Group Representations and Harmonic Analysis from Euler to Langlands, Part I

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Group representations and harmonic analysis play a critical role in subjects as diverse as number theory, probability, and mathematical physics. A representation-theoretic theorem of Langlands is a vital ingredient in the work of Wiles on Fermat's Last Theorem, and representation theory provided the framework for the prediction that quarks exist. What are group representations, why are they so pervasive in mathematics, and where is their theory headed?

Euler and His Product Expansions

Like much of modern mathematics, the field of group representations and harmonic analysis has some of its roots in the work of Euler. In 1737 Euler made what Weil [4] calls a "momentous discovery", namely, to start with the function that we now know as the Riemann ζ function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and to realize that the sum could be written as a product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \tag{1}$$

for $s > 1$. In fact, if each factor $(1-p^{-s})^{-1}$ on the right side of (1) is expanded in geometric series as $1 + p^{-s} + p^{-2s} + \cdots$, then the product of the factors for $p \le N$ is the sum of those terms $1/n^s$ for which n is divisible only by primes $\le N$; hence a passage to the limit yields (1). Euler well knew that the sum for $\zeta(s)$ exceeds the integral

$$\int_1^\infty \frac{dx}{x^s} = \frac{1}{s-1}.$$

This expression is unbounded as s decreases to 1, but the product (1) cannot be unbounded unless there are infinitely many factors. Hence (1) yielded for Euler a new proof of Euclid's theorem that there are infinitely many primes. In fact, (1) implies, as Euler observed, the better theorem that $\sum 1/p$ diverges.

Euler later went on from this proof to deduce that there are infinitely many primes $4n+1$ and infinitely many primes $4n+3$, and that is where the story of harmonic analysis really begins. To understand why the above analysis does not handle these cases, it is helpful to see in more detail how $\sum 1/p$ enters the above kind of argument. Consideration of series expansions of the exponentiated functions shows that

$$\log(1+x) < x < \log \frac{1}{1-x}$$

for $0 < x < 1$, and it is easy to see that the right side is no more than twice the left side if $0 < x \le \frac{1}{2}$. Consequently it follows from (1) that

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The author expresses his appreciation to Sigurdur Helgason, Hugo Rossi, John Tate, David Vogan, and the Institut Mittag-Leffler for help in the preparation of this article. Part II of this article will appear in the May 1996 issue of the Notices.

$$\text{(2)} \qquad \log \zeta(s) = \sum_{p \text{ prime}} \frac{1}{p^s} + \text{bounded term}$$

as s decreases to 1. Meanwhile, multiplication of the series for $\zeta(s)$ by 2^{-s} reproduces the even-numbered terms of the series, and therefore

$$\text{(3a)} \qquad \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \cdots$$

and

$$\text{(3b)} \qquad \left(1 - \frac{2}{2^s}\right)\zeta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \cdots.$$

The left side of (3b) is $(s-1)\zeta(s)$ times something that tends to $\log 2$ as s tends to 1. Euler knew Leibniz's test for convergence and could see in (3b) that the series on the right is convergent for $s > 0$ with a positive sum. It follows that $\zeta(s)$ near $s = 1$ is the product of $(s-1)^{-1}$ and a function with a finite nonzero limit. Combining this result with (2) gives

$$\text{(4)} \qquad \sum_{p \text{ prime}} \frac{1}{p^s} = \log \frac{1}{s-1} + \text{bounded term}$$

as s decreases to 1.

In handling primes congruent to 1 or 3 modulo 4, it is tempting to replace the sum over all primes of $1/p^s$ in the above argument by

$$\text{(5)} \qquad \sum_{p \equiv 1 \bmod 4} \frac{1}{p^s} \qquad \text{or} \qquad \sum_{p \equiv 3 \bmod 4} \frac{1}{p^s},$$

trace backwards, and see what happens. What happens is that the expansion of the corresponding product of $(1 - p^{-s})^{-1}$ as a sum does not yield anything very manageable. Euler's key new idea was to work with the sum and difference of the two terms in (5), rather than the two terms separately, and then to recover the two terms (5) at the end. This is full-fledged harmonic analysis on a 2-element group.

The essence of harmonic analysis is to decompose complicated expressions into pieces that reflect the structure of a group action when there is one; the goal is to make some difficult analysis manageable. Tracing backwards with the earlier argument as a model, Euler found two manageable series with product expansions. The first was

$$\text{(6)} \qquad \begin{aligned} \left(1 - \frac{1}{2^s}\right)\zeta(s) &= 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots \\ &= \sum_{n=1}^{\infty} \frac{\chi^+(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - \chi^+(p)p^{-s}}, \end{aligned}$$

where $\chi^+(n)$ is 0 for n even and 1 for n odd. The second series was

$$\text{(7)} \qquad \begin{aligned} L(s) &= 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots = \sum_{n=1}^{\infty} \frac{\chi^-(n)}{n^s} \\ &= \prod_{p \text{ prime}} \frac{1}{1 - \chi^-(p)p^{-s}}, \end{aligned}$$

where $\chi^-(n)$ is 0 for n even, 1 for $n \equiv 1 \bmod 4$, and -1 for $n \equiv 3 \bmod 4$. The log of $\frac{1}{1-\chi(p)p^{-s}}$ is approximately $\chi(p)p^{-s}$ even if $\chi(p)$ is negative. Arguing by taking the log of the product formula in (6) (or simply copying the result from (4)) gives

$$\text{(8)} \qquad \text{sum of terms in (5)} = \log \frac{1}{s-1} + \text{bounded term}$$

as s decreases to 1. Meanwhile application of the Leibniz test to the series in (7) proves that $L(s)$ converges for $s > 0$ and in particular is finite at $s = 1$. In addition, the test shows that $L(1) > 0$. Consequently taking the log of the product formula in (7) yields

$$\text{(9)} \qquad \text{difference of terms in (5)} = \text{bounded term}$$

as s decreases to 1. Comparing (8) and (9) shows that each of the series in (5) is unbounded as s decreases to 1. Hence there are infinitely many primes congruent to 1 modulo 4 and also infinitely many primes congruent to 3.

Role of a Group in Euler's Products

Where is the group and what is its role? The property of the two functions χ^+ and χ^-, call either of them χ, that allows the sums in (6) and (7) to be rewritten as products is that $\chi(mn) = \chi(m)\chi(n)$ for all positive integers m and n. Nowadays such functions are called *Dirichlet characters* modulo 4. We can think of χ^+ and χ^- as lifts to the integers of functions on the multiplicative group $\{1, 3\}$ of integers modulo 4 and prime to 4, with 0 used as the value on integers that are not prime to 4. The two functions on the group $\{1, 3\}$ are

$$\omega^+(1) = \omega^+(3) = +1 \qquad \text{and}$$
$$\omega^-(1) = +1, \ \omega^-(3) = -1.$$

These functions ω on this 2-element group are *multiplicative characters*, i.e., homomorphisms to the multiplicative group of nonzero complex numbers, and they are the only multiplicative characters for this group. They form a basis for the complex vector space of all complex-valued functions on the 2-element group. Essentially Euler had two functions to study, the characteristic function of each 1-element set for this group:

Euler

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Leonhard Euler

$$I_1(1) = 1,\ I_1(3) = 0 \quad \text{and} \quad I_3(1) = 0,\ I_3(3) = 1.$$

The series under study in (5) may be written as

$$\sum_{p \text{ prime}} \frac{I_1(p)}{p^s} \qquad \text{and} \qquad \sum_{p \text{ prime}} \frac{I_3(p)}{p^s}.$$

Euler's proof worked because he expanded the functions I_1 and I_3 in terms of the basis of multiplicative characters

$$I_1 = \tfrac{1}{2}(\omega^+ + \omega^-) \qquad \text{and} \qquad I_3 = \tfrac{1}{2}(\omega^+ - \omega^-),$$

succeeded at some computations for the individual terms

$$\sum_{p \text{ prime}} \frac{\omega^+(p)}{p^s} \qquad \text{and} \qquad \sum_{p \text{ prime}} \frac{\omega^-(p)}{p^s}$$

of the expansion, and reassembled his functions by means of

$$\omega^+ = I_1 + I_3 \qquad \text{and} \qquad \omega^- = I_1 - I_3.$$

This is the process of harmonic analysis.

Although the harmonic analysis in this work may be regarded as trivial linear algebra, the point is that the particular linear algebra is a vehicle for taking advantage of a group structure. This example is in a way too simple for understanding the basic principle. In fact, it was more than one hundred years before Dirichlet saw through it and proved his own theorem about primes in arithmetic progressions.

The above work of Euler is of more than historical interest. It is the direct ancestor of a large amount of current research in algebraic number theory, including the representation-theoretic input from the Langlands program into Fermat's Last Theorem. In addition, it illustrates the principle that although harmonic analysis may be at the core of the solution of a problem, several layers of ingenious ideas may lie between the statement of the problem and the use of harmonic analysis.

Multiplicative characters played a rather incidental role in mathematics from 1737 until about 1807. Cramer introduced determinants in 1750, defining the sign of a permutation and proving what we now call Cramer's Rule. The sign of a permutation is a multiplicative character on the permutation group on n letters, and determinant is a multiplicative character on nonsingular matrices of a fixed size. But the harmonic analysis aspect of these characters played no role in Cramer's work. Gauss, in expanding on Euler's work representing integers by binary quadratic forms, introduced his own notion of character, which corresponds roughly to what we now call a Dirichlet character. But again harmonic analysis was not involved.

Fourier Series

The next big development in the subject of group representations was the subject of Fourier series. The account here is taken from Grattan-Guinness [1]. In 1747 d'Alembert presented his work on the vibrating string problem: He found the differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2},$$

specified initial conditions, and obtained the solution $y = \frac{1}{2}(f(x+ct) + f(x-ct))$. Although Euler in 1748 considered trigonometric functions as examples of solutions of the equation, no one expected that such functions would offer any generality until d'Alembert in 1750 introduced the method of separation of variables into the solution of partial differential equations. Daniel Bernoulli argued philosophically that a trigonometric sine series should be general enough to express all solutions, but Euler rejected that argument on other philosophical grounds. Research into trigonometric series made little progress for the remainder of the eighteenth century. In 1777 Euler did discover the familiar motivating argument for the formula for Fourier coefficients, consisting of multiplying a trigonometric expansion through by a sine or cosine and integrating term by term, but this work was not published until 1798. And in 1799 Parseval published a formula for the sum of the squares of the coefficients of a trigonometric series in terms of integrals; his formula is reasonably close to what is now called Parseval's formula.

Then came Fourier, who was interested in the diffusion of heat. Fourier derived the heat equation, made a systematic study of cases that could be investigated with separation of variables, offered his own insights into the vibrating-string problem, introduced what we now know as Fourier series, and addressed the representability of certain discontinuous functions by such series, all in a single paper submitted in 1807. The paper ran into objections, particularly from Lagrange and Poisson, and was blocked from publication. At base the objections were that Fourier's results were inconsistent with the prevailing intuition about functions. Functions were supposed to be of an algebraic character. If a trigonometric series $\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots$ were to sum to a function like $\frac{1}{2}x$ on an interval $(-\pi, \pi)$, the algebraic character of the limit should force the limit to be $\frac{1}{2}x$ everywhere, and then the limit would not be periodic, contradiction. Fourier submitted a revision, including the 1807 material and also the inversion formula for the Fourier transform, in 1811 and won a prize. But publication of the revision too was blocked. Fourier's work finally appeared in 1822 in his celebrated book *Théorie analytique de la chaleur*.

The solution to the vibrating-string problem involves Fourier sine series. No group of symmetries is involved, and this work does not foreshadow harmonic analysis with group representations. Instead, this example is motivation for Sturm-Liouville theory, which began in 1836. Similarly Fourier's work with the heat equation does not automatically carry a group along with it. A group occurs only in examples having some symmetry, one such example being the case of an annulus, which has circular symmetry. In the case of the annulus, Fourier was led to series involving both cosines and sines, which nowadays are customarily written with complex exponentials:

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \tag{10}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx. \tag{11}$$

The group lying behind these formulas is the circle group $\mathbf{R}/2\pi\mathbf{Z}$. The functions e^{inx} are exactly the (continuous) multiplicative characters for this group, and (10) suggests that $f(x)$ is to be expanded in terms of these functions. Fourier was troubled by the issue we nowadays refer to

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Joseph Fourier

as L^2 completeness of the orthogonal set $\{e^{inx}\}$, and for this reason he preferred a more complicated method than (11) for obtaining the coefficients c_n. Fourier obtained the Parseval formula,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \tag{12}$$

by a manipulation of series, evidently without connecting the validity of his argument with the same completeness question.

Wiener's View of Harmonic Analysis

The expansion (10) enters after the variables have been separated in the partial differential equation. A linear ordinary differential operator D is to be applied to $f(x)$, with the result 0. This operator commutes with translations $\tau_x(y) = x + y$ by the circle group. If ω is a multiplicative character, then $(\tau_x\omega)(y) = \omega(x+y) = \omega(x)\omega(y)$, and we compute that

$$D\omega(x+y) = (\tau_x D\omega)(y) = (D\tau_x\omega)(y) = (D\omega(x)\omega)(y) = (D\omega)(y)\omega(x).$$

Dirichlet

Photograph courtesy of the Institut Mittag-Leffler

G. Lejeune Dirichlet

Putting $y = 0$ shows that $D\omega$ is a multiple of ω. In other words, the differential operator D sends each multiplicative character into a multiple of itself, and the effect on $f(x)$ is that its Fourier coefficients are multiplied by various constants. The result is to be 0 term-by-term, and we obtain necessary and sufficient conditions on $f(x)$.

The work of Fourier provides a second illustration of the principle that although harmonic analysis may be at the core of the solution of a problem, several layers of ingenious ideas may lie between the statement of the problem and the use of harmonic analysis.

To Wiener [5] in the twentieth century, the above method for treating constant-coefficient differential operators is the stuff of harmonic analysis. We have a linear operator T carrying periodic functions to periodic functions and commuting with translations. The operator T must carry e^{inx} to a multiple $b_n e^{inx}$, and linearity yields the formula

$$T\left(\sum c_n e^{inx}\right) = \sum b_n c_n e^{inx} \tag{13}$$

on trigonometric polynomials. Under a suitable condition of boundedness or closed graph for T, (13) extends to all functions in the domain of T. Thus Fourier series provide a tool for understanding linear operators that commute with translations, i.e., that respect the group of translations as symmetries.

Further Use of Multiplicative Characters

The remainder of the nineteenth century saw a few other developments with harmonic analysis related to multiplicative characters. Cauchy, in the course of his investigations of water waves, began to work with integral solutions of partial differential equations and published in 1817 the reciprocal formulas

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(q) \cos qx \, dq$$

and

$$g(q) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xq \, dx.$$

It is not known whether Cauchy saw Fourier's 1811 manuscript containing a version of the Fourier inversion formula for the Fourier transform. In today's notation the Fourier transform and inversion formula are often written

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixy} \, dx \tag{14a}$$

and

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(y) e^{2\pi ixy} \, dy. \tag{14b}$$

Further work with the Fourier transform in the nineteenth century included the Poisson Summation Formula, which on a formal level relates the two functions of (14) by

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) = \sum_{n=-\infty}^{\infty} f(n),$$

and the development of the Mellin transform, which is a version of the Fourier transform or Fourier-Laplace transform written with the multiplicative positive reals in place of the additive reals. In an important application Riemann used the Poisson Summation Formula in one of his proofs of the functional equation for the ζ function. (The functional equation itself is due to Euler.) But more serious theoretical work with the Fourier transform itself had to wait for Lebesgue integration, which is a twentieth-century invention.

In 1840 Dirichlet published his theorem that an arithmetic progression $an + b$ with a and b relatively prime contains infinitely many primes as n runs through the positive integers. The proof is a generalization of Euler's argument for $4n + 1$ and $4n + 3$, replacing multiplicative characters on the multiplicative group prime to

4 by multiplicative characters ω on the multiplicative group G prime to a. To each ω corresponds a Dirichlet character χ modulo a, defined as the lift to the integers of ω, with 0 used as the value on integers that are not prime to a. In place of the two functions (6) and (7) are the Dirichlet L functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-\chi(p)p^{-s}},$$

one for each Dirichlet character χ modulo a. Harmonic analysis is contained most visibly in an inversion formula for multiplicative characters of this kind: If f is a complex-valued function on G, then

$$f(x) = \frac{1}{|G|} \sum_{\omega} \left[\sum_{y \in G} f(y)\overline{\omega(y)} \right] \omega(x). \tag{15}$$

There is another key ingredient in the proof, and harmonic analysis is present there also, albeit less visibly. Euler's proof for $a = 4$ used that $L(1) \neq 0$, and Dirichlet had to prove that all $L(s,\chi)$ are nonvanishing at $s = 1$. One proof of this assertion works with the product of all the L functions for fixed a and relates it to the ζ function of the field generated by the rationals and $e^{2\pi i/a}$; harmonic analysis is involved in this identification. The nonvanishing of the $L(s,\chi)$ at $s = 1$ then follows by showing that the ζ function of this field has a pole at $s = 1$.

Later Dedekind worked with multiplicative characters of the (finite abelian) ideal class group of the ring of algebraic integers of a number field (finite extension of the rationals), and in 1882 Weber introduced multiplicative characters for an arbitrary finite abelian group G. The multiplicative characters of G form a group $\widehat{G}$ under pointwise multiplication of their values. Distinct multiplicative characters are orthogonal in the sense that $\sum_{x \in G} \omega(x)\overline{\omega'(x)} = 0$, and the inversion formula (15) is valid.

Multiplicative characters are less helpful in exploiting a nonabelian group of symmetries. A multiplicative character must send every commutator $xyx^{-1}y^{-1}$ into 1. For a group that is generated by its commutators, as for example a nonabelian simple group, it follows that 1 is the only multiplicative character. To be able to do harmonic analysis with nonabelian groups, one introduces a multidimensional generalization of multiplicative character, the group representation. In Part II we shall examine group representations and their role in harmonic analysis.

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