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Connection between Brownian Motion and Potential Theory

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1. INTRODUCTION

One of the fruitful achievements of probability theory in recent years has been the recognition that two seemingly unrelated theories in physics—one for Brownian motion and one for potentials—are mathematically equivalent. That is, the results of the two theories are in one-to-one correspondence and any proof of a result in one theory can be translated directly into a proof of the corresponding result in the other theory. Although this connection has been known to some mathematicians for a number of years, the details have not been collected in any one source, and an exposition seems in order.

Historically the first theorem indicating any connection was discovered by Kakutani in 1944. It was known that if D is a sufficiently nice bounded domain in the plane, then to each point x in D there is a measure μ_x on the boundary of D such that for any continuous function f on the boundary the unique harmonic function in D with f as boundary values has the value $\int f\, d\mu_x$ at x. In potential theory it turns out that if K is a nice compact subset of $Bd(D)$, then $\mu_x(K)$ is the value at x of a potential which is one on K and whose charge has all its mass on K. It was Kakutani's observation in [13] that $\mu_x(K)$ is also equal to the probability that a Brownian motion particle started at x reaches K before hitting $Bd(D) - K$. Doob and Hunt in [4–6, 9, 10] noticed other parallels between the subject of Brownian motion and the subject of potential theory and harmonic functions, and they extended Kakutani's work. From their results it gradually became clear that in a certain sense Brownian motion and potential theory were really the same and that the key to the connection was in the potential operator and its inverse, the Laplacian. Later Hunt in [11] exploited this connection by defining potential theories associated with a wide class of Markov processes.

The connection can be shown in several ways and we shall consider only one of them. After briefly describing three-dimensional Brownian motion and potential theory separately, we shall show that there is a natural way in terms of Brownian motion of obtaining the operator mapping charges into potentials and that, conversely, from the potential operator it is possible to recover the family $\{P^t\}$ of transition operators which define Brownian motion. These facts will make it clear that in a technical sense the two theories are identical. In the last section we shall discuss what minor modifications in the argument need to be made for dimensions greater than three and we shall see in what sense the correspondence fails in dimensions one and two.

2. Brownian Motion

Brownian motion physically is the constant movement that microscopic particles undergo because of molecular bombardment when left alone in a liquid. The first step in setting up a probabilistic model for this movement is to replace the known statistical estimates of what happens to a large number of particles by a probability for what happens to one particle. We are then to require two things:

(1) Pr [particle started at u is in E at time t] $= \int_E [1/(2\pi t)^{3/2}]\, e^{-|u-y|^2/2t}\, dy$, where E is any Borel set in R^3 and $|u - y|$ is the Euclidean distance from u to y. We shall abbreviate the left side as $\Pr_u[x_t \in E]$.

(2) If $t_1 < t_2 < \cdots < t_n$, then $\{x_{t_1}, x_{t_2} - x_{t_1}, \ldots, x_{t_n} - x_{t_{n-1}}\}$ is a set of independent random variables and $x_{t+s} - x_t$ has the same distribution as $x_s - u$. That is,

$$\begin{aligned}\Pr_u[x_{t_1} \in E_1, \ldots, (x_{t_n} - x_{t_{n-1}}) \in E_n] &= \Pr_u[x_{t_1} \in E_1] \cdot \cdots \cdot \Pr_u[x_{t_n} - x_{t_{n-1}} \in E_n] \\ &= \Pr_u[x_{t_1} \in E_1] \cdot \cdots \cdot \Pr_u[x_{(t_n - t_{n-1})} - u \in E_n].\end{aligned}$$

The abstract setup for Brownian motion is a measure space whose underlying set is the set of functions with domain the t-axis for positive rational t and with range in R^3, whose sigma-algebra will be specified shortly, and whose measure $\Pr_u$ is to satisfy the two conditions above. Before sketching a construction of this measure, we shall state a few consequences of the two conditions. From (1) we have

$$\Pr_u[x_t \in E] = \int_E p^t(u, dy),$$

where $p^t(u, \cdot)$ is the measure $(2\pi t)^{-3/2} \exp(-|u - y|^2/2t)dy$. With this notation, condition (2) implies that we must have

$$\begin{aligned}&\Pr_u[x_q \in E, x_r \in F, \ldots, x_s \in G, x_t \in H] \\ &\quad = \int_E p^q(u, dw) \int_F p^{r-q}(w, dx) \int \cdots \int_H p^{t-s}(y, dz).\end{aligned} \tag{1}$$

Equation (1) gives several definitions for $\Pr_u[x_t \in E]$ and we must check that they are consistent. For example, we must check that

$$\Pr_u[x_s \in R^3, x_t \in E] = \Pr_u[x_t \in E]$$

and

$$\Pr_u[x_s \in F, x_t \in R^3] = \Pr_u[x_s \in F].$$

Such identities can be verified by direct calculation.

The consistency of these definitions implies that if Ω is the set of all functions from the positive rational t-axis to R^3 and if $\{x_t, t \text{ rational}\}$ is the set of projections, then $\Pr_u$ defines a finitely additive set function on the least algebra containing the cylinder sets of Ω. By Kolmogorov's Extension Theorem (see [3, pp. 10 ff.]), $\Pr_u$ is completely additive and extends to the generated sigma-algebra $\mathbf{F}$. It can then be shown that the set of functions in Ω which are uniformly continuous on every bounded time interval has measure one. For such points ω we define

$$x_t(\omega) = \lim_{\substack{r \to t \\ r \text{ rational}}} x_r(\omega)$$

and we define $x_t(\omega)$ arbitrarily otherwise. The completion of the space $(\Omega, \mathbf{F}, \Pr_u)$ is the desired measure space: In it $x_t^{-1}(E)$ is measurable when E is a Borel set of R^3, conditions (1) and (2) are satisfied, and almost all elements of Ω are continuous.

The details of this construction may be found in [3] or [14]. We do not give them here because we shall not use any facts about $\Pr_u$ that are any deeper than Eq. (1).

Brownian motion need not be started deterministically at position u. If we start the particle according to probabilities assigned by a measure μ on R^3, then we have

$$\Pr_\mu[x_t \in E] = \int_{R^3} \int_E \frac{1}{(2\pi t)^{3/2}} e^{-|u-y|^2/2t} \, dy \, d\mu(u) = \int_{R^3} \int_E p^t(u, dy) \, d\mu(u).$$

A similar expression holds for $\Pr_\mu[x_s \in E, ..., x_t \in F]$.

If μ is a finite signed Borel measure on R^3, we define a measure μP^t by

$$(\mu P^t)(E) = \int_E \left[\int_{R^3} \frac{1}{(2\pi t)^{3/2}} e^{-|x-y|^2/2t} \, d\mu(x)\right] dy. \tag{2}$$

If the finite signed Borel measures on R^3 are considered as a Banach space M under the norm $\|\mu\|_1 = \mu^+(R^3) + \mu^-(R^3)$, then P^t is a continuous linear operator of norm one from M into itself. The effect of the operator P^t is determined by the facts that P^t is linear and that for measures $\mu \geqslant 0$ with $\mu(R^3) = 1$, $(\mu P^t)(E) = \Pr_\mu[x_t \in E]$. Moreover if $(\mu P^t)\chi_E$ denotes the measure whose value on the set F is $(\mu P^t)(E \cap F)$, then

$$\Pr_\mu[x_q \in E, x_r \in F, ..., x_s \in G, x_t \in H] = ((\cdots(((\mu P^q)\chi_E)P^{r-q})\chi_F \cdots)P^{t-s})(H).$$

Consequently the formulation we have just given for Brownian motion determines and is determined by the family of operators $\{P^t\}$. The family $\{P^t\}$ will be called the set of *transition operators* for Brownian motion. It is what we shall use in Section 4 when we discuss the connection with potential theory.

3. Potential Theory

In physics potential theory begins as a study of Coulomb's law of attraction of electrical charges. This law states that every two charges in the universe attract (or repel) each other with a force whose direction is the line connecting them and whose magnitude is proportional to the magnitude of each of them and inversely proportional to the square of the distance between them. That is,

$$F = \epsilon_0 \frac{Qq}{r^2},$$

where ϵ_0 is a constant depending on the units. As an aid in the study one introduces the notion of potential: The potential at a point x due to a charge q is the work (or energy) required to bring a unit charge from infinity to the point x. It can be shown that this potential is independent of the path along which the charge is brought to the point x and that its value is

$$\frac{1}{2\pi} \frac{q}{| x - x_0 |},$$

where x_0 is the position of the charge and where the constant $1/2\pi$ has been fixed after a certain choice of units.

More generally one defines a charge distribution to be any finite signed measure on the Borel sets of R^3. The potential at x due to the charge distribution is again the work required to bring a unit charge from infinity to the point x. Since force (and hence work) are additive, the potential due to a charge distribution consisting of charges $q_1, ..., q_n$ at points $x_1, ..., x_n$ is

Passing to the limit in an appropriate sense, we would expect the potential due to an arbitrary charge distribution μ to be

$$\frac{1}{2\pi}\int_{R^3}\frac{d\mu(y)}{|x-y|}.$$

We first check that such an expression is always well-defined, and then we shall define a potential to be any function of this form.

LEMMA 3.1. *If μ is a charge distribution, then*

$$g(x)=\frac{1}{2\pi}\int_{R^3}\frac{1}{|x-y|}\,d\mu(y)$$

is finite everywhere except possibly on a set of Lebesgue measure zero.

PROOF. It suffices to prove the lemma for the case $\mu \geqslant 0$ since the general case follows by taking differences. Letting K_n denote the closed ball about the origin of radius n, we have

$$\int_{K_n} g(x)\,dx=\frac{1}{2\pi}\int_{K_n}\int_{R^3}\frac{1}{|x-y|}\,d\mu(y)\,dx$$
$$=\frac{1}{2\pi}\int_{R^3}\left[\int_{K_n}\frac{1}{|x-y|}\,dx\right]d\mu(y).$$

The inside integral on the right is bounded by its value when $y=0$, which is some finite number c. Thus the right side does not exceed $c\mu(R^3)/2\pi<\infty$, and g must be finite a.e. in K_n. Hence g is finite a.e. in R^3. Q.E.D.

DEFINITION 3.2. If μ is a charge distribution, then the function

$$\frac{1}{2\pi}\int_{R^3}\frac{1}{|x-y|}\,d\mu(y)$$

is called the *potential* of μ. The operator transforming a charge into its potential is called the *potential operator.*

Potential theory is the class of theorems relating charges and potentials and quantities definable in terms of them. It includes the subject known in physics as electrostatics since the quantities commonly arising in electrostatics are all definable in terms of distances, charges, and potentials. It is readily verified, for instance, that capacity, energy, field, and force are all definable at least dimensionally in terms of distance, charge, and potential. As a further indication that the class of theorems relating charges and potentials is large, we give four examples of how concepts may be defined explicitly (and not just dimensionally) in terms of distance, charge, and potential.

1. The *total charge* of a charge distribution μ is $\mu(R^3)$. The support of μ is the complement of the union of all open sets U with the property that μ vanishes on every measurable subset of U.

2. We can reasonably ask what the total amount of work required to assemble a charge distribution is if only an "infinitesimal" amount of charge is brought into position at one time. The way to compute this amount of work is to integrate the potential function against the charge distribution, provided the integral exists. Thus we define the *energy of a charge distribution* to be the integral of its potential with respect to the charge, provided the integral exists.

3. If a total amount of charge q is put on a piece of conducting metal in R^3, the charge will redistribute itself in such a way that the potential is a constant on the set where the metal is. The situation where the potential is constant on the metal is the one which minimizes energy among all charges μ with total charge q and with support on the set where the metal is, and this situation is referred to as equilibrium. We define an *equilibrium potential* for a compact set E to be a potential which is 1 on E and which arises from a charge with support in E. An *equilibrium set* is a set which has an equilibrium potential. One can show that such a potential is unique and that the charge producing it is also unique.

4. The capacity of a conductor in R^3 is defined as the total amount of charge needed to produce a unit potential on the set where the conductor is. We thus define the *capacity* of any equilibrium set to be the total charge of the charge distribution which produces the equilibrium potential.

We shall not need these auxiliary definitions or any theorems about potentials in the remaining sections, and we shall therefore not state the classical theorems in the subject. Such results can be found in [1] and [2].

4. Connection between the Theories

The essence of the connection between Brownian motion and potential theory is that the potential operator can be obtained in a simple way from the transition operators P^t for Brownian motion, a result we state as Theorem 4.1 (see [10]). It is also true that the operators P^t can be obtained from the potential operator, and we shall prove this fact afterward. We recall from Eq. (2) that a Radon-Nikodym derivative of μP^t with respect to Lebesgue measure is $\int_{R^3} (2\pi t)^{-3/2} \exp(-|x-y|^2/2t)\, d\mu(y)$. We denote this function by $(\mu P^t)'(x)$.

THEOREM 4.1. *If μ is a charge, then the potential g of μ satisfies*

$$g(x) = \lim_{T\to\infty} \int_0^T (\mu P^t)'(x)\, dt$$

for every x for which $g(x)$ is defined.

PROOF. We may assume that $\mu \geqslant 0$ without loss of generality. Then

$$\int_0^T (\mu P^t)'(x)\, dt = \int_0^T \frac{1}{(2\pi t)^{3/2}} \left[\int_{R^3} e^{-|x-y|^2/2t}\, d\mu(y)\right] dt$$

$$= \int_{R^3} \left[\int_0^T \frac{1}{(2\pi t)^{3/2}} e^{-|x-y|^2/2t}\, dt\right] d\mu(y).$$

If we make the change of variable on t which sends $|x-y|^2/t$ into u^2, the above expression becomes

$$= \int_{R^3} \left[\int_{|x-y|/\sqrt{T}}^{\infty} \frac{2}{(2\pi)^{3/2}} |x-y|^{-1} e^{-u^2/2}\, du\right] d\mu(y).$$

By monotone convergence

$$\lim_{T\to\infty} \int_0^T (\mu P^t)'(x)\, dt = \int_{R^3} \left[\int_0^\infty \frac{2}{(2\pi)^{3/2}} |x-y|^{-1} e^{-u^2/2}\, du\right] d\mu(y)$$

$$= \frac{1}{2\pi} \int_{R^3} \frac{1}{|x-y|}\, d\mu(y) = g(x). \qquad \text{Q.E.D.}$$

Formally, therefore, the potential operator is $\lim_{T\to\infty} \int_0^T P^t\, dt$. To show in the converse direction that potential theory is at least as strong as Brownian motion, what we shall do is recover P^t from the potential operator and some elementary properties of P^t. The argument is long and will be broken into several steps:

(1) introduction of an operator $\underset{\sim}{Q}^t$ whose adjoint is P^t
(2) development of the infinitesimal generator A
(3) recovery of Q^t from A and some properties of Q^t
(4) definition of the inverse $-G$ of A
(5) identification of A and the Laplacian operator
(6) use of G to recover A from the potential operator.

First, we introduce $\underset{\sim}{Q}^t$. Let C_0 be the Banach space of all continuous real-valued functions on R^3 which vanish at infinity; we choose as usual $\|f\|_\infty = \sup|f(x)|$. Every element of C_0 is uniformly continuous on R^3, and the space of continuous linear functionals on C_0 is the set of finite signed Borel measures. Norm convergence in C_0 is uniform convergence, and we shall use the notation "strong lim" to denote such convergence. Define Q^t to be the restriction to C_0 of the adjoint of P^t; Q^t is determined by the equation $\mu(Q^t f) = (\mu P^t) f$.

PROPOSITION 4.2. *Q^t maps C_0 into C_0 and satisfies*

$$(Q^t f)(y) = \int_{R^3} \frac{1}{(2\pi t)^{3/2}} e^{-|x-y|^2/2t} f(x)\, dx.$$

Moreover, Q^t satisfies the following properties:

(1) $\|Q^t\| = 1$

(2) $Q^{s+t} = Q^s Q^t$

(3) $\operatorname{strong\,lim}_{t\downarrow 0} Q^t f = f$

(4) $\operatorname{strong\,lim}_{t\to\infty} Q^t f = 0$

(5) *If $f \geqslant 0$, then $Q^t f \geqslant 0$.*

PROOF. For every μ,

$$\mu(Q^t f) = (\mu P^t) f = \int_{R^3} f(x) \int_{R^3} (2\pi t)^{-3/2} e^{-|x-y|^2/2t}\, d\mu(y)\, dx$$

$$= \int_{R^3} \left[\int_{R^3} (2\pi t)^{-3/2} e^{-|x-y|^2/2t} f(x)\, dx \right] d\mu(y).$$

Hence $(Q^t f)(y)$ equals the integrand on the right side. Then (1) and (5) are obvious.

Next we prove that Q^t has range in C_0. We have

$$|Q^t f(y) - Q^t f(y_0)| \leqslant \|f\|_\infty \frac{1}{(2\pi t)^{3/2}} \int_{R^3} | e^{-|x-y|^2/2t} - e^{-|x-y_0|^2/2t} |\, dx.$$

As $y \to y_0$, the right side tends to zero by Lebesgue's Theorem, and hence $Q^t f$ is continuous. In addition,

$$|Q^t f(y)| \leqslant \int_{|x|\geqslant N} (2\pi t)^{-3/2} e^{-|x-y|^2/2t} |f(x)|\, dx$$

$$+ \|f\|_\infty \int_{|x|<N} (2\pi t)^{-3/2} e^{-|x-y|^2/2t}\, dx.$$

The first term is small for large N since f vanishes at infinity. For fixed N the second term tends to zero as $y \to \infty$. Hence $Q^t f$ vanishes at infinity.

We still have to prove (2), (3), and (4). For (2) we have

$$Q^sQ^tf(z) = (2\pi s)^{-3/2} (2\pi t)^{-3/2} \int_{R^3} \int_{R^3} \exp\left[-\frac{|y-z|}{2s}\right]^2$$

$$\times \exp\left[-\frac{|x-y|^2}{2t}\right] f(x)\, dx\, dy$$

$$= (2\pi s)^{-3/2} (2\pi t)^{-3/2} \int_{R^3} \int_{R^3} \exp\left[-\frac{|y-(zt+xs)/(s+t)|^2}{2st/(s+t)}\right]$$

$$\times \exp\left[-\frac{x-z|^2}{2(s+t)}\right] f(x)\, dy\, dx$$

$$= (2\pi(s+t))^{-3/2} \int_{R^3} \exp\left[-\frac{|x-z|^2}{2(s+t)}\right] f(x)\, dx$$

$$= Q^{s+t}f(z).$$

For (3) let $\epsilon > 0$ be given. Choose $\delta > 0$ so that $\sup_y |f(x+y) - f(y)| < \epsilon$ for all x with $|x| < \delta$. Then

$$\|Q^tf - f\|_\infty \leqslant \sup_y \int_{|x|<\delta/2} (2\pi t)^{-3/2} \exp(-|x|^2/2t)|f(x+y) - f(y)|\, dx$$

$$+ \sup_y \int_{|x|\geqslant\delta/2} (2\pi t)^{-3/2} \exp(-|x|^2/2t)|f(x+y) - f(y)|\, dx$$

$$\leqslant \epsilon + 2\|f\|_\infty \int_{|x|\geqslant\delta/2} (2\pi t)^{-3/2} e^{-|x|^2/2t}\, dx$$

$$= \epsilon + 2\|f\|_\infty \int_{|z|\geqslant\delta/2t^{1/2}} (2\pi)^{-3/2} e^{-|z|^2/2}\, dz.$$

For small enough t, the right side is $\leqslant 2\epsilon$. In (4) let $\epsilon > 0$ and choose M so that $|f(x)| < \epsilon$ when x is outside the ball $B = \{x| \, |x| \leqslant M\}$. Then

$$\|Q^tf\|_\infty \leqslant \epsilon + \sup_y \int_B (2\pi t)^{-3/2} e^{-|x-y|^2/2t}\, dx$$

$$\leqslant \epsilon + \sup_y \|f\|_\infty \int_{t^{-1/2}(B-y)} (2\pi)^{-3/2} e^{-z^2/2}\, dz$$

$$\leqslant \epsilon + \|f\|_\infty \int_{t^{-1/2}B} (2\pi)^{-3/2} e^{-z^2/2}\, dz$$

$$\leqslant 2\epsilon \text{ for large } t. \qquad \text{Q.E.D.}$$

From now until after Proposition 4.13, we assume that $\{Q^t\}$ is any family of operators mapping C_0 into C_0 and satisfying statements (1), (2), and (3) of Proposition 4.2. The procedure will be to introduce the infinitesimal generator A in terms of Q^t and then to show that Q^t can be recovered from it in the presence of the three properties of Q^t. This method originally was discovered independently by Yosida [15] and Hille [8].

The infinitesimal generator A of the family $\{Q^t\}$ is the operator

$$Af = \operatorname{strong}\lim_{t\downarrow 0}\left(\frac{Q^t - I}{t}\right)f$$

defined on the vector subspace of all $f \in C_0$ for which the limit exists.

LEMMA 4.3. *If f is in the domain of A, then Q^tf is in the domain of A and $AQ^tf = Q^tAf$.*

PROOF.

$$\begin{aligned} Q^t(Af) &= Q^t\left(\operatorname{strong}\lim \frac{Q^hf - f}{h}\right) \\ &= \operatorname{strong}\lim \frac{Q^h(Q^tf) - Q^tf}{h} \quad \text{by (1) and (2) for } Q^t \\ &= A(Q^tf). \end{aligned}$$

Q.E.D.

LEMMA 4.4. *If f is in the domain of A, then $(Q^tf)'(x)$, the derivative of $Q^tf(x)$ with respect to t, exists and is in C_0, and the difference quotients $(Q^{t+h}f(x) - Q^tf(x))/h$ converge to it uniformly in x. Moreover $(Q^tf)' = Q^tAf$.*

PROOF. For $h > 0$ we have by Lemma 4.3 and property (2) of $\{Q^t\}$

$$Q^tAf = AQ^tf = \operatorname{strong}\lim_{h\downarrow 0} \frac{Q^h(Q^tf) - Q^tf}{h} = \operatorname{strong}\lim_{h\downarrow 0} \frac{Q^{t+h}f - Q^tf}{h}.$$

A similar result as $h \uparrow 0$ is enough to establish the lemma. Let $h < 0$ and put $k = -h$. Then

$$\begin{aligned} \left\|\frac{Q^{t+h}f - Q^tf}{h} - Q^tAf\right\| &= \left\|Q^{t-k}\left(\frac{Q^kf - f}{k}\right) - Q^t(Af)\right\| && \text{by (2)} \\ &\leqslant \left\|\frac{Q^kf - f}{k} - Q^k(Af)\right\| && \text{by (1)} \\ &\leqslant \left\|\frac{Q^kf - f}{k} - Af\right\| + \|Af - Q^kAf\|. \end{aligned}$$

The first term on the right tends to zero as $k \downarrow 0$ since $f \in$ domain A, and the second term tends to zero by (3). Q.E.D.

LEMMA 4.5. *If $f \in C_0$, then $Q^tf(x)$ is continuous in t and hence $\int_0^T Q^tf(x)\,dt$ is defined.*

PROOF. We apply (3) twice and use (1). If $h > 0$,

$$\| Q^{t+h}f - Q^t f \| \leqslant \| Q^t \| \| Q^h f - f \| \leqslant \| Q^h f - f \| \to 0.$$

If $h = -k < 0$,

$$\| Q^{t+h}f - Q^t f \| \leqslant \| Q^{t-k} \| \| Q^k f - f \| \leqslant \| Q^k f - f \| \to 0. \qquad \text{Q.E.D.}$$

LEMMA 4.6. *If $f \in C_0$, then so is $\int_0^T Q^t f \, dt$.*

PROOF. First we prove continuity:

$$\begin{aligned} \left| \int_0^T Q^t(f(y+h) - f(y)) \, dt \right| &\leqslant \sup_y \left| \int_0^T Q^t(f(y+h) - f(y)) \, dt \right| \\ &\leqslant \int_0^T \| Q^t(f(y+h) - f(y)) \| \, dt \\ &\leqslant \int_0^T \| f(y+h) - f(y) \| \, dt \qquad \text{by (1)} \\ &= T \| f(y+h) - f(y) \|, \end{aligned}$$

and the right side tends to zero as $h \to 0$ by uniform continuity of f. Next we prove that $\int_0^T Q^t f \, dt$ vanishes at infinity. Let $\epsilon > 0$. Choose by (3) a $\delta > 0$ such that if $0 \leqslant t < \delta$, then $\| Q^t f - f \| < \epsilon$. Then for every n and for $0 \leqslant t < \delta$,

$$\| Q^{n\delta + t} f - Q^{n\delta} f \| = \| Q^{n\delta}(Q^t f - f) \| \leqslant \| Q^t f - f \| < \epsilon.$$

For each n such that $n\delta \leqslant T$, let r_n be a real number such that if $|y| \geqslant r_n$, then $|(Q^{n\delta} f)(y)| < \epsilon$. Let r be the maximum of the r_n's. Then $|y| \geqslant r$ implies $|Q^t f(y)| < 2\epsilon$ for every $t \in [0, T]$ and hence $| \int_0^T Q^t f(y) \, dt | < 2\epsilon T$.
Q.E.D.

LEMMA 4.7. *If $f \in C_0$, then*

$$\operatorname*{strong\,lim}_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} Q^s f \, ds = Q^t f.$$

PROOF. Let $\epsilon > 0$ and choose $h > 0$ by (3) so that $r \leqslant h$ implies $\| Q^r f - f \| < \epsilon$. Then

$$\begin{aligned} \left\| \frac{1}{h} \int_t^{t+h} Q^s f \, ds - Q^t f \right\| &= \frac{1}{h} \left\| \int_t^{t+h} (Q^s f - Q^t f) \, ds \right\| \\ &\leqslant \frac{1}{h} \int_t^{t+h} \| Q^s f - Q^t f \| \, ds \\ &\leqslant \frac{1}{h} \int_t^{t+h} \| Q^{s-t} f - f \| \, ds \\ &< \epsilon. \qquad \text{Q.E.D.} \end{aligned}$$

LEMMA 4.8. *A is a closed operator. That is, if f_n is a sequence of functions in the domain of A converging uniformly to f and if Af_n converges uniformly, then f is in the domain of A and*

$$Af = \text{strong lim } Af_n .$$

PROOF. Set $g = \text{strong lim } Af_n$. Since $\| Q^t(Af_n) - Q^t g \| \leq \| Af_n - g \|$, $Q^t Af_n$ converges in norm uniformly in t to $Q^t g$ as $n \to \infty$. Therefore for fixed h we have by Lemma 4.4 and by (1)

$$\begin{aligned} \int_0^h Q^t g \, dt &= \text{strong lim} \int_0^h Q^t A f_n \, dt \\ &= \text{strong lim} \int_0^h (Q^t f_n)' \, dt \\ &= \text{strong lim} \, (Q^h f_n - f_n) = Q^h f - f. \end{aligned}$$

Dividing by h, letting $h \downarrow 0$, and applying Lemma 4.7, we obtain $g = Af$.

Q.E.D.

LEMMA 4.9. *For every $f \in C_0$, $\int_0^T Q^t f \, dt$ is in the domain of A. Furthermore*

$$A\left(\int_0^T Q^t f \, dt\right) = Q^T f - f.$$

PROOF. By (1) and (2)

$$\begin{aligned} \frac{Q^h\left(\int_0^T Q^t f \, dt\right) - \int_0^T Q^t f \, dt}{h} &= \frac{1}{h}\int_0^T Q^{h+t} f \, dt - \frac{1}{h}\int_0^T Q^t f \, dt \\ &= \frac{1}{h}\int_h^{T+h} Q^t f \, dt - \frac{1}{h}\int_0^T Q^t f \, dt \\ &= \frac{1}{h}\int_T^{T+h} Q^t f \, dt - \frac{1}{h}\int_0^h Q^t f \, dt. \end{aligned}$$

As $h \downarrow 0$, the right side tends to $Q^T f - f$ by Lemma 4.7. Q.E.D.

LEMMA 4.10. *The domain of A is a dense subspace.*

PROOF. Let $h_n \downarrow 0$ and let f be given. Set $f_n = h_n^{-1} \int_0^{h_n} Q^t f \, dt$. Then $f_n \in \text{domain } A$ by Lemma 4.9 and $\text{strong lim} f_n = f$ by Lemma 4.7.

Q.E.D.

LEMMA 4.11. *The limit*

$$R_\lambda f = \operatorname*{strong\,lim}_{T\to\infty} \int_0^T e^{-\lambda t} Q^t f\, dt$$

exists for each $\lambda > 0$ *and for all* $f \in C_0$, *and* R_λ *is a bounded linear operator from* C_0 *into* C_0 *of norm* $\leqslant 1/\lambda$.

PROOF. The limit exists because

$$\left| \int_T^{T'} e^{-\lambda t} Q^t f\, dt \right| \leqslant \int_T^{T} e^{-\lambda t} \| Q^t \| \| f \| \, dt \leqslant \| f \| \int_T^\infty e^{-\lambda t}\, dt \to 0 \text{ as } T \to \infty.$$

Moreover

$$\| R_\lambda f \| \leqslant \int_0^\infty e^{-\lambda t} \| f \| \, dt = \| f \| / \lambda. \qquad \text{Q.E.D.}$$

LEMMA 4.12. *For each* $\lambda > 0$, $R_\lambda = (\lambda - A)^{-1}$.

PROOF. First we show $A(R_\lambda f) = \lambda R_\lambda f - f$ if $f \in \operatorname{domain} R_\lambda$; that is, $(\lambda - A) R_\lambda = I$. We have

$$\begin{aligned}
\frac{Q^h - I}{h} R_\lambda f &= \frac{1}{h} \int_0^\infty e^{-\lambda t} (Q^{h+t} - Q^t) f\, dt \\
&= \frac{1}{h} \int_h^\infty e^{-\lambda(t-h)} Q^t f\, dt - \frac{1}{h} \int_0^\infty e^{-\lambda t} Q^t f\, dt \\
&= \frac{e^{\lambda h} - 1}{h} \int_0^\infty e^{-\lambda t} Q^t f\, dt - \frac{1}{h} \int_0^h e^{-\lambda(t-h)} Q^t f\, dt \\
&\to \lambda R_\lambda f - f \text{ as } h \downarrow 0.
\end{aligned}$$

Secondly we show that $R_\lambda(Af) = -f + \lambda R_\lambda f$ if $f \in \operatorname{domain}(\lambda - A) = \operatorname{domain} A$; that is, $R_\lambda(\lambda - A)$ is the identity on domain A. We have

$$\begin{aligned}
R_\lambda(Af) &= \int_0^\infty e^{-\lambda t} Q^t A f = \int_0^\infty e^{-\lambda t} (Q^t f)'\, dt \\
&= e^{-\lambda t} Q^t f \Big|_0^\infty + \lambda \int_0^\infty e^{-\lambda t} Q^t f\, dt = -f + \lambda R_\lambda f. \quad \text{Q.E.D.}
\end{aligned}$$

PROPOSITION 4.13. *The operator* $Q_\lambda^t = \exp[t\lambda A(\lambda - A)^{-1}]$ *is well-defined by the exponential's power series for each* $\lambda > 0$, *is bounded, and satisfies*

$$\operatorname*{strong\,lim}_{\lambda\to\infty} \exp[t\lambda A(\lambda - A)^{-1}] f = Q^t f \tag{3}$$

for all $f \in C_0$.

PROOF. We have range $(\lambda - A)^{-1} = \text{domain}\,(\lambda - A) = \text{domain}\,A$, and thus $A(\lambda - A)^{-1}$ is defined everywhere on C_0. It satisfies

$$A(\lambda - A)^{-1} = (A - \lambda)(\lambda - A)^{-1} + \lambda(\lambda - A)^{-1} = \lambda(\lambda - A)^{-1} - I, \tag{4}$$

and the operator on the right is bounded; hence Q_λ^t is defined and bounded. Its norm is

$$\begin{aligned} \| \exp[t\lambda A(\lambda - A)^{-1}] \| &= \| \exp[t(\lambda^2(\lambda - A)^{-1} - \lambda I)] \| \\ &\leqslant e^{-\lambda t} \Big\| \sum_{m=0}^{\infty} \frac{1}{m!} [t\lambda^2(\lambda - A)^{-1}]^m \Big\| \\ &\leqslant e^{-\lambda t} \sum_{m=0}^{\infty} \frac{1}{m!} \| t\lambda^2 R_\lambda \|^m \\ &\leqslant e^{-\lambda t} \sum_{m=0}^{\infty} \frac{1}{m!} (\lambda t)^m = 1. \end{aligned}$$

For the rest of the proof it is sufficient to prove Eq. (3) for the dense subspace domain A (see Lemma 4.10). In fact, let f be given and choose $f_n \in \text{domain}\,A$ with $f = \text{strong}\lim f_n$. Then

$$\begin{aligned} \| Q_\lambda^t f - Q^t f \| &\leqslant \|(Q_\lambda^t - Q^t)(f - f_n)\| + \| Q_\lambda^t f_n - Q^t f_n \| \\ &\leqslant 2 \| f - f_n \| + \| Q_\lambda^t f_n - Q^t f_n \|. \end{aligned} \tag{5}$$

If we first choose n and then choose λ, we can make the right side $< \epsilon$.

We shall need two commutativity relations. If f is in domain A, then

$$\begin{aligned} \lambda A(\lambda - A)^{-1} f &= \lambda(A - \lambda)(\lambda - A)^{-1} f + \lambda^2(\lambda - A)^{-1} f \\ &= \lambda(\lambda - A)^{-1}(A - \lambda) f + \lambda(\lambda - A)^{-1}\lambda f = \lambda(\lambda - A)^{-1} A f. \end{aligned} \tag{6}$$

Since $Q^t A f = A Q^t f$ for $f \in \text{domain}\,A$ (Lemma 4.3) and hence since $(\lambda - A) Q^t = Q^t(\lambda - A)$ on domain A, the relation range $(\lambda - A)^{-1} = \text{domain}\,(\lambda - A) = \text{domain}\,A$ implies that

$$\begin{aligned} (\lambda - A)^{-1} Q^t &= (\lambda - A)^{-1}[Q^t(\lambda - A)](\lambda - A)^{-1} \\ &= (\lambda - A)^{-1}[(\lambda - A) Q^t](\lambda - A)^{-1} = Q^t(\lambda - A)^{-1}. \end{aligned} \tag{7}$$

We use Eqs. (4) and (6) together as follows. For $f \in \text{domain}\,A$

$$\begin{aligned} \| \lambda(\lambda - A)^{-1} f - f \| &= \| A(\lambda - A)^{-1} f \| = \| (\lambda - A)^{-1} A f \| \\ &\leqslant \|(\lambda - A)^{-1}\| \| A f \| \leqslant \lambda^{-1} \| A f \|. \end{aligned}$$

Hence $\|\lambda(\lambda - A)^{-1}f - f\| \to 0$ for $f \in$ domain A. The same argument as in Eq. (5) except with Q_λ^t and Q^t replaced by $\lambda(\lambda - A)^{-1}$ and I, respectively, shows that consequently $\|\lambda(\lambda - A)^{-1}f - f\| \to 0$ for all $f \in C_0$. Applying this result to Af for $f \in$ domain A, we have

$$\limsup_\lambda \| Af - \lambda A(\lambda - A)^{-1}f \| = \limsup_\lambda \| Af - \lambda(\lambda - A)^{-1}(Af) \| = 0. \quad (8)$$

Now we can prove Eq. (3). By Eq. (6) the second step of the following calculation is justified:

$$\begin{aligned}
&\limsup_\lambda \| Q^t f - \exp[t\lambda(\lambda - A)^{-1}]f \| \\
&\quad = \limsup_\lambda \left\| \int_0^t \frac{d}{dr} (\exp[(t - r)\lambda A(\lambda - A)^{-1}]Q^r f)\, dr \right\| \\
&\quad = \limsup_\lambda \left\| \int_0^t (\exp[—]Q^r Af - \exp[—]Q^r \lambda A(\lambda - A)^{-1}f)\, dr \right\| \\
&\quad \leqslant \limsup_\lambda \int_0^t \| \exp[—]\| \, \| Q^r \| \, \| Af - \lambda A(\lambda - A)^{-1}f \|\, dr \\
&\quad \leqslant \limsup_\lambda t \| Af - \lambda A(\lambda - A)^{-1}f \|.
\end{aligned}$$

The right side is zero by Eq. (8). Q.E.D.

The content of Proposition 4.13 for the present purpose is that A, the three formal properties of $\{Q^t\}$, and the definition of A in terms of $\{Q^t\}$ completely determine $\{Q^t\}$.

For the fourth step in the recovery of $\{P^t\}$ from the potential operator, we introduce an operator G and prove that its inverse is $-A$. We define

$$Gf = \operatorname*{strong\,lim}_{T\to\infty} \int_0^T Q^t f\, dt$$

on the subspace of all f in C_0 for which the right side exists.

Proposition 4.14. *If $f \in$ domain A, then $Af \in$ domain G, and $G(-Af) = f$. Conversely, if $f \in$ domain G, then $Gf \in$ domain A, and $(-A)Gf = f$.*

Proof. Let $f \in$ domain A. By Lemma 4.4

$$\int_0^T Q^t(-Af)\, dt = - \int_0^T (Q^t f)'\, dt = -Q^T f + f.$$

By conclusion (4) of Proposition 4.2, the right side converges uniformly to f as $T \to \infty$. Conversely, let $f \in$ domain G. Set $g_T = \int_0^T Q^t f\, dt$. By hypothesis g_T converges in norm. Also by Lemma 4.9

$$Ag_T = A\left(\int_0^T Q^t f\, dt\right) = Q^T f - f,$$

and the right side converges uniformly to $-f$ by (4) of Proposition 4.2. Hence by Lemma 4.8, $Gf \in$ domain A and $A(Gf) = -f$. Q.E.D.

We shall now identify A as a constant multiple of the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

when each is restricted to a suitable domain. This result is due to Yosida [16].

LEMMA 4.15. *If $Q^t f \in$ domain A for every $t > 0$ and if $A(Q^t f) = Q^t g$, then $f \in$ domain A and $Af = g$.*

PROOF. We have strong $\lim_{t \downarrow 0} Q^t f = f$ and strong $\lim_{t \downarrow 0} A(Q^t f) = g$. Apply Lemma 4.8. Q.E.D.

We use the notation L^1 to denote the space of all Lebesgue integrable functions on R^3, and we denote $\|f\|_1 = \int |f(x)|\, dx$.

LEMMA 4.16. *If $f \in C_0 \cap L^1$, then $Q^t f \in$ domain A for $t > 0$, and*

$$A(Q^t f)(y) = \int_{R^3} \frac{\partial}{\partial t}\left((2\pi t)^{-3/2} e^{-|x-y|^2/2t}\right) f(x)\, dx. \tag{9}$$

PROOF. We have to show that

$$\frac{1}{h} \int_{R^3} [(2\pi(t+h))^{-3/2} e^{-|x-y|^2/2(t+h)} - (2\pi t)^{-3/2} e^{-|x-y|^2/2t}] f(x)\, dx$$

tends uniformly in y to the right side of (9) as $h \downarrow 0$. Set $g(s, x) = (2\pi s)^{-3/2} e^{-|x|^2/2s}$, and let $g_s(s, x)$ be the partial derivative of g with respect to s. Let $\epsilon > 0$. Restricted to the domain where $s > t/2$, $g_s(s, x)$ is uniformly continuous in the two variables jointly. Choose $\delta > 0$ corresponding to ϵ. By the Mean Value Theorem if $h < t/2$ and $h < \delta$, then there is a k with $0 < k < h < \delta$ and

$$\left| \frac{g(t+h, x) - g(t, x)}{h} - g_t(t, x) \right| = |g_t(t+k, x) - g_t(t, x)| < \epsilon.$$

Therefore for such h,

$$\begin{aligned} &\left| \frac{1}{h} \left[\int_{R^3} g(t+h, x-y) f(x)\, dx \right.\right. \\ &\qquad \left.\left. - \int_{R^3} g(t, x-y) f(x)\, dx \right] - \int_{R^3} g_t(t, x-y) f(x)\, dx \right| \\ &\qquad \leqslant \int_{R^3} \left| \frac{g(t+h, x-y) - g(t, x-y)}{h} - g_t(t, x-y) \right| |f(x)|\, dx \\ &\qquad \leqslant \epsilon \|f\|_1 . \end{aligned}$$

Q.E.D.

Let S be the set of testing functions of Schwartz; that is, the set of real-valued infinitely differentiable functions on R^3 which, together with their partial derivatives of all orders, remain bounded when multiplied by any polynomial. We have $S \subseteq C_0 \cap L^1$. The Fourier transform is a one-one map of S onto itself.

PROPOSITION 4.17. *Any function $f \in S$ is in the domain of A and satisfies $Af = 2\nabla^2 f$.*

PROOF. By direct calculation

$$\frac{\partial}{\partial t}[(2\pi t)^{-3/2}e^{-|x-y|^2/2t}] = 2(2\pi t)^{-3/2}\left[-\frac{3}{2t} + \frac{|x-y|^2}{2t}\right]$$

$$= 2\nabla^2[(2\pi t)^{-3/2}e^{-|x-y|^2/2t}].$$

Hence by Lemma 4.16

$$A(Q^t f)(y) = \int_{R^3} 2\nabla^2[(2\pi t)^{-3/2}e^{-|x-y|^2/2t}]f(x)\,dx.$$

Applying Green's identity twice and using the fact that $f \in S$, we get

$$A(Q^t f)(y) = \int_{R^3} 2(2\pi t)^{-3/2}e^{-|x-y|^2/2t}\nabla^2 f(x)\,dx = Q^t(2\nabla^2 f)(y).$$

The result then follows from Lemma 4.15. Q.E.D.

Finally we use G to recover A from the potential operator.

LEMMA 4.18. *If $f \in C_0 \cap L^1$, then f is in the domain of G and*

$$Gf(x) = \frac{1}{2\pi}\int_{R^3}\frac{f(x)}{|x-y|}\,dx.$$

PROOF. Without loss of generality, let $f \geqslant 0$. Then $\int_0^T Q^t f\,dt$ increases with T, so that by Dini's Theorem the convergence is uniform if $\int_0^\infty Q^t f\,dt$ is a continuous function vanishing at infinity. As in the case of Theorem 4.1 we find

$$g(y) = \int_0^\infty Q^t f(y)\,dt = \frac{1}{2\pi}\int_{R^3}\frac{f(x)}{|x-y|}\,dx.$$

For continuity we have for any $\delta > 0$

$$|g(y) - g(y_0)| = \frac{1}{2\pi}\left|\int_{R^3}\frac{f(y+x) - f(y_0+x)}{|x|}\,dx\right|$$

$$\leqslant \frac{1}{2\pi}\,2\,\|f\|_\infty \int_{|x|<\delta}|x|^{-1}\,dx$$

$$+ \frac{1}{2\pi\delta}\int_{|x|\geqslant\delta}|f(y+x) - f(y_0+x)|\,dx.$$

Choose δ so that the first term on the right side is $< \epsilon/2$ and then choose y (by Lebesgue's Theorem) close enough to y_0 to make the second term $< \epsilon/2$.

It remains to be proved that $g(y) \to 0$ as $y \to \infty$. Choose R large enough so that $(2\pi)^{-1}R^{-1}\|f\|_1 < \epsilon$, and choose $r < R$ small enough so that

$$(2\pi)^{-1} \|f\|_\infty \int_{|x|<r} |x|^{-1}\, dx < \epsilon.$$

Then choose k large enough so that $|y| \geqslant k$ implies

$$(2\pi)^{-1}r^{-1} \int_{|x|<R} f(x+y)\, dx < \epsilon.$$

For any such y, $|g(y)| < 3\epsilon$. Q.E.D.

LEMMA 4.19. *If $f \in S$, then the differential equation*

$$(I - \nabla^2)\, y = f \tag{10}$$

has a solution in S.

PROOF. Let $\hat{f}$ be the Fourier transform of f. Then $\hat{f} \in S$. If r denotes the distance-from-the-origin function, then $\hat{f}/(1 + r^2)$ is infinitely differentiable. By induction on the order of the derivative, we see that any polynomial times any derivative of it is bounded. Hence $\hat{f}/(1 + r^2)$ is in S. Its inverse Fourier transform is in S and is a solution of (10).

THEOREM 4.20. *The transition operators $\{P^t\}$ of Brownian motion are determined by the potential operator K in the following sense: Let K_S be the restriction of K to the subspace of signed measures whose densities exist and are in S. Then K_S is one-one. Let $-L$ be the operator which transforms a potential in range K_S into the density of the corresponding charge in domain K_S. There is a unique family of linear operators $Q^t : C_0 \to C_0$ such that*

(1) $\|Q^t\| = 1$

(2) $Q^{s+t} = Q^s Q^t$

(3) $\text{strong}\lim_{t\downarrow 0} Q^t f = f$ for all $f \in C_0$

(4) $Lf = \text{strong}\lim_{h\downarrow 0} \dfrac{Q^h - I}{h} f$ *for all $f \in$ domain L.*

The operator P^t is the adjoint of Q^t.

PROOF OF EXISTENCE. By Lemma 4.18, K_S is a restriction of the composition of two maps: one which sends a signed measure with density in S into its density function and the other one equal to G. Each of these maps is one-one (Proposition 4.14) and hence K_S is one-one. By Proposition 4.14, L is a restriction of A. The operators $\underset{\sim}{Q}^t$ and P^t of Brownian motion then certainly satisfy the rest of the conditions.

PROOF OF UNIQUENESS. Suppose there is another family $\bar{Q}^t$ satisfying (1), (2), (3), and (4). Then its infinitesimal generator $\bar{A}$ is well-defined and is such that L is a restriction of $\bar{A}$ (by (4)) and that $(\lambda - \bar{A})^{-1}$ is a bounded operator defined on all of C_0 (Lemmas 4.11 and 4.12). Hence $(\lambda - L)^{-1}$ is a restriction of both $(\lambda - A)^{-1}$ and $(\lambda - \bar{A})^{-1}$. But $L = 2\nabla^2$ on S by Proposition 4.17 and $(2 - 2\nabla^2)^{-1}$ is defined on the dense subset S of C_0 by Lemma 4.19. Therefore for $\lambda = 2$, $(\lambda - L)^{-1}$ is densely defined. Therefore $(2 - A)^{-1} = (2 - \bar{A})^{-1}$ and $A = \bar{A}$. By Proposition 4.13, $Q^t = \bar{Q}^t$. Q.E.D.

5. BROWNIAN MOTION AND POTENTIAL THEORY IN n DIMENSIONS

Both Brownian motion and potential theory are often studied in other domains than all of R^3, such as all of R^n or some open set of R^n. In this section we shall point out what parts of the preceding carry over to the n-dimensional theory, but we shall treat only the case where the underlying set is all of R^n.

The n-dimensional Brownian motion transition operator P^t is defined by

$$(\mu P^t)(E) = \int_E \left[\frac{1}{(2\pi t)^{n/2}} \int_{R^n} e^{-|x-y|^2/2t}\, d\mu(y)\right] dx.$$

The potential operator differs in appearance from dimension to dimension more than the Brownian motion operators do, but its kernel is still a constant multiple of the integral of $|x|^{-(n-1)}$; the potential $g(x)$ of the signed measure μ is defined by

$$g(x) = \begin{cases} -\int_{R^1} |x-y|\, d\mu(y) & \text{in dimension } 1 \\ 2\int_{R^2} \log|x-y|\, d\mu(y) & \text{in dimension } 2 \\ c_n \int_{R^n} \frac{1}{|x-y|^{n-2}}\, d\mu(y) & \text{in dimension } n \geqslant 3, \end{cases}$$

where

$$c_n = \tfrac{1}{2}\pi^{-n/2}\Gamma(\tfrac{1}{2}(n-2)).$$

In dimension $n \geq 3$, $g(x)$ is necessarily finite a.e., but in dimensions 1 and 2 we shall need to assume that g is finite a.e.

In one or two dimensions a Brownian motion particle started at the origin returns to any neighborhood of the origin after any specified amount of time with probability one, but in three or more dimensions it returns with probability less than one. Thus the two cases differ sharply for Brownian motion and the same thing may be expected for potential theory and for the connection between the theories. The fact that the potential kernels are unbounded at infinity in dimensions one and two is a clue to the situation.

Indeed, all of the results of Section 4 are valid word-for-word in all dimensions $n \geq 3$, except that an occasional constant must be changed. The formula

$$g(x) = \lim_{T\to\infty} \int_0^T (\mu P^t)'(x)\, dt$$

of Theorem 4.1 is still the heart of the connection, and the theories are still equivalent.

But the situation is different in dimensions 1 and 2. Proposition 5.1, whose proof is omitted, is what replaces Theorem 4.1, and it is not sufficient to demonstrate that the potential operator can be obtained in a natural way from the family $\{P^t\}$. It asserts that the exact analog of Theorem 4.1 holds for a charge precisely when its total charge is zero and its potential is finite a.e.

PROPOSITION 5.1. *In R^n for $n = 1$ or 2 let μ be a signed measure, and suppose that*

$$\int_{R^1} |x-y|\, d|\mu|(y) < \infty \qquad \textit{a.e. (Lebesgue) if } n = 1$$

or

$$\int_{R^2} |\log|x-y||\, d|\mu|(y) < \infty \qquad \textit{a.e. if } n = 2.$$

If $\mu(R^n) \neq 0$, then

$$\lim_{T\to\infty} \int_0^T (\mu P^t)'(x)\, dt = +\infty \quad \textit{or} \quad -\infty \quad \textit{a.e.}$$

If $\mu(R^n) = 0$, then

$$g(x) = \lim_{T\to\infty} \int_0^T (\mu P^t)'(x)\, dt$$

exists a.e., is finite a.e., and satisfies

$$g(x) = \begin{cases} -\int_{R^1} |x-y|\, d\mu(y) & \textit{if } n = 1 \\ 2\int_{R^2} \log|x-y|\, d\mu(y) & \textit{if } n = 2. \end{cases}$$

In the converse direction most of the results of Section 4 are still valid after changes in certain constants. Everything from Proposition 4.2 through Proposition 4.17 goes over with little change. Lemma 4.19 is still valid, and only Lemma 4.18 and Theorem 4.20 break down. The result that replaces Lemma 4.18 is Proposition 5.2, and it is insufficient to prove directly any analog of Theorem 4.20, even for charges of total charge zero.

PROPOSITION 5.2. *In R^n for $n = 1$ or 2, let $f \in C_0 \cap L^1$. Suppose that*

$$\int_{R^1} |f(y)|\,|x-y|\,dy < \infty \qquad \text{if} \quad n = 1$$

or

$$\int_{R^2} |f(y)|\,|\log|x-y||\,dy < \infty \qquad \text{if} \quad n = 2.$$

If $\int_{R^n} f(y)\,dy = 0$, then $\int_0^T (Q^t f)(x)\,dt$ tends pointwise *to a finite limit $g(x) = \int_0^\infty (Q^t f)(x)\,dt$ as $T \to \infty$. The function g is continuous and bounded (but possibly not in C_0) and satisfies*

$$g(x) = \begin{cases} -\int_{R^1} f(y)|x-y|\,dy & \text{if} \quad n = 1 \\ 2\int_{R^2} f(y)\log|x-y|\,dy & \text{if} \quad n = 2. \end{cases}$$

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