Riemannian symmetric spaces begun by Harish-Chandra in 1958 and extended by Helgason in the 1960s and 1970s, and Harish-Chandra's 1976 theory for the group case $(G \times G)/\mathrm{diag}(G)$.

Langlands Program

The other application is to the Langlands program in number theory and arithmetic geometry, for which Langlands received the Wolf Prize in 1996 (*Notices*, February 1996, p. 221). The Langlands program has several aspects and we mention just one, which is the search for a generalization of Artin reciprocity to nonabelian Galois extensions of number fields k. The hope is to identify all Artin L functions with other kinds of L functions that have analytic continuations and functional equations. A source of some L functions with good analytic behavior is a further class of L functions introduced by Hecke and given in terms of cusp forms. In the Langlands theory they get incorporated into a generalization of Tate's thesis, whose results were discussed above. Tate's thesis is regarded in the Langlands theory as handling 1-by-1 matrices, because the ideles may be regarded as $GL_1(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k. The Hecke L functions arise in the 2-by-2 case of the Langlands theory. A little more specifically, a Grossencharacter is a character of the ideles trivial on the diagonally imbedded number field k, and Langlands associated an L function to each irreducible "admissible" representation of $GL_n(\mathbb{A})$ that occurs discretely in L^2 of $GL_n(\mathbb{A})/GL_n(k)Z$ (with k imbedded diagonally and Z equal to the center) and satisfies a condition generalizing the vanishing at the cusps for classical modular forms. Work by Tamagawa, Godement, Jacquet-Langlands, then Godement-Jacquet, and finally Jacquet established good analytic behavior for these L functions.

Artin reciprocity amounts to the theorem that the Artin L function of any 1-dimensional Galois-group representation is an L function obtained from $GL_1(\mathbb{A})$ in this way. *Langlands reciprocity* is the conjecture that the Artin L function of any n-dimensional Galois-group representation is an L function obtained from $GL_n(\mathbb{A})$ in this way. When the number field is $\mathbb{Q}$, Langlands reciprocity amounts to an assertion of what the pattern is for how an arbitrary irreducible monic polynomial in one variable with $\mathbb{Z}$ coefficients factors when reduced modulo a prime. In work for which he was awarded the Cole Prize in 1982, Langlands proved this conjecture for a class of 2-dimensional Galois-group representations that could not be handled by previous methods. Tunnell extended the Langlands theorem to handle all 2-dimensional Galois-group representations whose image in $PGL_2(\mathbb{C})$ is a subgroup of the symmetric group on four letters.

This profound theorem of Langlands and Tunnell, although phrased in terms of group representations, L functions, and adeles, is ultimately a theorem about prime numbers. It is the representation-theoretic cornerstone of the work of Wiles on Fermat's Last Theorem.

As Langlands said in [9] in 1990 about his whole program, "...we are dealing with a tissue of conjectures that cannot be attacked frontally. The aesthetic tension between the immediate appeal of concrete facts and problems on the one hand and, on the other, their function as the vehicle to express and reveal not so much universal laws as an entity of a different kind, of which these laws are the very mode of being, is perhaps more widely acknowledged in physics, where it has long been accepted that the notions needed to understand perceived reality may bear little resemblance to it, than in mathematics, where oddly enough, especially among number theorists, conceptual novelty has frequently been deprecated as a reluctance to face the concrete and a flight from it. Developments of the last half-century have matured us, as an examination of Gerd Faltings's proof of the Mordell conjecture makes clear, but there is a further stage to reach."

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Editor's Note: Book by Knapp and Vogan Receives Award

The book *Cohomological Inducation and Unitary Representations*, by Anthony W. Knapp and David A. Vogan, Jr., has been selected as one of the outstanding books of 1995 by the Professional/Scholarly Publishing Division of the Association of American Publishers. Published by Princeton University Press, the book was the winner in the mathematics category; there are 28 subject categories in all. A panel of judges from the publishing industry and the industrial, medical, and scientific communities chose the winners from among 380 professional and scholarly works nominated across the spectrum of science, technology, business, and the humanities.