The author is greatly indebted to Professor Charles Misner of the University of Maryland for suggesting this problem and helping in the execution through encouragement and discussion, and to Professor John A. Wheeler of Princeton University for encouragement and many helpful discussions.

* National Science Foundation predoctoral fellow.

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[11] The lemma, stated more fully, says that either the determinant of the metric is constant or else it goes to zero in a finite proper time; the option of a constant determinant is not allowed in a dust-filled space. The approach to zero determinant may be either toward the past or toward the future, and not necessarily in both directions. This lemma is proved by Lifshitz and Khalatnikov (ref. 5), who remark that this fact was originally pointed out by L. D. Landau (no ref. given). This lemma was proved by A. Komar, *Phys. Rev.*, **104**, 544 (1956), and in the special case of zero rotation by Raychaudhuri (ref. 7).

DISTAL FUNCTIONS

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Let G be a fixed abelian topological group. A pair (X,G) is a flow if X is a compact Hausdorff space and if G acts on X in a jointly continuous manner. The flow is minimal[1] if every orbit is dense. It is distal[2] if whenever $\{g_n\}$ is a net (i.e., a generalized sequence indexed by any directed set) in G with $g_nx \rightarrow z$ and $g_ny \rightarrow z$, then $x = y$. Auslander and Hahn[3] introduced classes of distal and minimal functions defined on G. A distal function is any complex-valued function defined on G which is equivalent to the restriction to a single orbit of a continuous function defined on the space X of some distal flow (X,G), and a minimal function on G arises in this manner from a minimal flow. Among other things, they proved that the distal functions on G form a Banach algebra (with the supremum norm), but, for instance, the minimal functions on the additive group of reals are not closed under addition. Earlier work of Ellis[2] implied that almost periodic functions are distal and that distal functions are minimal.

The purpose of the present note is to announce two structure theorems for distal functions (Theorems 2 and 3) when G is locally compact abelian and has a countable dense set.

We shall give our own definition of distal function, which is equivalent to Auslander's and Hahn's. Following an idea suggested by the approach in a paper of Bochner's,[4] we consider nets $\{g_n\}$ in G such that, for every f in the set UC of bounded uniformly continuous complex-valued functions on G, the limit

$$\lim f(g + g_n)$$

exists for all g in G. (Here, plus denotes group addition.) Two such nets are equivalent if they yield the same limit for every such f and for all g. Then each equivalence class defines a *shift operator* which maps f into its limit function. Shift operators are denoted T_α, T_β, etc. Each shift operator is a homomorphism of norm one of UC into itself. With this notation f is *distal* if $T_\alpha T_\beta f = T_\alpha T_\gamma f$ always implies $T_\beta f = T_\gamma f$, and f is *minimal* if for any T_α there is a T_β such that $T_\beta T_\alpha f = f$. A *distal algebra* is a group-invariant conjugate-closed Banach algebra of distal functions containing the constants.[5]

We call a shift operator T_u idempotent if $T_u T_u = T_u$, and we call T_u minimal if $T_u f$ is minimal for every f in UC.

THEOREM 1. *If f is in UC, then f is distal if and only if $T_u T_\alpha f = T_\alpha f$ for every T_α and for every T_u which is both minimal and idempotent.*

Theorem 1 gives an alternate proof of Auslander's and Hahn's observation that the distal functions form a Banach algebra and appears to be useful in relating distal functions to differential equations.

A standard argument with nets shows that the set S of shift operators is closed under composition. We topologize S by saying that $\lim T_{\alpha_n} = T_\alpha$ if $\lim T_{\alpha_n} f = T_\alpha f$ pointwise for every f in UC. Then S is a semigroup and a compact Hausdorff space, and the map $T_\alpha \to T_\alpha T_\beta$ is continuous for fixed T_β.

We say that f is *almost periodic* (a.p.) over a distal algebra B if f is distal and if furthermore

$$\lim T_{\alpha_n} T_{\delta_n} f = T_\alpha T_\delta f \tag{1}$$

pointwise whenever

$$T_{\alpha n} \to T_\alpha \text{ and } T_{\delta_n} \to T_\delta \tag{2a}$$

and

$$T_{\delta_n} h = T_\delta h = h \tag{2b}$$

for all h in B and for all n. A distal algebra A is a.p. over B if $A \supseteq B$ and if every function in A is a.p. over B. The functions a.p. over B form a distal algebra containing B. The functions a.p. over the constants are exactly the ordinary almost periodic functions; in fact, when B is the algebra of constants, condition (2*b*) is trivially fulfilled by every shift operator, and equation (1) is then equivalent with the characterization

$$T_\alpha T_\beta f = T_{\beta+\alpha} f$$

given by Bochner.[4]

The proof of the following theorem involves (1) the main theorem in a paper of Furstenberg's,[6] (2) a suitable elaboration of this theorem, and (3) the equivalence between almost periodicity of one separable distal algebra over another and the notion of isometric extension discussed in the same paper.[6]

THEOREM 2. *If A is a distal algebra, then there exist an ordinal ν and a system $\{A_\xi\}$ of distal subalgebras of A, indexed by all ordinals $\xi \leq \nu$, such that*

(1) *A_0 is the algebra of constant functions, and $A_\nu = A$;*

(2) *if $\xi \leq \xi'$, then $A_\xi \leq A_{\xi'}$;*

(3) *if $\xi < \nu$, then $A_{\xi+1}$ is a.p. over A_ξ;*

(4) *if ξ is a limit ordinal $\leq \nu$, then $A_\xi = \text{closure}\left(\bigcup_{\eta<\xi} A_\eta\right)$.*

The effect of Theorem 2 is to reduce the problem of total analysis of a distal algebra to a problem of relative analysis, namely, of a distal subalgebra A a.p. over a smaller distal subalgebra B. Theorem 3 will show that for this situation there exists a sort of Fourier analysis of A with coefficients in B.

Let A and B be distal algebras with A a.p. over B. The Fourier analysis that we shall do will be an analysis of the action of G on A, but in describing it we shall introduce some auxiliary groups I_z, which will disappear before the statement of Theorem 3. Let $M(A)$ and $M(B)$ be the maximal ideal spaces of A and B, let π be the projection of $M(A)$ onto $M(B)$, let x_0 be the evaluation-at-zero element of $M(A)$, and let $z_0 = \pi(x_0)$. Then A is canonically identified with the set of all continuous functions on $M(A)$. Any net $\{g_n\}$ that defines a shift operator T_α on functions also defines a shift operator on points of $M(A)$ under the definition $T_\alpha x = \lim g_n x$,[7] and again the shift operators have a semigroup structure and a compact topological structure. Let $z \epsilon M(B)$. The definition of almost periodicity of A over B implies that the set of all shift operators which map $\pi^{-1}(z)$ into itself forms a compact group called I_z, provided we identify shift operators which have the same action on $\pi^{-1}(z)$. (Although G is, by assumption, abelian, the groups I_z are ordinarily nonabelian.) There exists at least one T_γ with $T_\gamma x_0 \epsilon \pi^{-1}(z)$. For a fixed such T_γ and for varying T_δ, the map $\varphi_\gamma(T_\delta) = T_\gamma T_\delta T_\gamma^{-1}$ induces a topological isomorphism of I_{z_0} onto I_z, and any two such maps (for different choices of T_γ) differ by an inner automorphism of I_z. Let $\Lambda = \{\lambda\}$ be the set of equivalence classes of irreducible finite-dimensional unitary representations of I_{z_0}, let their traces be $\chi_\lambda(t)$, $t \epsilon I_{z_0}$, and let their degrees be α_λ. The map φ_λ of I_{z_0} into I_z defined above maps these representations into the corresponding set for I_z. Now if $T_\gamma x_0 = T_{\gamma'} x_0$, then the facts that φ_γ and $\varphi_{\gamma'}$ differ by an inner automorphism and that equivalence classes of representations are invariant under conjugates imply that the mappings of representations induced by φ_γ and $\varphi_{\gamma'}$ are identical. Thus for fixed λ, we can speak unambiguously of χ_λ and α_λ as associated to all I_z.

If f is a continuous function on $M(A)$, i.e., if f restricts to a member of A, we define $P_\lambda f$ on $M(A)$ by

$$P_\lambda f(x) = \alpha_\lambda \int_{t \epsilon I_{\pi(x)}} \overline{\chi_\lambda(t)} f(tx) dt,$$

where dt is normalized Haar measure on $I_{\pi(x)}$. When $\lambda = \lambda_0$ is the class of the trivial representation, $\chi_{\lambda_0}(t)$ is identically one, and $P_{\lambda_0} f$ reduces to a constant on each fiber $\pi^{-1}(z)$ because I_z acts transitively on $\pi^{-1}(z)$. We define

$$E(f|B)(z) = P_{\lambda_0} f(x) \quad \text{if} \quad z \epsilon M(B) \quad \text{and} \quad \pi(x) = z.$$

The functional $E(\cdot|B)$ has all the properties of a conditional integral, and $E(f|B)$ turns out to be a continuous function on $M(B)$ and hence restricts to a member of B.

THEOREM 3. *If A is a.p. over B, then for each λ in Λ:*

(1) *P_λ maps the set of continuous functions on $M(A)$ into itself, and $P_\lambda^2 = P_\lambda$;*

(2) *image (P_λ) is a G-invariant unitary finitely-generated B-module, and image (P_λ) contains no proper nonzero G-invariant B-submodule.*

If, in addition, f and g are continuous on $M(A)$, then

(3) *(Orthogonality relation) If $\lambda \neq \sigma$, then $E((P_\lambda f)\overline{(P_\sigma g)}|B) = 0$.*

(4) *(Parseval's equality) $P_\lambda f$ is identically zero except for countably many λ, say $\lambda_1, \lambda_2, \ldots.$ Moreover,*

$$E(|f|^2|B) = \sum_{k=1}^{\infty} E(|P_{\lambda_k} f|^2|B),$$

where the series on the right converges uniformly on $M(B)$.

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FUNDAMENTAL PROPERTIES OF GENERALIZED SPLINES

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1. *Introduction.*—A spline function of a single variable (more simply, a spline), defined on an interval $[a,b]$ of the real line, is composed of segments of polynomial functions of degree $2n - 1$ so joined that the resulting composite function is of class C^{2n-2} $[a,b]$. The cubic spline function ($n = 2$) represents the analytic counterpart of the draftsman's spline in consequence of the small deflection property of beams.

In 1946, Schoenberg[1] studied the use of splines in the smoothing of equidistant