SALOMON BOCHNER

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Salomon Bochner was a mathematician whose research profoundly influenced the development of a wide area of analysis in the last three quarters of the twentieth century. He contributed to the fields of almost periodic functions, classical Fourier analysis, complex analysis in one and several variables, differential geometry, Lie groups, probability, and history of science, among others.

He did not often write long papers. Instead he would typically distill the essence of one or more topics he was studying, begin a paper with a treatment not far removed from axiomatics, show in a few strokes how some new theorem followed by making additional assumptions, and conclude with how that theorem simultaneously unified and elucidated old results while producing new ones of considerable interest. Part of the power of his method was that he would weave together his different fields of interest, using each field to reinforce the others. The effect on the body of known mathematics was often to introduce a completely new point of view and inspire other mathematicians to follow new lines of investigation at which his work hinted.

His early work on almost periodic functions on the line illustrates this approach. Harald Bohr of Copenhagen, younger brother of Niels, had established himself as a notable mathematician by writing two papers[1] in the *Acta Mathematica* in 1924 and 1925, each about 100 pages long, introducing these functions and establishing basic theorems about them. The *Acta* at that time was the premier international journal in mathematics, and Bohr's work was considered to be of top quality. One way of viewing Bohr's theory was that the definition was arranged to give an abstract characterization of the functions on the line that are uniform limits of finite linear combinations of exponentials $e^{i\lambda x}$, the exponents λ not necessarily all being integer multiples of a single exponent. The actual definition is not particularly memorable, and there is no need to reproduce it here. The almost periodic functions are closed under addition, multiplication, and uniform limits; periodic functions provide examples. Bohr showed that any such function has what is now called a Bohr mean—in other words, that $B(f) = \lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T} f(x)\,dx$ exists. Armed with this mean, Bohr defined a kind of Fourier expansion for these functions, writing $f(x) \sim \sum_\lambda a_\lambda e^{i\lambda x}$, where $a_\lambda = B(f(x)e^{-i\lambda x})$. Only countably many of the coefficients a_λ can be different from 0. The main theorem of Bohr's first *Acta* paper is that f is determined by its coefficients a_λ. In the second *Acta* paper the main theorem is the desired result that any almost periodic function can be approximated uniformly by finite linear combinations of functions $e^{i\lambda x}$.

Bohr's results had been announced in 1923, and Bochner went to work on almost periodic functions while the second of these 100-page papers of Bohr's was still in press. In the three-page announcement (1925) Bochner observed first that the function f is almost periodic if and only if every sequence of translates has a subsequence that converges uniformly; in modern terminology, f is almost periodic if and only if its set of translates has compact closure in the metric of uniform convergence. This definition was much easier to work with than Bohr's definition. Bochner's next observation was that the approximation theorem in Bohr's second paper could readily be deduced from the main theorem of the first paper by constructing what is now called an approximate identity, a step that Bochner carried out in short order. In effect Bochner reduced Bohr's second 100-page paper to an argument that is so short that it can be described in a conversation. The notion of an approximate identity, not just with almost periodic functions but throughout real analysis, has become a standard tool for reducing problems about arbitrary functions to problems about nicer functions. The Bochner definition made sense on any group, not just the additive group of the line, and Bochner had opened an avenue of investigation for someone. Indeed, John von Neumann[2] in 1934 published a generalization to all groups that used Bochner's definition and combined it with techniques from the work of Hermann Weyl. Bochner and von Neumann combined forces to write a sequel (1935) that extended the theory to vector-valued functions, no doubt motivated by the theory of vector-valued integration for the Lebesgue integral—what is now called the Bochner integral—that Bochner had introduced in (1933).

The announcement of 1925 was only Bochner's third paper. The first two, which appeared in 1921 and 1922, dealt with the subject of his thesis, a combination of Fourier analysis and complex-variable theory. In this thesis, Bochner constructed, before Stefan Bergman, what is now called the Bergman kernel.[3] Bochner did not pursue the subject, while Bergman did, and thus the kernel came to be named for Bergman.

Pursuing his interest in complex analysis, Bochner wrote several further papers in the theory of functions of one complex variable. The paper (1928) in this direction, dealing with maximal extensions of noncompact Riemann surfaces, is of unusual interest not so much because of its topic but rather because it contains a comprehensive version of what has come to be known as Zorn's lemma, which Zorn apparently discovered[4] as late as 1933 and published[5] in 1935.

As a classical Fourier analyst, Bochner soon took an interest in the Fourier transform on the line and studied the multidimensional extension of it. He paid particular attention to convergence questions and to the Poisson summation formula, which relates the Fourier transform to Fourier series and is used in proving the "modular relation" that connects the values of a theta function at z and $-1/z$ in the upper half plane. His book *Vorlesungen über Fouriersche Integrale* (1932) is a classic in the subject and established his stature as an analyst once and for all. This book contains what is now often known simply as Bochner's Theorem,[6] characterizing continuous positive definite functions on Euclidean space. A continuous complex-valued function f is defined to be positive definite if $\iint \varphi(x)f(x-y)\overline{\varphi(y)}\,dx\,dy$ is ≥ 0 for every continuous function φ supported inside a finite cube. According to the theorem, such functions are characterized as the Fourier transforms of nonnegative finite measures.

In its Euclidean setting, Bochner's Theorem has rather few applications outside of probability. The power of the theorem comes through its generalizations to other settings in harmonic analysis. One such setting is the theory of locally compact groups. The definition of positive definite function makes sense for such a group G if $f(x-y)$ is replaced by $f(xy^{-1})$. I. Gelfand and D. Raikov[7] observed that if U is a unitary representation of G, then $x \mapsto (U(x)v, v)$ is positive definite for any v in the underlying Hilbert space, even if U is infinite-dimensional. Combining this observation and the Krein–Milman Theorem, they proved that G has enough irreducible unitary representations to separate points. Their result was part of the impetus for including infinite-dimensional representations in representation theory, a subject that has continued to grow in importance to the present day.

Another such setting is the special case in which G is abelian. For this special case the foundational duality theory of L. Pontrjagin went through at least three incarnations, carried out successively by Pontrjagin, by A. Weil, and by H. Cartan and R. Godement. The Cartan–Godement approach[8] starts from the results of Gelfand and Raikov. A version of Bochner's Theorem is established simultaneously with the proofs of Pontrjagin duality, the Fourier inversion formula, and the Plancherel formula for the group's Fourier transform. All four of these results have to be established together; none can be omitted in the Cartan–Godement approach. Thus Bochner's Theorem becomes part of the foundation of the theory of locally compact abelian groups. The adele and idele groups of a number field furnish important examples of locally compact abelian groups, and these theorems for such groups are essential underpinnings in the modern understanding of class field theory.

Bochner's initial multidimensional investigations of convergence questions in Fourier analysis mostly concerned rectangular partial sums. Then, beginning with the classic paper (1936), he addressed in earnest the natural question of summing Fourier series and Fourier integrals in spherical fashion. The question had been considered earlier by other authors, but Bochner brought to the question a new summability method that has come to be called Bochner–Riesz summability. This results in helpful simplifications that do not occur with related summability methods. In dimension $k > 1$, let x and y denote real k-tuples and let n denote an integer k-tuple. The Fourier series of f is $f(x) \sim \sum_n c_n e^{in\cdot x}$, where $c_n = (2\pi)^{-k} \int_{[-\pi,\pi]^k} f(y) e^{-in\cdot y}\, dy$ and the dot in the exponents indicates the dot product. The Bochner–Riesz sums are $S_{R,\delta}(x) = \sum_{|n|\leq R} (1 - (|n|/R)^2)^\delta c_n e^{in\cdot x}$ with $\delta > 0$. We are to think of letting R tend to infinity. Ordinary spherical convergence is the case of $\delta = 0$, and the cases $\delta > 0$ are to be viewed as easier to handle. Bochner examined the validity of the localization property, i.e., the extent to which the existence of $\lim_{R\to\infty} S_{R,\delta}(x)$ depends only on the values of f near x. He showed that localization holds for $\delta > \frac{1}{2}(k-1)$ and fails for $\delta < \frac{1}{2}(k-1)$. A similar conclusion holds for Fourier transforms.

Bochner returned to these matters in the early 1950s. Spherical summation inevitably leads one to Bessel functions, and Bochner was led to combine his knowledge of Bessel functions with that of the "modular relation" in (1951) to give a complete analysis of the effect of rotations on the Fourier transform in k-dimensional space R^k. Any function on R^k is a suitable kind of limit of linear combinations of functions $g(|x|)H(x)$, where $H(x)$ is a harmonic polynomial. Bochner showed that the Fourier transform of such a product is of the form $(Tg)(y)H(y)$, where Tg is given in terms of g by an explicit one-dimensional integral involving a Bessel function, a so-called "Hankel transform." In a paper the next year he extended his work on Bessel functions by obtaining transformation formulas for what have come to be called Bessel functions of a matrix argument.

These topics were taken up by Bochner's student Carl Herz. For the case of a radial function f with Fourier transform $\widehat{f}$, Herz examined the sense in which f could be recovered as the limit on R of the inverse Fourier transform of the product of $\widehat{f}$ by the characteristic function of the ball of radius R centered at the origin. He showed[9] that if f is in L^p and $2k/(k+1) < p \leq 2$, then the approximations converge to f in L^p. In the direction of positive generalizations, E. M. Stein later obtained analogous results for convergence when f is not necessarily radial but the truncated Fourier transforms are replaced by Bochner–Riesz approximations to the truncations; Stein obtained norm convergence for an interval of p's that depends on the Bochner–Riesz index δ. For $\delta = 0$, Stein obtained nothing new—only the convergence in L^2. Stein made critical use of an observation that, although the restriction of the Fourier transform to a hyperplane does not make sense for an L^p function when $p > 1$, there is a nontrivial interval $1 \leq p < p_0$ such that restriction to a sphere makes sense for the Fourier transform of an L^p function. The interplay between curvature of a set and the meaningfulness of the restriction of a Fourier transform to the set was studied extensively by later authors and continues to be a subject of investigation. In the direction of negative generalizations of the work on spherical summability, C. Fefferman ultimately proved that the Herz approximations for a nonradial function f need not converge in L^p except for $p = 2$. Thus the use of Bessel functions is an essential aspect of the theory. Herz[10] took up another topic of Bochner's and developed a substantial theory of Bessel functions of a matrix argument. Later K. Gross and R. Kunze generalized aspects of the Herz theory and related these matters to the subject of analysis on semisimple Lie groups.

In the subject of differential geometry, Bochner is best known for his stunning quantification of the century-old idea that the curvature of a compact Riemannian manifold can force global topological conclusions about the manifold. This curvature-topology work was initially encapsulated in a single formula (1946) and its variations and applications. Of Bochner's formula, M. Berger writes:[11]

> "The Bochner article [(1946)] will remain an unavoidable cornerstone of transcendental methods linking the local geometry to global properties of the underlying space. Bochner calculated the Laplacian of the norm squared of a differential 1-form ω on a Riemannian manifold [in terms of the covariant

derivative of ω, the Hodge Laplacian $d\delta + \delta d$ of ω, and the Ricci curvature tensor applied to ω]."

In reviewing this paper, S. Myers lists some consequences of the formula and its variations for compact manifolds:[12]

> "For example, (1) a compact M with positive mean [=Ricci] curvature has no vector field whose divergence and curl both vanish, (2) a compact M with negative mean curvature has no continuous group of isometries, (3) a compact H with negative mean curvature has no continuous group of analytic homeomorphisms, (4) a compact H with negative (positive) mean curvature has no analytic contravariant (covariant) tensor field, (5) if a compact H with positive mean curvature is covered by a finite number of neighborhoods, if a meromorphic functional element is defined in each neighborhood and if the difference of meromorphic elements is holomorphic whenever the elements overlap, then there exists one meromorphic function on H which differs by a holomorphic function from each meromorphic element given."

Bochner pursued this topic for five or six years, writing several papers, one of them joint with K. Yano, and ultimately publishing the book *Curvature and Betti Numbers* (1953) jointly with Yano.

Other mathematicians developed this topic in two quite distinct directions. K. Kodaira worked with complex Kähler manifolds, which include all nonsingular projective algebraic varieties, and arrived at the celebrated Kodaira Vanishing Theorem. In the paper[13] in which this theorem is proved, Kodaira writes, "In the present note we shall prove by a differential-geometric method due to Bochner some sufficient conditions for the vanishing of [the sheaf cohomology spaces] $H^q(V; \Omega^p(F))$ in terms of the characteristic class of the bundle F." This theorem is fundamental in modern algebraic geometry. Sixteen years later, P. Griffiths and W. Schmid[14] adapted to infinite-dimensional representation theory the idea that curvature conditions can imply vanishing of sheaf cohomology, and sheaf cohomology became a tool for realizing interesting infinite-dimensional representations of noncompact semisimple Lie groups.

A. Lichnerowicz took up aspects[15] of the theory for noncomplex manifolds. He obtained different applications of Bochner's original formula and also obtained additional formulas of his own. One of the latter applied the Bochner technique to the spinor fields on a spin manifold, yielding a formula[16] relating the square of the Dirac operator, the covariant derivative, and the scalar curvature. M. Gromov and H. B. Lawson[17] combined this formula with work of A. Borel and F. Hirzebruch and with the Atiyah-Singer Index Theorem and were able to classify all simply-connected compact manifolds admitting a Riemannian metric with positive scalar curvature.

In the late 1930s Bochner began a systematic investigation of functions of several complex variables. Robert Gunning, the editor of Bochner's collected papers and a student of Bochner's from the 1950s, summarizes this work as follows:[18]

> "Bochner's interest in functions of several complex variables began with

their Fourier analysis, leading to his characterization of the envelopes of holomorphy of tube domains [(1938)]. He later wrote on generalizations of Cauchy's integral formula for functions of several variables (including what is known as the Bochner–Martinelli integral formula [(1943)]) and applications of these formulas to analytic continuation on singularities of analytic spaces [(1953)] and on conditions for the analytic and linear dependence of complex analytic functions in various cases. The book *Several Complex Variables* ... (1948), written jointly with W. T. Martin, summarized much of his earlier work and his own outlook on the subject."[19]

About the book (1948), S. Krantz says,[20] in reviewing the volumes of collected papers, "The book by Bochner and Martin ... was among the first on the subject of several complex variables; although there are now many books on the subject, that volume is frequently cited in the modern literature." About Bochner's work as a whole, Krantz continues, "Not only did Bochner touch many areas of mathematics, but his ideas are so profound that they are still of great interest today."

Salomon Bochner, son of Joseph and Rude Bochner, was born August 20, 1899, into a Jewish family of modest means in the Polish city of Cracow, which was then part of the Austro-Hungarian Empire. His brilliance was already evident to the teachers in his Jewish elementary school, and when Bochner was nine years old, one of them predicted that he would make his living as a mathematician. In 1915, shortly after the outbreak of World War I, the threat of a Russian invasion of Austria-Hungary led the Bochner family to flee to Germany, which at that time was seen as more hospitable to Jews than was Russia, or even Austria-Hungary. One example of this greater openness was the fact that, unlike in Cracow, the state schools, including the prestigious gymnasia, made accommodations for orthodox Jewish children whose religious practices did not allow them to write on Saturdays, which was a school day. When his family arrived in Berlin, Bochner immediately took the entrance examination for a gymnasium, without having studied much German, and he received the highest score in the city, which garnered him financial support from a wealthy Berlin Jew. At the gymnasium he developed a great love for classics and history, which he maintained throughout his life, but he chose to pursue mathematics professionally, because he felt that it was a surer career path.

He received his doctor of philosophy degree from the University of Berlin in 1921. The elder Constantin Carathéodory and he became good friends during this time. According to an online mathematics genealogy project[21], Bochner's thesis adviser was Erhard Schmidt. In later years Bochner would not say much about Schmidt. Instead he would occasionally say, with a little smile, that, in his observation, a mathematician often took after his mathematical grandfather. In Bochner's case this was David Hilbert.

The time when Bochner got his degree was a time of hyperinflation in Germany, and his family was in desperate straits financially. As a consequence Bochner did not immediately take an academic job but instead went into the family import-export business, doing mathematics only recreationally. Over a period of four years, he did extremely well at the business. Despite this success his family could see that his real interest was in mathematics, and they encouraged him to return to mathematics full time. He did so, and, on the basis particularly of his paper (1925), he became an International Education Board fellow at Oxford University, Cambridge University, and the University of Copenhagen for 1925–27. In England he became good friends with G. H. Hardy, and they wrote one paper together. In 1927 at the end of the fellowship, he became a lecturer at the University of Munich.

Like many untenured academic Jews in Germany, Bochner was dismissed from his position during the 1932–33 year. For the second time he became a refugee; he went to England, a country he had come to love during his stay there in the 1920s, and asked Hardy for help in getting a position. Meanwhile, an offer arrived from Solomon Lefschetz, who was the first Jewish professor to have been hired by Princeton, and Hardy encouraged Bochner to accept the offer rather than to try to stay in England, which was rapidly becoming overcrowded with German academic refugees. Bochner accepted the offer and left for America alone, becoming an "associate" at Princeton for the 1933–34 year and an assistant professor starting in 1934.

During the 1930s he would travel every summer to Germany to visit his family, and in 1938 he helped his family immigrate to England and get properly settled. It was on one of these voyages that he met Naomi, his wife-to-be, an American traveling to Europe on a vacation. They were married on Thanksgiving Day in 1938 with John von Neumann as best man. After their marriage the Bochners developed lifelong friendships with Marston Morse and his wife Louise, as well as with Eugene Wigner and his wife Mary.

Bochner was promoted to associate professor in 1939, and to professor in 1946. During this period in his life, Bochner was a part-time Member of the Institute for Advanced Study for 1945–48, a lecturer at Harvard for the spring semester of 1947, a consultant to the Los Alamos Project in Princeton in 1951, and, for 1952–53, a visiting professor in the Department of Statistics at the University of California at Berkeley.

In 1959 Bochner was appointed Henry Burchard Fine Professor of Mathematics, and he held that position until his mandatory retirement from Princeton in 1968. He was then immediately appointed E. O. Lovett Professor of Mathematics at Rice University, a position he held until his death in 1982. For the interval 1969–76 he was chairman of the department. The atmosphere at the two institutions was quite different. At Princeton younger people in the department who knew him would refer to him in the third person as "The Master" or sometimes "Himself." The staff called him "Professor Bochner" in recognition of his endowed chair; ordinary professors were simply "Mr." At Rice, however, the environment was more relaxed, and a number of people called him "Sal." While still at Princeton, Bochner himself commented that "Princeton has more prima donnas per square foot than any other place in the world."

Bochner was elected to the National Academy of Sciences in 1950. He was an invited speaker at the International Congress of Mathematicians in 1950, gave the Colloquium Lectures of the American Mathematical Society in 1956, and was keynote speaker at the AAAS Symposium in 1971 on the "Role of Mathematics in the Development of Science." In January 1979 the American Mathematical Society awarded him the first Leroy P. Steele Prize for Lifetime Achievement, citing him for "his cumulative influence on the fields of probability theory, Fourier analysis, several complex variables, and differential geometry."

As Bochner grew older, he partly turned from mathematics to classics, philosophy, and history of science and of mathematics. He regarded this move not as a forced retreat from his chosen field but rather as an opportunity to return to the humanistic interests that had engaged him in his youth. He was most proud of his book (1966), *The Role of Mathematics in the Rise of Science*, which went into paperback. During his Rice years, he became close colleagues of the historians of science, and received much acclaim for a public lecture on Einstein, delivered in honor of the centenary of Einstein's birth.

The Bochners had one child, a daughter Deborah, who became Deborah Bochner Kennel. Trained as a Renaissance historian, she is at this writing in 2003 working as a writer and editor for the Center for Medieval and Renaissance Studies at the University of California at Los Angeles. She has two children, who both chose Princeton for their undergraduate educations. Matthew Bochner Kennel is an assistant research physicist at the University of California at San Diego, and Sarah Alexandra Kennel is an assistant curator at the National Gallery of Art in Washington, DC. Deborah described Salomon Bochner as a very attentive father, who gave lifelong unconditional love and, as she matured, intellectual stimulation and companionship in a wide variety of humanistic subjects. She commented also that he was witty, with an intellectual formation typical of the the prewar continental academic mode, and was also a strong Anglophile. She said he enjoyed describing himself as having been "born under Victoria." Deborah added that he definitely had his idiosyncrasies: he disliked both picnics and barbecues, always repeating that "it took man millions of years to learn to cook and eat inside and I don't see why I should reverse the process." This attitude was consistent with various comments he made to his colleagues, such as "Scenery is for adolescents—of all ages."

At the time of the move to Rice, the Bochners rented a apartment in Houston but continued to keep their house in Princeton. They would travel from one place to another seasonally, and on occasion would visit their grandchildren in Los Angeles, where Deborah had settled with her family. While they were on a trip to Los Angeles in 1971, Naomi died unexpectedly, and her husband soldiered on alone at Rice. He developed both eye trouble and a heart condition. In 1981 he had successful cataract surgery on one eye, but in 1982 he had a heart attack during surgery on the other eye, and died a few days later, on May 2, 1982.

In his time at Princeton, Bochner took a few young faculty members under his wing as postdocs, officially or unofficially. One of these was K. Chandrasekharan, with whom Bochner jointly authored the book (1949). Another was a young functional analyst from Yale, Robert Langlands. Bochner pushed Langlands in the direction of algebraic number theory, arranging for him to teach a course in class field theory. One of Bochner's thesis students, William Veech, remembers passing Langlands in the hall one day in the 1960s and asking Langlands what he would do next. His response was "noncommutative class field theory." Indeed he did; aspects of the work by Langlands played a crucial role thirty years later in the proof of Fermat's Last Theorem. According to Veech, Atle Selberg thanked Bochner publicly at a banquet in 1969 in honor of Bochner's seventieth birthday for having sent him at an early stage some papers by Langlands, who Selberg said "is now one of the best mathematicians in the world."

Veech went on, saying that Selberg, in that same brief talk, mentioned that once in conversation with Hermann Weyl, Weyl remarked something close to "Now Bochner, he is really somebody." Veech recorded in his May 1982 eulogy of Bochner a further memory of that banquet: After all the banquet talks had been completed, Bochner himself "was invited to make some remarks, of which he had but one: In the 1930s, there was a trolley car that ran from Princeton to Trenton and back. Bochner's one regret in life, he confided to the hushed assembly, was that he had never ridden that trolley."

The online mathematics genealogy project[22] lists Bochner as having 38 doctoral students. I was one of the last, finishing in 1965. Bochner was not someone to whom students flocked, and he actually had no current students in the semester before my qualifying examination. Bochner's student Veech, who had recently graduated and had stayed on as an instructor, pointed out to me the advantages of seeking Bochner as adviser. I found that Bochner was awe-inspiring, yet approachable and not particularly intimidating in person. This man had had, after all, forty more years experience at mathematics than I had had, but he still made me feel that I could produce something new that would interest him.

After I had passed my qualifying examination, Bochner gave me a warm-up problem, which took two weeks to solve, and then I was on my own to produce a thesis. The advice he offered was more philosophical, or sometimes sociological, than mathematical. Mathematical advice was left to be supplied by another earlier Bochner student, Harry Furstenberg, who was visiting Princeton for a year.

The piece of philosophical advice that I remember most vividly, and would always pass along to my own students, was "Theorems come from theories, and not the other way around." On one occasion he said, "Young mathematicians work on theorems, mature mathematicians work on theories, and elderly mathematicians work on theories about theories."

At some point Bochner told me that part of his job was to keep me on an even keel emotionally, picking me up when I was down and knocking me down a bit when I was too confident. After I had produced a first theorem and cheerfully proposed to show it to him, he peered at me while walking with me toward his office and asked, "Is it earth shaking, earth shattering, or earth annihilating?" Later on, when I had assembled a body of my own mathematics and we were discussing it, I said dejectedly that it all seemed so trivial now. He responded, "Yours is experience number 13765972 of this kind [or perhaps it was some other large integer]. Everyone has this kind of experience. It means that you finally have understood what you have done."

At another time he said that he did not want to be a father figure to me. This was a comment whose complexity I still have not fully understood. Perhaps this was just a pithy comment of the kind that he would often make on the spur of the moment. Or perhaps he knew that my father had died unexpectedly a year before I arrived in Princeton.

At some point when I was well along toward a thesis, he and I had a conversation about his experience with different branches of mathematics. He said that he deliberately chose to avoid competitive areas. Only later would I understand that he had in fact created a number of areas and then left them when other people took them up.

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NOTES

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2. J. von Neumann. Almost periodic functions in a group, I. *Trans. Amer. Math. Soc.* 36 (1934):445–492.

3. Bergman spelled his name with two n's in German and French and with one n in English. His original paper on the kernel was in German.

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6. Sometimes the name Herglotz is attached also to the theorem because in retrospect it can be seen that an earlier theorem of Herglotz's was a version of Bochner's Theorem for Fourier series.

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15. See Note 11.

16. A. Lichnerowicz. Spineurs harmonique. *C. R. Acad. Sci. Paris* 257 (1963):7–9.

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19. *Ibid*, Part 3, p. 1. The dates [(1938)], [(1943)], and [(1953)] have been added to the quotation, and they and (1948) refer to the present selected bibliography.

20. S. G. Krantz. *Mathematical Reviews.* Items 92m:01093a to 92m:01093d.

21. http://www.genealogy.math.ndsu.nodak.edu

22. *Loc. cit.*

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