Notes on Covering Spaces

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A. Fundamental group.

Notation: \( X \) denotes a separable metric space, i.e., a regular Hausdorff space with a countable base.

Paths and loops:
- **Path in** \( X \): continuous function \( a: [0, \|a\|] \to X \).
  \( \|a\| = \text{stopping time} \).
  \( a(0) = \text{initial point} \).
  \( a(\|a\|) = \text{final point or endpoint} \).
- **Loop in** \( X \): path with \( a(0) = a(\|a\|) \); \( a(0) \) is the base point for the loop.
- **Identity path**: a path with \( \|a\| = 0 \).
- **Constant path**: a path with \( a(t) = a(0) \).
- **Inverse path** \( a^{-1} \): \( a^{-1}(t) = a(\|a\| - t) \) for \( 0 \leq t \leq \|a\| \).

Multiplication of paths: If \( a \) and \( b \) are paths with \( a(\|a\|) = b(0) \), then their product \( c = a \cdot b \) is defined to be the path traced out by \( a \) and then \( b \):

\[
c(t) = \begin{cases} 
a(t) & \text{for } 0 \leq t \leq \|a\| \\
b(t - \|a\|) & \text{for } \|a\| \leq t \leq \|a\| + \|b\| \end{cases}.
\]

Properties:
1. The product \( c \) is a path with stopping time \( \|a\| + \|b\| \).
2. If \( a \cdot b \) and \( b \cdot c \) are defined, then \( (a \cdot b) \cdot c \) and \( a \cdot (b \cdot c) \) are defined and \( (a \cdot b) \cdot c = a \cdot (b \cdot c) \).
3. If \( i \) is an identity path, then \( i \cdot a = a \) whenever \( i \cdot a \) is defined and \( b \cdot i = b \) whenever \( b \cdot i \) is defined.
4. \( a \cdot a^{-1} \) and \( a^{-1} \cdot a \) are always defined.
Equivalence: Two paths $a$ and $b$ with the same initial points and same final points are equivalent, written $a \simeq b$, if there are continuous functions $s : [0,1] \to [0,\infty)$ and $h : \{(u,t)\} \to X$ (for $u \in [0,1]$ and $t \in [0,s(u)]$) such that

\[
\begin{align*}
    h(0,t) &= a(t), & h(u,0) &= a(0) = b(0) \\
    h(1,t) &= b(t), & h(u,s(u)) &= a(\|a\|) = b(\|b\|).
\end{align*}
\]

Picture:

Properties:

1. "Equivalent" is an equivalence relation; denote a class by $[a]$.
2. If $a \simeq a'$ and $b \simeq b'$ and if $a \cdot b$ is defined, then $a' \cdot b'$ is defined and $a \cdot b \simeq a' \cdot b'$. Also $a^{-1} \simeq a'^{-1}$.
3. Constant paths are equivalent with identity paths.
4. $a \cdot a^{-1}$ and $a^{-1} \cdot a$ are equivalent with constant paths.

Class multiplication:

If $a \cdot b$ is defined, set

\[
[a][b] = [ab] \quad \text{and} \quad [a]^{-1} = [a^{-1}].
\]

Both are well defined by (2).

Properties:

1. If $[a][b]$ and $[b][c]$ are defined, then $([a][b])[c]$ and $[a]([b][c])$ are defined and equal.
2. $[1][a] = [a]$, where $1 = \text{identity path } a(0)$

$[a][1] = [a]$, where $l = \text{identity path } a(\|a\|)$. 

(3) \([a][a]^{-1} = [1]\), where \(1\) = identity path \(a(0)\)
\([a]^{-1}[a] = [1]\), where \(1\) = identity path \(a(\|a\|)\).

**Proposition 1.** Fix \(p\) in \(X\). The set of classes of loops with
base point \(p\) is a group under class multiplication, denoted
\(\pi(X,p)\). If also \(q\) is in \(X\), then any path \(\xi\) from \(p\) to
\(q\) canonically defines an isomorphism of \(\pi(X,p)\) with \(\pi(X,q)\).

**Proof.** In the properties above, all products are now defined.

So \(\pi(X,p)\) is a group, by the properties. Let \(a\) be a loop
based at \(p\). Then \(\xi^{-1}a\xi\) is a loop based at \(q\), and this
correspondence defines the isomorphism.

**Definition:** \(\pi(X,p)\) = **fundamental group** of \(X\) with base point \(p\).

If \(X\) is pathwise connected, then \(\pi(X,p)\) as an abstract group is
independent of \(p\). We say a pathwise connected \(X\) is **simply
connected** if \(\pi(X) = 1\).

Let \(f: X \to Y\) be continuous. Then \(f\) induces a homomorphism
\(f_*: \pi(X,p) \to \pi(Y,f(p))\)

by \(f_*([a]) = [f \circ a]\), which is independent of the representative.
The induced homomorphism has the important property
\((f \circ g)_* = f_* g_*\).
B. Properties of covering spaces.

Notation: \( X \) and \( Y \) denote separable metric spaces that are pathwise connected and locally pathwise connected (i.e., each point has arbitrarily small pathwise connected neighborhoods. Then every open set has open connected components.

Let \( e : X \to Y \) be continuous, and let \( V \) be open in \( Y \). We say \( V \) is evenly covered by \( e \) if each connected component of \( e^{-1}(V) \) is mapped by \( e \) homeomorphically onto \( V \). (Note this implies \( V \) is connected.)

Let \( e : X \to Y \) be continuous. We say \( e \) is a covering map if each \( y \) in \( Y \) has an open neighborhood \( V_y \) that is evenly covered by \( e \). (Note that this implies \( e \) is onto \( Y \).) In this case \( Y \) is called the base space and \( X \) is the covering space.

**Proposition 2 (Path-lifting theorem).** Suppose \( e : X \to Y \) is a covering map. If \( y(t), 0 \leq t \leq 1 \), is a path in \( Y \) and if \( x_0 \) is in \( e^{-1}(y(0)) \), then there exists a unique path \( x(t), 0 \leq t \leq 1 \), in \( X \) with \( x(0) = x_0 \) and \( e(x(t)) = y(t) \).

Proof. Let \( T \) be the set of \( t \) in \([0,1]\) such that \( y|_{[0,t]} \) lifts to a path from \( x_0 \). The set \( T \) is nonempty since \( 0 \) is in \( T \). \( T \) is open. (In fact, let \( t_0 \) be in \( T \). Form the connected component \( U_0 \) of \( e^{-1}(V_y(t_0)) \) containing \( x(t_0) \), so that \( e^{-1} : V_y(t_0) \to U_0 \) is continuous. Extend \( x(t) \) by the definition \( x(t) = e^{-1}(y(t)) \); then we see that \( T \) is open. (Note that this definition is forced since \( x(t) \) for \( t \) near
Let $t_0$ must be a connected subset of $e^{-1}(V_y(t_0))$ containing $x(t_0)$ and so must be in the component $U_0$. \] $T$ is closed. \\
[In fact, let $t_0$ be a limit point of $T$ not in $T$. Form $V_y(t_0)$ and choose $\delta > 0$ so that $y(t)$ is in $V_y(t_0)$ for $t_0 - \delta \leq t \leq t_0$. Find the component of $e^{-1}(V_y(t_0))$ containing $x(t_0 - \delta)$ and lift $y$ to this component. As above, $t_0$ is in $T$. Thus $T$ is closed.] Since $[0,1]$ is connected, we conclude $T = [0,1]$.

For uniqueness let $T'$ be the set of $t$ in $[0,1]$ such that all lifts of $y|_{[0,t]}$ starting at $x_0$ agree. Then $T'$ is nonempty, and it is closed by continuity. $T'$ is open by the argument for $T$ above. By connectedness, $T' = [0,1]$.

**Lemma.** Let $e : X \to Y$ be a covering map and let $V$ be an open subset of $Y$ that is evenly covered. Then the components of $e^{-1}(V)$ are open.

**Proof.** Let $U_0$ be a component of $e^{-1}(V)$ and let $x_0$ be in $U_0$. Since $X$ is locally connected, choose a connected open neighborhood $U$ of $x_0$ contained in the open set $e^{-1}(V)$. Then $U \subseteq U_0$ since $U_0$ is a component, and so $U_0$ is open.

**Proposition 3** (Covering homotopy theorem). Let $e : X \to Y$ be a covering map, let $K$ be a compact space, and let $f_0 : K \to X$ be continuous. If $g : K \times [0,1] \to Y$ is continuous and satisfies $g(\cdot, 0) = ef_0$, then there is a unique continuous $f : K \times [0,1] \to X$ such that $f(\cdot, 0) = f_0$ and $g = ef$.

**Proof.** For each $k$ in $K$, Proposition 2 shows there is a unique path $f|_{k \times [0,1]}$ starting at $f_0(k)$ and covering the
path $g|_{k \times [0,1]}$. This defines $f$ and proves uniqueness.

We must prove $f$ is continuous as a function of two variables. Let $T$ be the set of $t_0$ in $[0,1]$ such that $f(k,t)$ is continuous at $(k,t)$ for all $k$ in $K$ and all $t \leq t_0$.

Then 0 is in $T$. [In fact, fix $k_0$ in $K$. Form the component $U_0$ of $e^{-1}(V_{g_0}(k_0))$ containing $f_0(k_0)$, so that $e^{-1} : V_{g_0}(k_0) \rightarrow U_0$ is continuous. $U_0$ is open by the lemma.

Choose a neighborhood $N_{k_0}$ of $k_0$ so that $f_0(N_{k_0}) \subseteq U_0$ and $g(N_{k_0} \times [0,\varepsilon)) \subseteq V_{g_0}(k_0)$. Then $f(N_{k_0} \times [0,\varepsilon)) \subseteq U_0$. Hence $f = e^{-1}g$ on $N_{k_0} \times [0,\varepsilon)$ and is continuous.]

$T$ is open. [In fact, let $t_0$ be in $T$. Fix $k_0$ in $K$. Form the component $U_0$ of $e^{-1}(V_g(k_0,t_0))$ containing $f(k_0,t_0)$; then $e^{-1} : V_g(k_0,t_0) \rightarrow U_0$ is continuous. Choose a neighborhood $N_{k_0}$ of $k_0$ and an $\varepsilon_{k_0}$ so that $f(N_{k_0} \times (t_0 - \varepsilon_{k_0},t_0 + \varepsilon_{k_0})) \subseteq U_0$. Then $f = e^{-1}g$ on $N_{k_0} \times (t_0 - \varepsilon_{k_0},t_0 + \varepsilon_{k_0})$ and so is continuous on this set. The $N$'s cover $K$. Extract a finite subcover and use the minimum of the $\varepsilon$'s to see that $T$ is open.]

$T$ is closed. [In fact, let $t_0$ be a limit point of $T$ not in $T$. Fix $k_0$ in $K$. Form $V = V_g(k_0,t_0)$, and choose $N'$ (a neighborhood of $k_0$) and $\varepsilon$ so that $g(N' \times [t_0 - \varepsilon,t_0 + \varepsilon]) \subseteq V$. Form the component $U_0$ of $e^{-1}(V)$ containing $f(k_0,t_0 - \varepsilon)$; then $e^{-1} : V \rightarrow U_0$ is continuous. Choose a neighborhood $N \subseteq N'$ so that $f(N,t_0 - \varepsilon) \subseteq U_0$. Then $f(N \times [t_0 - \varepsilon,t_0 + \varepsilon]) \subseteq U_0$, $f = e^{-1}g$ on this set, and $f$ is continuous at $(k_0,t_0)$. So $T$ is closed.] By connectedness, $T = [0,1]$. 
Proposition 4. If \( e : X \to Y \) is a covering map, then a path \( x(t) \) in \( X \) is a contractible loop (i.e., a loop equivalent with a constant map) if and only if the projected path \( e \circ x(t) \) is a contractible loop. Consequently \( e_* \) is one-one.

Proof. \( \Rightarrow \) is trivial. For \( \Leftarrow \), put \( f_0(t) = x(t) \), and suppose \( e \circ x(t) \) is contractible. Then we can find a continuous \( g(t,s) \) for \( s \in [0,1] \) such that \( g(\cdot,0) = e \circ x(\cdot) \), \( g(0,s) = e(0) \), \( g(1,s) = e(1) = e(0) \), and \( g(\cdot,1) = e(0) \). Find \( f \) as in Proposition 3. Then \( f(\cdot,0) = x(\cdot) \). Also \( f(0,s) \) is continuous into the discrete space \( e^{-1}(e(0)) \) and so is constant and must be \( x(0) \). Similarly \( f(\cdot,1) = x(0) \), and then \( f(1,s) = x(0) \). So \( x(0) = x(1) \), and \( x(t) \) is equivalent with the constant \( x(0) \).

Proposition 5. If \( e : X \to Y \) is a covering, if \( y_0 \) is in \( Y \), and if \( x_0 \) is in \( e^{-1}(y_0) \), then the lift of a loop \( y(t) \) based at \( y_0 \) is a loop if and only if \( [y(t)] \) is in \( e_* \pi(X,x_0) \).

Proof. \( \Rightarrow \) is trivial. For \( \Leftarrow \), choose a loop \( x'(t) \) based at \( x_0 \) in \( X \) such that \( [e'(t)] = [y(t)] \), and let \( g \) exhibit \( e' \) and \( y \) as equivalent: \( g(\cdot,0) = e'(\cdot) \), \( g(\cdot,1) = y(\cdot) \), \( g(0,s) = g(1,s) = y_0 \). Produce \( f \) as in Proposition 3. As above, \( f(\cdot,1) \) is a loop based at \( x_0 \), and it is the lift of \( y(t) \).

Theorem 1 (Map-lifting theorem). If \( e : X \to Y \) is a covering, if \( P \) is a pathwise connected, locally pathwise connected separable metric space, if \( g : P \to Y \) is continuous, and if \( p_0 \) is in \( g^{-1}(y_0) \) and \( x_0 \) is in \( e^{-1}(y_0) \), then there exists a continuous \( f : P \to X \) with \( f(p_0) = x_0 \) and \( g = ef \) if and only if \( e_* (\pi(P,p_0)) \subseteq e_* (\pi(X,x_0)) \). When \( f \) exists, it is unique.
Proof. If $f$ exists, then
\[ e_* \pi(P, p_0) = e_* \pi^*(P, p_0) \subseteq e_* \pi(X, x_0). \]
Conversely, suppose the inclusion holds. Let $p$ be in $P$ and let $w$ be a path from $p_0$ to $p$. Then $gw$ is a path from $y_0$ to $y = g(p)$, and we let $u$ be the lift of $gw$ to a path from $x_0$ to some $x$ in $X$. Define $f(p) = x$. Another choice of $w$ leads to a loop in $P$ that is mapped under $g$ to a loop in $Y$. Since the class of the loop in $Y$ is in $e_* \pi(P, p_0) \subseteq e_* \pi(X, x_0)$, Proposition 5 shows the loop lifts to a loop in $X$; therefore $x$ is uniquely determined by this definition.

To see that $f$ is continuous at $p$, let $U_0$ be the component of $e^{-1}(V_g(p))$ containing $f(p)$, so that $e^{-1} : V_g(p) \to U_0$ is continuous. Choose a pathwise connected neighborhood $N$ of $p$ in $P$ so that $g(N) \subseteq V_g(p)$. Then a path from $p$ within $N$ is mapped by $g$ and lifts to a path from $f(p)$ within $U_0$. Hence $f(N) \subseteq U_0$. Then $f = e^{-1}g$ on $N$, and $f$ is continuous on $N$.

If $f$ exists, then the conditions in the definition above must be satisfied. So $f$ is unique. This completes the proof.

Notation for uniqueness theorem: Let $e : X \to Y$ and $e' : X' \to Y$ be coverings. We say $e$ and $e'$ are equivalent coverings if there is a homeomorphism $t$ of $X$ onto $X'$ such that $e' \circ t = e$.
Theorem 2 (Uniqueness theorem). Let $e : X \to Y$ and $e' : X' \to Y$ be coverings, and let $y_0$ be in $Y$. Then $e$ and $e'$ are equivalent coverings if and only if base points $x_0$ in $X$ and $x'_0$ in $X'$ can be chosen so that $e(x_0) = e'(x'_0) = y_0$ and

$$e_*(\pi(X,x_0)) = e'_*(\pi(X',x'_0)).$$

Proof. If $e$ and $e'$ are equivalent, choose $x_0'$ in $e^{-1}(y_0)$ and define $x'_0 = \iota(x_0)$. Then $e_*\pi(X,x_0) = e'_*\iota_*\pi(X,x_0) = e'_*\pi(X',x'_0)$.

Conversely suppose $e_*\pi(X,x_0) = e'_*\pi(X',x'_0)$. By Theorem 1, we can cover $e : X \to Y$ by $i : X \to X'$ and we can cover $e' : X' \to Y$ by $j : X' \to X$. Now $jt$ leaves $x_0$ fixed and $e(jt) = (e) = e'$, so $j$ covers the identity map of $Y$ into itself and by the uniqueness in Theorem 1 is the identity map of $X$. Similarly $ij$ is the identity, and therefore $ij$ is a homeomorphism. By construction $e' = e$, and therefore $i$ defines an equivalence.
C. Existence of covering spaces.

Notation: \( X \) and \( Y \) are separable metric spaces, pathwise connected, and locally pathwise connected.

In proving existence of covering spaces, the next proposition will guide the choice of open sets in \( Y \) that are to be evenly covered by \( e \). In particular, we shall want to assume that such sets exist.

**Proposition 6.** If \( e : X \to Y \) is a covering, then any pathwise connected open subset \( Q \) of \( Y \) such that any loop in \( Q \) is contactible in \( Y \) is evenly covered.

**Proof.** Fix \( y_0 \) in \( Q \), \( x_0 \) in \( e^{-1}(y_0) \). Lift the paths in \( Q \) from \( y_0 \) to paths in \( e^{-1}(Q) \) from \( x_0 \). Let \( P_{x_0} \) be the union of the images of these paths, and let \( e' = e|_{P_{x_0}} \). Then \( e' \) is continuous from \( P_{x_0} \) onto \( Q \).

We show \( e' \) is one-one. Thus let \( p_1 \) and \( p_2 \) be in \( e'^{-1}(q) \). Connect \( x_0 \) to \( p_1 \) and \( p_2 \) by paths in \( e^{-1}(Q) \). The projections to \( Q \) yield a loop in \( Q \), which is in \( e_*(\pi(X,x_0)) \) since it is contactible in \( Y \) (by hypothesis). By Proposition 5 the lift is a loop and \( p_1 = p_2 \). Thus \( e' \) is one-one.

We show \( e' \) is onto \( Q \). If \( q \) is given, join \( y_0 \) to \( q \) by a path and lift to a path from \( x_0 \) to some \( x \). Then \( e'(x) = q \).

We show \( e'^{-1} \) is continuous. If \( y \) is in \( Q \), let
\( V \subseteq Q \) be a pathwise connected evenly covered neighborhood of \( y \), let \( x \) be in \( P_{x_0} \cap e^{-1}(y) \), and let \( U \) be the component of \( e^{-1}(V) \) containing \( x \), so that \( e^{-1} : V \to U \) is continuous. Then \( e^{-1} \circ e^{-1} = e^{-1} \) on \( V \), and \( e^{-1} \) is thus continuous at \( y \). Hence \( e^{-1} \) is a homeomorphism of \( P_{x_0} \) onto \( Q \).

Clearly \( P_{x_0} \) is connected. If \( x_0 \neq x \), then \( P_{x_0} \) and \( P_x \) coincide or are disjoint since a point of intersection joins \( x_0 \) to \( x \). Also \( P_{x_0} \) is open: In fact, let \( x \) be in \( P_{x_0} \) and let \( U \subseteq e^{-1}(Q) \) be a pathwise connected open neighborhood of \( x \). Then \( U \) is path connected to \( P_{x_0} \) and so \( U \subseteq P_{x_0} \). Thus \( P_{x_0} \) is open.

Finally the union of all \( P_x \) for \( x \) in \( e^{-1}(Q) \) is \( e^{-1}(Q) \), and so \( P_{x_0} \) is a component of \( e^{-1}(Q) \). Thus \( Q \) is evenly covered.

We say \( Y \) is **locally simply connected** if each \( y \) in \( Y \) has an open pathwise connected, simply connected neighborhood. In this case, each \( y \) in \( Y \) has arbitrarily small open pathwise connected neighborhoods with the following property: Any loop in the neighborhood is contractible in \( Y \).

**Lemma.** If \( Y \) is locally simply connected, then \( \pi(Y, y_0) \) is countable.

**Proof.** Let \( \mathcal{V} \) be a countable base of open sets of \( Y \) that are pathwise connected and contain only loops that are contractible in \( Y \). If \( V_j \) and \( V_k \) are in \( \mathcal{V} \), then \( V_j \cap V_k \) has a countable number of components (since \( Y \) is separable and the
components are open). Let $V^*$ be the collection of all components of all $V_j \cap V_k$. By the separability we can cover each $V_j$ by countably many open sets $V_j'$ such that $V_j' \subseteq V_j$. Let $f(t), 0 \leq t \leq 1$, be a loop based at $y_o$. Cover $[0,1]$ with open intervals such that $f$ of each interval is in some $V_j'$, and, by compactness of $[0,1]$, extract a finite subcover. If $V_j' \subseteq V_j$ and if $f$ maps some open interval into $V_j'$, then $f$ maps the closure of the interval into $V_j$, by continuity. As a result, we can choose points $t_i$ and sets $V_i$, $0 \leq i \leq n$, with $t_0 = 0, t_n = 1$, $f(t_i) \in V_i \cap V_{i-1}$ for $i \geq 1$, and $f(t) \in V_1$ for $t_i \leq t \leq t_{i+1}$. Let $V^*_1$ be the component of $V_1 \cap V_{i-1}$ to which $f(t_i)$ belongs. To $f$ we can associate the finite sequence

$$V_1, V^*_1, V_2, V^*_2, \ldots, V^*_{n-1}, V_n$$

and this sequence determines $[f]$. (To see this, suppose $g$ is given with the same sequence. We may assume without loss of generality that $g(t_i) = f(t_i)$ for all $i$ since $V^*_i$ is pathwise connected. Then $f \simeq g$ for each interval $t_i \leq t \leq t_{i+1}$, because taken in succession they give a loop in $V_i$.) The set of sequences is countable, and so $\pi(Y,y_o)$ is countable.

**Theorem 3** (Existence theorem). If $Y$ is locally simply connected, if $y_0$ is in $Y$, and if $H$ is a subgroup of $\pi(Y,y_0)$, then there exists a covering space $X$ with covering map $e : X \to Y$ and with point $x_0$ in $X$ such that $e(x_0) = y_0$ and $e_* (\pi(X,x_0)) = H$. 


Proof. Let \( X \) be the set of equivalence classes of paths in \( Y \) from \( y_0 \) under the equivalence relation that \( f \sim f' \) if
1. \( f \) and \( f' \) have the same final point
2. \( [f'.f^{-1}] \) is in \( H \).

This is an equivalence relation because \( H \) is a group.

Typical equivalence classes will be denoted \( x \) or \( [f] \). Let \( x_0 \) be the class of the constant path at \( y_0 \). For \( x \) in \( X \), let \( e(x) \) be the endpoint of a path in the class \( x \); then \( e(x_0) = y_0 \).

Let \( V \) be a base of open sets in \( Y \) that are pathwise connected and contain only loops that are contractible in \( Y \).

For \( V \) in \( V \) and \( x \) in \( e^{-1}(V) \), define \( U(x,V) \subseteq X \) by

\[
U(x,V) = \{x' \in e^{-1}(V) \mid x' = \text{class of some } f \cdot h, \text{ where } [f] = x \text{ and } h \text{ is a path starting at } e(x) \text{ and remaining within } V \}.
\]

The sets \( U(x,V) \) have the following properties:

1. A class \( x' \) in \( U(x,V) \) is not affected by using a different representative \( f \), nor is it affected by using a different \( h \) as long as the endpoint of \( h \) is not changed. [In fact, \( [fh] = [f'h] \) because \( fh \) and \( f'h \) have the same endpoint and \( [f'h^{-1}.f^{-1}] = [f.f^{-1}] \in H \). Also, if \( h \) and \( h' \) both have the same endpoint, then so do \( f'h \) and \( f'h' \); since \( h'h^{-1} \) is contractible in \( Y \), we have \( [f'h^{-1}.f^{-1}] = [f.f^{-1}] = [1] \in H \) and hence \( [fh] = [f'h'] \).]

2. If \( x' \) is in \( U(x,V) \), then \( U(x',V) = U(x,V) \).
[In fact, let \( x'' \) be in \( U(x,V) \), and write \( x' = [f_0 \cdot h_0] \) and \( x'' = [f \cdot h] \). Applying (1), we have...
\[ x'' = [f \cdot h] = [f \cdot h_0 \cdot h_0^{-1} \cdot h] = [(f \cdot h_0) \cdot (h_0^{-1} \cdot h)]. \] 

On the right side, \([f \cdot h_0] = x'\) because \(f \cdot h_0\) has endpoint the same as for \(f_0 \cdot h_0\) and because \(x = [f] = [f_0]\) implies \([f_0 \cdot h_0 \cdot h_0^{-1} \cdot f^{-1}] = [f_0 \cdot f^{-1}] \in H\).

Also \(h_0^{-1} \cdot h\) is a path within \(V\) starting at \(e(x')\). Thus (*) shows that \(x''\) is in \(U(x', V)\). Hence \(U(x, V) \subseteq U(x', V)\).

In particular, \(x\) is in \(U(x', V)\). Then we can repeat the above argument with \(x\) and \(x'\) interchanged to conclude \(U(x', V) \subseteq U(x, V)\).

(3) If \(x\) is in \(U(x_1, V_1) \cap U(x_2, V_2)\), choose \(V\) so that \(e(x)\) is in \(V\) and \(V \subseteq V_1 \cap V_2\). Then \(x\) is in \(U(x, V)\) and \(U(x, V) \subseteq U(x_1, V_1) \cap U(x_2, V_2)\).

[In fact, \(U(x, V) \subseteq U(x, V_1) = U(x_1, V_1)\), with the equality holding by (2). Similarly, \(U(x, V) \subseteq U(x_2, V_2)\), and the assertion follows.]

Now let \(\mathcal{U} = \{U(x, V)\}\). \(\mathcal{U}\) is a base for a topology of \(X\), according to (3), and \(e\) is continuous since \(e^{-1}(V)\) is the union of all \(U(x, V)\) for \(x\) in \(e^{-1}(V)\). Next, \(e\) is one-one on \(U(x, V)\). [In fact, suppose \(x\) and \(x'\) are in \(U(x, V)\), \(e(x) = e(x')\), and \([f] = x\). The path \(h\) that exhibits \(x'\) as in \(U(x, V)\) is then a loop in \(V\), hence contractible. Thus \(x = [f] = [f \cdot h] = x'\).]

Also \(e\) maps \(U(x, V)\) onto \(V\). [In fact, let \(v\) be given in \(V\), and join \(e(x)\) to \(v\) by a path \(h\) in \(V\). Then \(f \cdot h\) defines a point \(x'\) of \(X\) with \(e(x') = v\) and exhibits \(x'\) as in \(U(x, V)\).]

Moreover, \(e^{-1} : V \rightarrow U(x, V)\) is continuous because \((e^{-1})^{-1}(U(x', V')) = V'\) is open. Thus \(e : U(x, V) \rightarrow V\) is a
homeomorphism. Since \( V \) is connected, \( U(x,V) \) is connected. The set \( U(x,V) \) is open by definition, and \( e^{-1}(V) \) is the union of the \( U(x,V) \) for \( x \) in \( e^{-1}(V) \), with the \( U(x,V) \) disjoint or equal, by (2). Consequently the sets \( U(x,V) \) are the connected components of \( e^{-1}(V) \). Hence \( V \) is evenly covered by \( e \).

Now we prove the appropriate topological properties of \( X \). Since \( U(x,V) \) is open and is homeomorphic with \( V \), \( X \) is locally pathwise connected. To see that \( X \) is pathwise connected, let \( x \) in \( X \) be given. Then \( x \) is a class of paths in \( Y \) starting at \( y_0 \). Pick such a path \( f \). Then for \( 0 \leq t \leq \|f\| \), \( \Upsilon(t) = f\left|[0,t]\right| \) is a path from \( x_0 \) to \( x \) in \( X \). The space \( X \) is Hausdorff and regular because these properties are local properties and they hold in \( Y \). To complete the argument that \( X \) is a separable metric space, it is enough, in view of the metrization theorem, to prove that \( X \) has a countable base.

To prove that \( X \) has a countable base, we may assume that \( U \) is countable. For each \( V \) in \( U \), select one \( y \) in \( V \). Then the number of sets \( U(x,V) \) with \( e(x) = y \) is the same as the number of elements of \( e^{-1}(y) \), which is the number of classes of paths from \( y_0 \) to \( y \), modulo \( H \). Since \( \pi(Y,y_0) \) is countable by the lemma, \( U \) is countable.

Finally we are to show that \( e_*(\pi(X,x_0)) = H \). Let \( \Upsilon \) be a loop based at \( x_0 \) and let \( f = e\Upsilon \). If \( x \) is the point in \( X \) corresponding to \( f \), then \( f'(t) = f\left|[0,t]\right| \) is a path from \( x_0 \) to \( x \) covering \( f \), and so \( f' = \Upsilon \). Thus \( x \) is the endpoint of \( \Upsilon \), which is \( x_0 \). Consequently \( f \) and the
constant path represent the same point in $X$, and $[f]$ must be in $H$. Thus $e_\ast(\pi(X; x_0)) \subseteq H$. In the reverse direction, let $[f]$ be in $H$, and lift $f$ to $\tilde{f}$. Again $f$ represents $x_0$ since $[f]$ is in $H$, and it represents the endpoint of $\tilde{f}$. Thus $\tilde{f}$ is a loop, and Proposition 5 shows that $[f]$ is in $e_\ast(\pi(X; x_0))$. Hence $e_\ast(\pi(X; x_0)) = H$. This completes the proof.

By Theorems 3 and 2, if $Y$ is locally simply connected, $Y$ has a simply connected covering space that is unique up to equivalence. This space is called the universal covering space of $Y$. 
D. Computation of fundamental groups.

We shall establish formulas for \( \pi(X) \) for some basic spaces \( X \) and then show how \( \pi(Y) \) can often be computed when a covering \( e : X \rightarrow Y \) is given and \( \pi(X) \) is known. This will allow us to compute \( \pi(X) \) in all cases of interest for Lie group theory.

**Proposition 7.** \( \mathbb{R}^n \) is simply connected.

*Proof.* Without loss of generality, let the base point be 0.

Let \( f(t) \) be a loop based at 0. Then \( h(u,t) = (1-u)f(t), \)
\( 0 \leq u \leq 1, \) exhibits \( f(t) \) as equivalent with a constant path.

**Proposition 8.** If \( X \) and \( Y \) are pathwise connected separable metric spaces with \( x_o \in X \) and \( y_o \in Y \), then \( \pi(X \times Y, (x_o,y_o)) \) is canonically isomorphic with \( \pi(X,x_o) \oplus \pi(Y,y_o) \).

*Proof.* We map \( \pi(X \times Y, (x_o,y_o)) \) to \( \pi(X,x_o) \oplus \pi(Y,y_o) \) by mapping a loop \((f(t),g(t))\) based at \((x_o,y_o)\) to
\( ([f(t)],[g(t)]). \) If \( p_X \) and \( p_Y \) denote the projection maps of \( X \times Y \) onto \( X \) and \( Y \), respectively, then this map can be written as \( ((p_X)_*,(p_Y)_*)^* \); hence it is a well-defined map on the fundamental group and is a group homomorphism. To see it is onto, let \([f(t)] \in \pi(X,x_o)\) and \([g(t)] \in \pi(Y,y_o)\).

Without loss of generality, we may assume \( 0 \leq t \leq 1 \) in both cases. Then \([ (f(t),g(t)) \] maps onto \([ [f(t)],[g(t)] \). Thus the map is onto. To see the map is one-one, suppose \([f(t)]=1\)
and \([g(t)]=1\). Again suppose \( 0 \leq t \leq 1 \) in both cases.

Find \( h_f(u,t) \) with \( h(u,0) = h(u,1) = x_o, \) \( h(0,t) = f(t), \)
and \( h(l,t) = x_o \). Find \( h_g(u,t) \) similarly. Then 
\[
h(f,g)(u,t) = (h_f(u,t), h_g(u,t)) \text{ exhibits } [(f(t), g(t))]
\]
as equal to 1.

**Proposition 9.** Let \( X = \mathbb{R}^1 \) = line and \( Y = \{z \in \mathbb{C} \mid |z| = 1\} \) = circle. Let \( e : X \to Y \) be the map \( e(x) = e^{ix} \). Then \( e \) is a covering map, \( \pi(Y) = \mathbb{Z} \), and \( y(t) = e^{it}, 0 \leq t \leq 2\pi \), is a generator of \( \pi(Y) \).

**Proof.** Let \( e^{i\theta} \) in \( Y \) be given, \( 0 \leq \theta \leq 2\pi \). Choose 
\[
V_\theta = \{e^{i\varphi} \mid |\varphi - \theta| < \pi/2\}.
\]
Then 
\[
e^{-1}(V_\theta) = \bigcup_{n=-\infty}^{+\infty} \{\varphi \in X \mid |\varphi - \theta - 2\pi n| < \pi/2\}
\]
disjointly, with each set homeomorphic to \( V_\theta \). Hence \( e \) is a covering map.

We use \( x_o = 0 \) as base point in \( X \). By Proposition 7, \( \pi(X,0) = \{1\} \). Let \( y_n(t) = e^{it}, 0 \leq t \leq 2\pi n \). The lift of \( y_n \) is \( x_n(t) = t, 0 \leq t \leq 2\pi n \), which is not a loop. By Proposition 5, \([y_n]\) is not in \( e_* \pi(X,0) = \{1\} \). By definition \([y_n] = [y_1]^n\). Hence \( y_1 \) generates an infinite cyclic group contained in \( \pi(Y,1) \).

Now suppose \( y^*(t), 0 \leq t \leq 1 \), is any loop based at 1 in \( Y \). Lift to a path \( x^*(t), 0 \leq t \leq 1 \), based at 0. Then \( e(x^*(1)) = 1 \) implies \( x^*(1) = 2\pi n \) for some integer \( n \).

Form \( y^* \cdot y_n^{-1} \). This path lifts to a loop in \( X \) (necessarily contractible) and so is contractible in \( Y \), by Proposition 4. Thus \([y^*] = [y_n] = [y_1]^n\). Hence \( y_1 \) generates all of \( \pi(Y,1) \).
Proposition 10. The n-sphere $S^n$ is simply connected if $n > 1$.

Proof. This follows from the next lemma if we take

$$X_1 = S^n - \{(1,0,\ldots,0)\} \quad \text{and} \quad X_2 = S^n - \{(-1,0,\ldots,0)\}.$$ 

Lemma. Let $X$ be pathwise connected and locally pathwise connected. If there exist connected, simply connected open subsets $X_1$ and $X_2$ of $X$ with $X = X_1 \cup X_2$ and with $X_1 \cap X_2$ connected, then $\pi(X) = 1$.

Proof. Fix $x_0$ in $X_1 \cap X_2$. (If $X_1 \cap X_2$ is empty, one of $X_1$ and $X_2$ must be empty, and the result follows.) Let $a(t)$ be a loop based at $x_0$. Without loss of generality, we may assume $0 \leq t \leq 1$. To each point in $(0,1)$, we can associate an open interval centered at the point such that it maps the closure of this interval completely into $X_1$ or completely into $X_2$. For $t = 0$, similarly, a maps the closure of some (relatively) open interval $[0,\epsilon)$ into one of $X_1$ and $X_2$, and for $t = 1$, we similarly obtain a (relatively) open interval $(\epsilon',1]$. By compactness of $[0,1]$, we extract a finite subcover and obtain a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $a([t_{i-1},t_i]) \subset X_{k_i}$ for $1 \leq i \leq n$. Construct a path $b_i$, $0 \leq i \leq n$, from $x_0$ to $a(t_i)$ such that $b_i$ remains in whatever $X_j$'s $a(t_i)$ is in. This is possible since $X_1 \cap X_2$ is pathwise connected. Let $a_i(t) = a(t+t_{i-1})$ for $0 \leq t \leq t_i$, $1 \leq i \leq n$. Then

$$[a] = [b_0 a_1 b_1^{-1}] [b_1 a_2 b_2^{-1}] \cdots [b_{n-1} a_n b_n^{-1}] .$$

Fix $i$. Then $a_i(t) \subset X_{k_i}$, in particular for $t = 0$ and $t = t_i - t_{i-1}$. So $b_i^{-1} \subset X_{k_i}$ by construction. Thus
\[ b_{i-1}a_{i-1} \] lies in \( X_{k_i} \). Since \( X_{k_i} \) is simply connected, 
\[ [b_{i-1}a_{i-1}] = 1. \] Therefore \([a] = 1\).

Let \( X \) be the universal covering space of a pathwise connected, locally pathwise connected, locally simply connected, separable metric space \( Y \), and let \( e : X \to Y \) be the covering map. A deck transformation of \( X \) is a homeomorphism \( f \) of \( X \) that satisfies \( ef = e \).

**Theorem 4.** Let \( e : X \to Y \) be a universal covering map (as above), and let \( e(x_0) = y_0 \). Then

1. \( \pi(Y, y_0) \) is in one-one correspondence with \( e^{-1}(y_0) \), the correspondence being that \( x_1 \in e^{-1}(y_0) \) corresponds to \([e(\text{any path from } x_0 \text{ to } x_1)]\).

2. the group of deck transformations \( H \) of \( X \) acts simply transitively on \( e^{-1}(y_0) \).

3. the correspondence that associates to a deck transformation \( f \) in \( H \) the member of \( \pi(Y, y_0) \) corresponding to \( f(x_0) \) is a group isomorphism of \( H \) onto \( \pi(Y, y_0) \).

**Proof.** (1) Let \([y(t)], y(0) = y_0, \) be in \( \pi(Y, y_0) \). Let \( x(t) \) be its lift with \( x(0) = x_0 \), and let \( x_y \) be the endpoint. We make \( y \to x_y \). This map is independent of the representative because a contractible loop based at \( y_0 \) lifts to a loop in \( X \), by Proposition 4. For the inverse correspondence, let \( x_1 \) be given. Then \([e(\text{any path from } x_0 \text{ to } x_1)]\) is well defined because any loop based at \( x_0 \) is contractible, \( X \) being simply connected. The result is in \( \pi(Y, y_0) \) and defines the inverse map.
(2) If $x_1$ is in $e^{-1}(y_0)$, we apply Theorem 1 to the diagram

```
   X
  /  \    \ f
 /    \   f(x_0) = x_1.
/      \  \ e
/        \ g=e
X ------> Y
```

Then $g_*([\pi(X,x_0)]) = [1]$, and so $f$ exists with $ef = g$. Arguing similarly, we see that $f$ is a homeomorphism. This proves transitivity. The transitivity is simple by the uniqueness in Theorem 1.

(3) The map in question is one-one onto by (1) and (2). We show it is a homomorphism. Let $f$ and $g$ be in $H$. Then $f \circ g$ corresponds to $[ex(t)]$, where $x$ is any path from $x_0$ to $f(g(x_0))$. If we choose $x(t)$ to pass through $f(x_0)$ on the way, we see that the problem is to show that $[eu(t)] = [ev(t)]$ if

- $u(t)$ is a path from $x_0$ to $g(x_0)$
- $v(t)$ is a path from $f(x_0)$ to $f(g(x_0))$.

Now $[ef(u(t))] = [eu(t)]$ since $f$ is a deck transformation, and $f(u(t))$ is a path from $f(x_0)$ to $f(g(x_0))$. Since $X$ is simply connected, $[ef(u(t))] = [ev(t)]$. Therefore $[eu(t)] = [ev(t)]$. 
E. Topological groups.

A topological group is a Hausdorff space that is a group such that multiplication and inversion are continuous.

Properties of a topological group $G$:

1. Left and right translations are homeomorphisms.
   
   Proof. The continuity of multiplication implies the continuity of translation, by restriction. Translation is a homeomorphism because translation by the inverse element is continuous.

2. To each neighborhood $V$ of the identity $1$ corresponds a neighborhood $U$ such that $UU^{-1} \subseteq V$.
   
   Proof. This is the statement of continuity of the map $(x,y) \rightarrow xy^{-1}$ at $(x,y) = (1,1)$.

3. $G$ is regular as a topological space.
   
   Proof. If a point and a closed set are given, we may assume the point is $1$, by (1). Then find $U$ in (2) for $V = G - F$. We claim that $U \subseteq V$. In fact, if $x$ is in $U - U$, then $xU$ is a neighborhood of $x$ and so meets $U$.
   
   If $y$ is in $xU \cap U$, then $y = xu$, $x = yu^{-1} \in UU^{-1} \subseteq V$.

4. Let $H$ be a closed subgroup, and let $G/H$ have the quotient topology. Then the projection $p : G \rightarrow G/H$ is open, and $G/H$ is a Hausdorff regular space such that the action of $G$ on $G/H$ is jointly continuous. If $G$ has a countable base, so does $G/H$. 
Proof. The set $E \subseteq G/H$ is open if and only if $p^{-1}(E)$ is open. Let $U$ be open in $G$. Then $p^{-1}(p(U)) = U \cup \{h \in H\}$ which is open. The action of $G$ on $G/H$ is the composition multiplication \[ G \times G \to G \to G/H, \] which is continuous.

If $x$ is in $G/H$ and $F$ is a disjoint closed subset, we may assume $x = 1$ by this continuity. Choose a neighborhood $U$ of 1 in $G$ and a neighborhood $N$ of 1 in $G/H$ such that $UN \subseteq G/H - F$. As in (3), we claim $\overline{N} \subseteq UN$. [In fact, if $y$ is in $\overline{N}$, then $U^{-1}y$ is a neighborhood of $y$ since $p$ is open. Hence $U^{-1}y \cap N$ is not empty; let $z$ be a member. Then $z = u^{-1}y$, and $y = uz \in UN$.] Consequently $G/H$ is regular. Next, $p^{-1}([x]) = xH$ is closed, and so $[x]$ is closed. Thus $G/H$ is a $T_1$ regular space and must be Hausdorff. If $\mathcal{U}$ is a base of $G$, $p\mathcal{U}$ is a base of $G/H$ since $p$ is open. Hence if $G$ has a countable base, so does $G/H$.

(5) Let $H$ be a closed subgroup. If $H$ and $G/H$ are connected, then $G$ is connected.

Proof. Let $G = U \cup V$ with $U$ and $V$ nonempty, open, and disjoint. For each $x$ in $G$, $xH$ is connected, by (1). Since $xH \subseteq U \cup V$, we must have $xH \subseteq U$ or $xH \subseteq V$. Let $A = \{x \in G \mid xH \subseteq U\}$ and $B = \{x \in G \mid xH \subseteq V\}$. We have just seen that $G = A \cup B$. Since $U$ and $V$ are complements, $A = \{x \in G \mid xH \cap V = \emptyset\}$. It follows that $A = \{x \in G \mid xH \cap p^{-1}pV = \emptyset\}$. Consequently (4) shows that $A$ is open in $G$. Similarly
B is open in G. Since \( p^{-1}pA = A \) and \( p^{-1}pB = B \),
we see that \( G/H = pA \cup pB \) is a disjoint decomposition
of \( G/H \) into open sets. By connectedness of \( G/H \), one
of \( pA \) and \( pB \) is empty, say \( pB \). Then \( B \) is empty,
\( A = G \), and \( U = G \). We conclude \( G \) is connected.

(6) Let \( H \) be a closed subgroup. If \( H \) and \( G/H \) are compact,
then \( G \) is compact.

Proof. Let \( \mathcal{U} \) be an open cover of \( G \). For each \( x \) in \( G \),
\( \mathcal{U} \) is an open cover of \( xH \). Let \( \mathcal{V}_x \) be a finite
subcover and let
\[
V_x = \{ y \in G \mid yH \text{ is covered by } \mathcal{V}_x \}.
\]
We prove \( V_x \) is open. Fix \( y \) in \( V_x \). For each \( h \in H \) we
can find \( U_h \in \mathcal{V}_x \) with \( yh \in U_{yh} \). By continuity of
multiplication, we can then find open neighborhoods \( M_h \) of \( y \)
and \( N_h \) of \( h \) such that \( M_h N_h \subseteq U_h \). The open sets \( N_h \)
cover \( H \), and we let \( \{N_{h_j}\} \) be a finite subcover. Then
\[
M = \bigcap_j M_{h_j}
\]
is an open neighborhood of \( y \). If \( z \) is in \( M \),
we show \( z \) is in \( V_x \). Let \( h \) be given and let \( h \in N_{h_j} \).
Then \( z \in M_{h_j} N_{h_j} \subseteq U_{h_j} \in \mathcal{V}_x \). Hence \( z \) is in \( V_x \), and we
conclude \( V_x \) is open. The sets \( pV_x \) cover \( G/H \). Since \( G/H \)
is compact, let \( \{pV_{x_1}, \ldots, pV_{x_n}\} \) be a finite subcover.

Since \( V_x = p^{-1}pV_{x_j} \) for all \( j \), \( \{V_{x_1}, \ldots, V_{x_n}\} \) covers \( G \).
Then \( \bigcup_j V_{x_j} \) is a finite subcover of \( \mathcal{U} \).

(7) If \( H \) is a closed normal subgroup, then \( G/H \) is a
topological group.

Proof. Let \( V \) be a neighborhood of \( 1 \) in \( G/H \). Choose by
(4) a neighborhood \( U \) of \( 1 \) in \( G \) and a neighborhood \( N \)
of \( 1 \) in \( G/H \) such that \( UN \subseteq V \). Then \( pU \) and \( N \) are
neighborhoods of 1 in G/H such that \((pU)^n \subseteq V\). Hence multiplication is continuous at (1,1), therefore everywhere. If V is a neighborhood of 1 in G/H and U is an open neighborhood of 1 in G with \(U^{-1} \subseteq p^{-1}(V)\), then \(pU^{-1} \subseteq V\), and so inversion is continuous at 1, hence everywhere. Finally G/H is Hausdorff by (4).

(8) Any open subgroup is closed.

Proof. If H is the open subgroup, then \(H = G - U \times H\) shows H is closed.

(9) The identity component \(G_0\) of 1 in G is a closed normal subgroup.

Proof. The image of \(G_0 \times G_0\) under multiplication is a connected set containing 1. Hence \(G_0 G_0 \subseteq G_0\). Similarly \(G_0^{-1} \subseteq G_0\). So \(G_0\) is a group. It is closed because components are closed in any topological space. It is normal because the same argument shows \(xG_0 x^{-1} \subseteq G_0\) for each x in G. If G is locally connected, its components are open, and \(G_0\) in particular is open. Hence so are the cosets of \(G_0\). Since \(p : G \rightarrow G/G_0\) is an open mapping, \(G/G_0\) has every subset open.

(10) If G is connected, then any neighborhood of 1 generates G.

Proof. Let V be a neighborhood of 1, and choose an open neighborhood U of 1, by continuity of inversion, such that \(U = U^{-1} \subseteq V\). Set \(H = U U U \cdots U\), with \(n\) factors in the \(n^{th}\) term. Then H is nonempty, is open, and is a subgroup of G, since \(U = U^{-1}\). By (8), H is closed. Therefore \(H = G\). Then V must generate G since \(U \subseteq V\).
(11) If $H$ is a discrete subgroup of $G$ (i.e., if every subset of $H$ is relatively open), then $H$ is a closed subgroup.

Proof. Choose a neighborhood $V$ of 1 in $G$ so that $H \cap V = \{1\}$, and choose an open neighborhood $U$ of 1 with $UU \subseteq V$, by (2). If $x$ is in $H - H$, then $U^{-1}x$ is a neighborhood of $x$ and so must contain a member $h$ of $H$. Write $u^{-1}x = h$. Then $u = xh^{-1}$ is in $H - H$ and is in $U$. Since it is a limit point of $H$, we can find $h' \neq 1$ such that $h' \in Uxh^{-1}$. Then $h'$ is in $U(xh^{-1}) \subseteq UU \subseteq V$, and $h' = 1$, contradiction.

(12) If $G$ is connected, then any discrete normal subgroup $H$ of $G$ lies in the center of $G$.

Proof. If $h$ is in $H$, then $ghg^{-1}$ is in $H$, and by continuity $ghg^{-1}$ is in the same component of $H$ as $1hl^{-1} = h$. Since $H$ is discrete, $ghg^{-1} = h$ and $h$ is central.

Notation for the remainder of the section:

$G =$ pathwise connected, locally pathwise connected, separable metric topological group

$H =$ closed subgroup of $G$, locally pathwise connected (but not necessarily connected)

Proposition 11. (a) The quotient $G/H$ is pathwise connected and locally pathwise connected.

(b) If $H_0$ is the identity component of $H$, then the natural map of $G/H_0$ onto $G/H$ is a covering map.
Proof. (a) If \( x \) and \( y \) are given in \( G/H \), take their preimages in \( G \), connect them by a path, and map back down to \( G/H \) to see that \( G/H \) is pathwise connected. If \( x \) is in an open subset \( U \) of \( G/H \), take a preimage \( \tilde{x} \) in \( p^{-1}(x) \), and choose a pathwise connected open subset \( V \subseteq p^{-1}(U) \) with \( \tilde{x} \) in \( V \). Then \( p(V) \) is a pathwise connected open subneighborhood of \( U \) and shows that \( G/H \) is locally pathwise connected.

(b) Let \( p_o : G \to G/H_o \), \( p : G \to G/H \), and \( q : G/H_o \to G/H \) be the projection maps. Then \( q^{-1}(U) = p_o(p^{-1}(U)) \) is open if \( U \) is open, and so \( q \) is continuous. Also \( q(U) = p(p_o^{-1}(U)) \) shows that \( q \) is open. Since \( H \) is locally connected, we can find an open neighborhood \( U \) of \( 1 \) in \( G \) with \( U \cap H = H_o \) (i.e., \( H_o \) is relatively open in \( H \)). Next, find an open connected \( V \) about \( 1 \) with \( V^{-1}V \subseteq U \); this is possible by (2). Then \( V^{-1}V \cap H \subseteq H_o \). Form the sets \( V_h H_o \), \( h \in H \).

These sets are open and connected in \( G/H_o \) and their union is \( q^{-1}(VH) \). If \( V_{h_1} H_o \cap V_{h_2} H_o \) is not empty, then the same thing is true first of \( V H_o h_1 \cap V h_2 h_1 \), then of \( V^{-1}V H_o \cap H_o h_2 h_1^{-1} \), and finally of \( V^{-1}V \cap H_o h_2 h_1^{-1} \). Since \( V^{-1}V \cap H \subseteq H_o \), \( h_2 h_1^{-1} \) is in \( H_o \), and so \( V_{h_1} H_o = V_{h_2} H_o \). In short, distinct sets \( V h H_o \) are disjoint. Thus \( q \) is one-one continuous open from each \( V h H_o \) onto \( VH \), and \( VH \) is evenly covered. By translation we see that each \( gVH \) is evenly covered. Hence \( q \) is a covering map.
Corollary. (a) If $G/H$ is simply connected, then $H$ is connected.

(b) If $H$ is discrete, then the quotient map of $G$ onto $G/H$ is a covering map.

Proof. (a) If $G/H$ is simply connected, then $q : G/H_0 \to G/H$ cannot be a nontrivial covering because the diagram

\[
\begin{array}{ccc}
G/H & \xrightarrow{f} & G/H_0 \\
\downarrow{g = \text{identity}} & & \downarrow{q} \\
G/H & \xrightarrow{g} & G/H
\end{array}
\]

says that $f$ exists and is $q^{-1}$. So $H_0 = H$, and $H$ is connected.

(b) This is the special case of Proposition 11b in which $H_0 = \{1\}$.

Proposition 12. Let $G$ be simply connected and let $H$ be a discrete subgroup of $G$, so that $p : G \to G/H$ is a covering map. Then the group of deck transformations of $G$ is exactly the group of right translations in $G$ by members of $H$.

Consequently $\pi(G/H, 1 \cdot H)$ is canonically isomorphic with $H$.

Proof. Let $f_h(g) = gh$. Then $p f_h(g) = ghH = gH = p(g)$, so that $p f_h = p$ and $f_h$ is a deck transformation. Since $p^{-1}(1 \cdot H) = H$, the group of translations $f_h$ is simply transitive on $p^{-1}(1 \cdot H)$ and by Theorem 4 is the full group of deck transformations.

Again by Theorem 4, $\pi(G/H, 1 \cdot H) \cong H$.

Proposition 13. Let $G$ be locally simply connected, let $\mathcal{G}$ be the universal covering space with covering map $e : \mathcal{G} \to G$, and let $I$ be in $e^{-1}(1)$. Then there exists a unique multiplication on $\mathcal{G}$ that makes $\mathcal{G}$ into a topological group in such a way that $e$ is a group homomorphism.
Proof. Let \( m : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) be multiplication, and let 
\( \varphi : \mathcal{G} \times \mathcal{G} \to \mathcal{G} \) be the composition \( m \cdot (e,e) \). Since 
\[
[1] = \varphi_\ast \left( \pi(\mathcal{G} \times \mathcal{G}, \mathfrak{I} \times \mathfrak{I}) \right) \subseteq e_\ast \left( \pi(\mathfrak{G}, \mathfrak{I}) \right),
\]
there exists a unique continuous \( \tilde{\varphi} : \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} \to \tilde{\mathcal{G}} \) such that 
\( \varphi = \tilde{\varphi} \) and \( \tilde{\varphi}(\mathfrak{I}, \mathfrak{I}) = \mathfrak{I} \). This \( \tilde{\varphi} \) is the multiplication on 
\( \tilde{\mathcal{G}} \) (and it is the only possible candidate for the multiplication). 

It is associative by unique lifting because 
\[
e(\tilde{\varphi}(x,y),z) = \varphi(\tilde{\varphi}(x,y),z) = \varphi(x,y)(ez) = ((ex)(ey))ez = ex((ey)(ez)) = e(\tilde{\varphi}(x,\tilde{\varphi}(y,z))
\]
and \( \tilde{\varphi}(\tilde{\varphi}(\mathfrak{I}, \mathfrak{I}), \mathfrak{I}) = \mathfrak{I} = \tilde{\varphi}(\mathfrak{I}, \tilde{\varphi}(\mathfrak{I}, \mathfrak{I})) \). It has \( \mathfrak{I} \) as identity 
because \( \tilde{\varphi}(\mathfrak{I}, \cdot) \) and \( \tilde{\varphi}(\cdot, \mathfrak{I}) \) cover the identity and send \( \mathfrak{I} \) 
to \( \mathfrak{I} \). To obtain existence of inverses, we lift the composition 
inversion \( e : \tilde{\mathcal{G}} \to \mathcal{G} \) to a map of \( \tilde{\mathcal{G}} \) to \( \mathcal{G} \) sending \( \mathfrak{I} \) to \( \mathfrak{I} \). 
Finally \( e \) is a group homomorphism because 
\[
e(\tilde{\varphi}(x,y)) = \varphi(x,y) = m(ex, ey).
\]

The group \( \tilde{\mathcal{G}} \) of Proposition 13 is called the \textbf{universal covering} 
group of \( \mathcal{G} \).

In the homework, we shall obtain information about rotation groups 
\( \text{SO}(n) \) and special unitary groups \( \text{SU}(n) \) and their fundamental 
groups. The information will be obtained by induction by means 
of the following proposition.

\textbf{Proposition 14.} If \( \mathcal{G} / \mathcal{H} \) is simply connected and if \( \mathcal{G} \) and \( \mathcal{H} \) are 
locally simply connected, then \( \pi(\mathcal{G}, \mathfrak{I}) \) is isomorphic to a 
quotient group of \( \pi(\mathcal{H}, \mathfrak{I}) \).

\textbf{Remarks:} In the homework, we shall see that the fundamental group 
of a group is necessarily abelian.
Proof. Let $\mathcal{G}$ be the universal covering group of $G$ and let $e : \mathcal{G} \to G$ be the covering homomorphism. Set $\mathcal{H} = e^{-1}(H)$. Define $e_0 : \mathcal{G}/\mathcal{H} \to G/H$ by $e_0(\mathcal{G}/\mathcal{H}) = e(g)H$. Then $e_0$ is well defined, one-one, and onto. An open set in $\mathcal{G}/\mathcal{H}$ is mapped under $e_0$ to its preimage in $\mathcal{G}$, to its image in $G$, and to its image in $G/H$. So $e_0$ is open. Similarly $e_0^{-1}$ is open, and so $e_0$ is a homeomorphism. Therefore $\mathcal{G}/\mathcal{H}$ is simply connected.

We claim that $\mathcal{H}$ is locally pathwise connected. In fact, if $U$ is a connected open neighborhood of $1$ in $\mathcal{G}$ mapped homeomorphically by $e$, then $e(U) \cap H$ contains a relatively open pathwise connected neighborhood $V$ of $1$ since $H$ is locally pathwise connected. Then $U \cap e^{-1}(V)$ is homeomorphic with $V$ and is the required neighborhood of $1$ in $\mathcal{H}$.

By (a) of the Corollary to Proposition 11, $\mathcal{H}$ is connected. By (b) of the Corollary, the map $e|_\mathcal{H} : \mathcal{H} \to H$ is a covering map. Now by Proposition 11, $\pi(G,1) \cong \ker e = \ker e|_\mathcal{H}$. Hence the result follows from the following lemma applied to $H$.

**Lemma.** If $G$ is locally simply connected and $e : G' \to G$ is a covering homomorphism and $\tilde{e} : \tilde{G} \to G'$ is the universal covering group and homomorphism, then $\ker e \cong \ker e\tilde{e}/\ker \tilde{e}$.

Proof. The map of $\ker e\tilde{e}/\ker \tilde{e}$ to $\ker e$ is induced by $\tilde{e}$. Namely if $\tilde{k}$ is in $\ker \tilde{e}$, then $\tilde{e}(\tilde{k}) \in \ker e$ and $g\tilde{k} \in \ker e\tilde{e}$ for $k \in \ker \tilde{e}$ map to the same member of $\ker e$ under $\tilde{e}$. Since $\tilde{e}$ is a homomorphism, so is the induced map. The map
is onto because if $k$ is in $\ker e$ and $\bar{k}$ is in $\tilde{e}^{-1}(k)$, then $\tilde{e}(\bar{k}) = k$. To see it is one-one, let $\tilde{e}(g) = 1$ in $\ker e$. Then $g$ is in $\ker \tilde{e}$; that is, $g$ is trivial in $\ker \tilde{e}/\ker e$. So the map is an isomorphism onto.