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Singular Integrals and the Principal Series, III

(intertwining operators/semisimple Lie groups)

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ABSTRACT The intertwining operators for nonunitary principal series representations of a real semisimple Lie group are used to construct intertwining operators for all the series of unitary representations appearing in the Plancherel formula. The resulting machinery reduces computations with Harish-Chandra's c-functions and the Plancherel measure to computations with those c-functions and measures obtained from a minimal parabolic subgroup.

In earlier work (1, 2) we presented a theory of intertwining operators that gives complete information about questions of reducibility for representations of the principal series of a connected semisimple Lie group of matrices. The representations in question are those induced from finite-dimensional irreducible unitary representations of a minimal parabolic subgroup.

In the present note we shall show how to use this theory to reduce some questions about the other continuous series representations contributing to the Plancherel formula of such a group, to questions about the principal series. Detailed proofs will appear later. Immediate applications are the following: (*a*) We obtain an extension of the theory of intertwining operators that deals with these other continuous series. (*b*) Formulas connecting the *c*-functions of Harish-Chandra (3) with intertwining operators allow one to compute all *c*-functions in terms of *c*-functions relative to minimal parabolic subgroups. (*c*) The contribution to the Plancherel measure from the other continuous series can be computed in terms of Plancherel factors for the principal series. Applications to irreducibility questions and existence of complementary series will be given in a subsequent announcement.

Let G be a connected semisimple Lie group of matrices with maximal compact subgroup K and Cartan involution θ. According to Harish-Chandra (3, 4), the unitary representations contributing to the Plancherel formula of G are constructed from the finitely many conjugacy classes of Cartan subgroups of G. To each conjugacy class one can associate a parabolic subgroup with Langlands decomposition $P = MAN$ such that M has a discrete series. The continuous series representations associated to the Cartan subgroup are parametrized by (ξ,λ), where ξ is a discrete series representation of M and λ is an imaginary-valued linear functional on the Lie algebra $\mathfrak{a}$ of A. The representation $U(\xi,\lambda,x)$, with x in G, is given as $\mathrm{ind}_{MAN\uparrow G}(\xi \otimes e^{\lambda} \otimes 1)$. We adopt the convention that G acts on the left in the induced representation.

The idea for studying $U(\xi,\lambda,\cdot)$ is very simple. Suppose we consider $U(\omega,\lambda,\cdot)$ instead, where ω is a principal series representation of M. It is an observation of Mackey's (5), by the double induction theorem, that $U(\omega,\lambda,\cdot)$ is unitarily equivalent with a principal series representation of G. Next, suppose that ω is in the nonunitary principal series of M. (See ref. 1 for the definition.) In the same way the induced representation $U(\omega, \lambda,\cdot)$ can be regarded as in the nonunitary principal series of G. If the discrete series ξ can be imbedded suitably in ω, then we should expect that $U(\xi,\lambda,\cdot)$ will be imbedded in $U(\omega,\lambda,\cdot)$ and the intertwining operators attached to $U(\xi,\lambda,\cdot)$ will be restrictions of those for $U(\omega,\lambda,\cdot)$.

Actually the group M need not be connected or semisimple, but it does have sufficient structure to allow one to define nonunitary principal series for it. A maximal compact subgroup is $K_M = K \cap M$, and M has a minimal parabolic subgroup $M_M A_M N_M$. The group $M_p A_p N_p$ is a minimal parabolic subgroup of G if $M_p = M_M$, $A_p = A_M A$, and $N_p = N_M N$. Induction from finite-dimensional irreducible representations of $M_M A_M N_M$ gives the nonunitary principal series of M. The theory of intertwining operators for *these* nonunitary principal series representations will play no role here except in *Theorem 1* below. Since the extension of the theory of ref. 2 to groups as general as M presents no difficulties or surprises, we shall not elaborate on it at this time.

Our arguments will apply to MAN whenever each discrete series representation of M can be imbedded infinitesimally (with no continuity assumed) in the nonunitary principal series of M. (The imbedding is assumed equivariant with respect to K_M and the universal enveloping algebra of M.) We are informed that W. Casselman has proved such a subrepresentation theorem for connected semisimple matrix groups. With his result it is an easy matter to extend his theorem to a class of groups rich enough to include all the groups M under consideration.

THEOREM 1. *Suppose that ξ and ω are, respectively, discrete series and nonunitary principal series representations of M in the Hilbert spaces H^{ξ} and H^{ω}. If i is an infinitesimal imbedding of ξ in ω and if $H^{\omega}{}_0$ denotes the Hilbert space closure of the image of i, then i^{-1} extends to a continuous map of the C^{∞} vectors of $H^{\omega}{}_0$ into the C^{∞} vectors of H^{ξ}.*

This theorem will play a key analytic role in questions later about irreducibility. Its proof uses the intertwining operators for M and the Cauchy Integral Formula.

The explanation of the construction of continuous series intertwining operators will be clearer if we present only the formalism. We are to think of ξ imbedded infinitesimally in

$$\omega = \mathrm{ind}_{M_M A_M N_M \uparrow M}(\sigma \otimes \exp \lambda_M \otimes 1),$$

which acts on

$$H^\omega = \{f: M \to H^\sigma | f(xm_M a_M n_M) = e^{-(\lambda_M+\rho_M)\log a_M}\, \sigma(m_M)^{-1} f(x)\}$$

by $(\omega(m)f(m') = f(m^{-1}m')$. The norm squared in H^ω is

$$\int_{K_M} |f(k)|^2_{H^\sigma}\, dk.$$

If λ is a linear functional on $\mathfrak{a}$, then $U(\omega,\lambda,\cdot) = \mathrm{ind}_{MAN\uparrow G}(\omega \otimes e^\lambda \otimes 1)$ acts on

$$H^U = \{F: G \to H^\omega | F(xman) = e^{-(\lambda+\rho)\log a}\, \omega(m)^{-1}F(x)\}$$

by $U(x)F(x') = F(x^{-1}x')$, and the norm squared is

$$\int_K |F(k)|^2_{H^\omega}\, dk.$$

The evaluation map $F(\cdot) \to F(\cdot)$ (1) exhibits $U(\omega,\lambda,\cdot)$ as equivalent via a unitary operator with the nonunitary principal series representation

$$\mathrm{ind}_{M_pA_pN_p\uparrow G}(\sigma \otimes \exp \lambda_p \otimes 1),$$

where λ_p is λ_M on $\mathfrak{a}_M$ and λ on $\mathfrak{a}$. (Here $\mathfrak{a}_M$ is the Lie algebra of A_M.)

Let $P_1 = MAN_1$ be another parabolic subgroup built from MA and let $V = \theta N$. We can induce $\omega \otimes e^\lambda \otimes 1$ from P or from P_1, and we distinguish the induced representations by writing $U_P(\omega,\lambda,\cdot)$ or $U_{P_1}(\omega,\lambda,\cdot)$, respectively. The formal intertwining operator from U_P to U_{P_1} is

$$(A(P_1:P:\omega:\lambda)F)\ (x) = \int_{V\cap N_1} F(x\bar{n})d\bar{n}. \qquad [1]$$

Now there exists a unique s in the Weyl group $W(\mathfrak{a}_p)$ such that $s(N_1N_M)s^{-1} = NN_M$. We write s also for a representative of this element in K. Letting $N_p = NN_M$ and $V_p = \theta N_p$, we have

$$V_p \cap s^{-1}N_p s = V \cap N_1.$$

The set on the left is the set for the integration defining a nonunitary principal series intertwining operator and suggests defining

$$(A(P_1:P:\omega:\lambda)F)\ (x)(m) = (A(s,\sigma,\lambda_p)F(\cdot)(1))(xms^{-1}) \qquad [2]$$

and taking the normalizing factor $\gamma(P_1:P:\omega:\lambda)$ for the left side to be the same as that for the right. The normalized operators are given by $\mathfrak{A} = \gamma^{-1}A$.

The right side of [2] refers to an operator of the kind considered earlier (2) and thus admits an analytic continuation to values of λ for which the integral [1] does not converge. The result is a definition, analytic continuation, and normalization for $A(P_1:P:\omega:\lambda)$ acting on smooth functions.

To bring in ξ, let $H^\omega{}_0$ be as in *Theorem 1*. Then $A(P_1:P:\omega:\lambda)$ has a sense when restricted to the smooth functions in

$$\mathrm{ind}_{MAN\uparrow G}(\omega(\cdot)|_{H^{\omega_0}} \otimes e^\lambda \otimes 1). \qquad [3]$$

The K-finite functions for the continuous series representation $U_P(\xi,\lambda,\cdot)$ are identified with the K-finite functions for [3], and we define $A(P_1:P:\xi:\lambda)$ on them to match $A(P_1:P:\omega:\lambda)$. The normalizing factor is taken unchanged. The result is a definition, analytic continuation, and normalization for $A(P_1:P:\xi:\lambda)$ acting on K-finite functions.

For w in the normalizer in K of $\mathfrak{a}$ (and thus representing an element of the Weyl group $W(\mathfrak{a})$), let $R(w)$ be right translation by w and let

$$A_P(w,\xi,\lambda) = R(w)A(w^{-1}Pw:P:\xi:\lambda)$$

$$\mathfrak{A}_P(w,\xi,\lambda) = R(w)\mathfrak{A}(w^{-1}Pw:P:\xi:\lambda).$$

The machinery above enables one to prove painlessly that the integral in [1] converges also in the sense of H^ξ-valued integrals if $Re\lambda$ is sufficiently far out in the positive Weyl chamber. Consequently the unnormalized operators depend on ξ but not the parameters σ and λ_M of the imbedding. The normalizing factors do depend on the imbedding, however. To simplify the remaining exposition, *we shall restrict attention to discrete series ξ that have an imbedding with the following invariance property:* Whenever w in K represents a member of $W(\mathfrak{a}_p)$ that permutes the positive $\mathfrak{a}_M$ roots and is such that $w\xi$ is equivalent with ξ, then $w\sigma$ is equivalent with σ and $w\lambda_M = \lambda_M$.

Making suitable conventions about Haar measures, normalizing factors, and compatibility of infinitesimal imbeddings, we arrive at the following result.

THEOREM 2. (*i*) *Let $\mathfrak{F}$ be a finite set of irreducible representations of K, and let $E_\mathfrak{F}$ be the orthogonal projection according to $\mathfrak{F}$. Then $A(P_1:P:\xi:\lambda)$ commutes with $E_\mathfrak{F}$ and satisfies* $E_\mathfrak{F}U_{P_1}(\xi,\lambda,\cdot)E_\mathfrak{F}A(P_1:P:\xi:\lambda) = A(P_1:P:\xi:\lambda)E_\mathfrak{F}U_P(\xi,\lambda,\cdot)E_\mathfrak{F}$.

(*ii*) $\mathfrak{A}(P_2:P_0:\xi:\lambda) = \mathfrak{A}(P_2:P_1:\xi:\lambda)\mathfrak{A}(P_1:P_0:\xi:\lambda)$. *The same identity holds for the unnormalized operators if* $N_2 \cap N_0 \subseteq N_1 \cap N_0$.

(*iii*) *If w in K represents a member of $W(\mathfrak{a})$, then* $A(P_1:w^{-1}Pw:\xi:\lambda) = R(w^{-1})A(wP_1w^{-1}:P:w\xi:w\lambda)R(w)$.

(*iv*) *If w_1 and w_2 in K represent elements of $W(\mathfrak{a})$, then* $\mathfrak{A}_P(w_1w_2,\xi,\lambda) = \mathfrak{A}_P(w_1,w_2\xi,w_2\lambda)\mathfrak{A}_P(w_2,\xi,\lambda)$.

(*v*) $EA(P_1:P:\xi:\lambda)E^{-1} = A(P_1:P:E\xi E^{-1}:\lambda)$ *if E is a unitary operator on the discrete series space H^ξ.*

(*vi*) $A(P_1:P:\xi:\lambda)^* = A(P:P_1:\xi:-\bar{\lambda})$ *if the adjoint is defined K-space by K-space. Moreover, in (i), (iii), and (v) the same identities hold also for the normalized operators.*

We now connect these intertwining operators with the c-functions of Harish-Chandra (3). In view of the above construction, one can then in principle reduce computations with c-functions (and hence with the Plancherel measure) to computations with minimal parabolic subgroups and ultimately to computations in groups of real-rank-one.

Let F be a finite set of irreducible representations of K, and let α_F be the sum of the degrees times characters of the members of F. Let V_F be the space of complex-valued functions f on $K \times K$ such that

$$\alpha_F * f(\cdot,k_2) = f(\cdot,k_2) \text{ and } \bar{\alpha}_F * f(k_1,\cdot) = f(k_1,\cdot).$$

Define a double representation τ of K on V_F by

$$\tau(k_1)f\tau(k_2)(k,k') = f(k_1^{-1}k, k_2k').$$

Let ${}^0\mathcal{C}_\xi(M,\tau_M)$ be the space of all functions ψ from M to V_F such that

$$\psi(k_1mk_2) = \tau(k_1)\psi(m)\tau(k_2) \text{ for } m \in M,\ k_1 \in K_M,\ k_2 \in K_M$$

and such that the entries of ψ are linear combinations of matrix entries of ξ. Finally let H_F be the subspace of functions f in the representation space of $U(\xi,\lambda,\cdot)$, regarded as H^ξ-valued functions on K, such that $\alpha_F * f = f$. Then the linear map $T \to \psi_T$ of $Hom(H_F,H_F)$ into ${}^0\mathcal{C}_\xi(M,\tau_M)$ given by

$$\psi_T(m)(k_1,k_2) = d_\xi Tr(e^*\xi(m)eL(k_2)TL(k_1)^{-1}),$$

where e is evaluation at 1 and d_ξ is the formal degree of ξ and L is the left regular representation, is an isomorphism onto.

Using notation that is off by a factor of i from Harish-Chandra's (3), we define Eisenstein integrals by

$$E(P:\psi:\lambda:x) = \int_K \psi(xk)\tau(k)^{-1}e^{(\lambda-\rho)H(xk)}\,dk$$

under the convention $\psi(kman) = \tau(k)\psi(m)$. It is an observation of N. Wallach (6) that

$$E(P:\psi_T:\lambda:x)(k_1,k_2) = d_\xi Tr(E_F U_P(\xi,\lambda,k_1{}^{-1}xk_2)TE_F). \quad [4]$$

The c-functions are defined in terms of the asymptotic expansions of the Eisenstein integrals for λ regular and imaginary by

$$E_{P_1}(P:\psi:\lambda:ma) = \sum_{s\in W(\mathfrak{a})} (c_{P_1|P}(s:\lambda)\psi)(m)e^{s\lambda\log a}$$

See ref. 3 for details.

THEOREM 3. $c_{P|P_1}(1:\lambda)\psi_T = \psi_{T'}$, where

$$T' = A(\bar{P}:P:\xi:\lambda)A(P:P_1:\xi:\lambda)TA(P:P_1:\xi:\lambda)^{-1}$$

and $\bar{P} = \theta P$. Conversely, if f and g are in H_F, then

$$(A(\bar{P}:P:\xi:\lambda)f,g) = d_\xi{}^{-1}\int_K (c_{P|P}(1:\lambda)\psi_T)(1)(k,k)dk,$$

where $Th = (h,g)f$.

Other identities involving $c_{P_1|P}(s:\lambda)$ can be given too, but their statements are more cumbersome. All the identities involve the same technique of proof: Establish an intertwining identity far out in the Weyl chamber, pass to Eisenstein integrals via [4], continue analytically to λ imaginary, and pass to c-functions by taking asymptotics. The special case $P_1 = P$ in *Theorem 3* requires separate proof and follows by comparing integral formulas for $c_{P|P}(1:\lambda)$ and $A(\bar{P}:P:\xi:\lambda)$. The identity in *Theorem 3'* below will be used in dealing with irreducibility questions in a subsequent announcement.

Let w be a representative in K of an element s in the Weyl group $W(\mathfrak{a})$. If $w\xi$ is equivalent with ξ, then one can define $\xi(w)$ in such a way that ξ extends to a representation of the smallest group containing M and w; the definition of $\xi(w)$ is unique up to a scalar factor equal to a root of unity.

THEOREM 3'. *If T is in $Hom(H_F,H_F)$, if w is a representative in K of the element s of $W(\mathfrak{a})$, and if $w\xi$ is equivalent with ξ, then*

$$c_{P|P}(s:\lambda)\psi_T = \psi_{T'},$$

where

$$T' = A(\bar{P}:P:\xi:s\lambda)(\xi(w)\mathfrak{A}_P(w,\xi,\lambda))T(\xi(w)\mathfrak{A}_P(w,\xi,\lambda))^{-1}.$$

Identities related to the ones in *Theorems 3* and *3'* have been obtained independently by J. Arthur.‡

Iterated application of two c-functions leads to the Plancherel measure (3). To obtain formulas for the Plancherel measure, one approach is to apply *Theorem 3* to get the measure as a product of normalizing factors and then use the results of ref. 2 to convert these normalizing factors to products of Plancherel factors for real-rank-one groups; this approach gives concrete analytic formulas for the measure. An alternate approach is to apply *Theorem 3* to compute Harish-Chandra's factor $\mu_{\xi,\beta}(\lambda)$ in ref. 3 in terms of normalizing factors; this approach will be useful in handling irreducibility questions subsequently. We give one example of each kind of identity in the following two corollaries to *Theorem 3*.

COROLLARY 1. *Let w be an element of the normalizer of A_p in K such that $w(VN_M)w^{-1} = NN_M$, and let ξ be imbedded at (σ,λ_M). Then there exists a constant c_ξ depending on ξ and the imbedding such that the Plancherel measure $\mu_\xi(\lambda)$ satisfies*

$$\mu_\xi(\lambda) = c_\xi \prod p_{\sigma|M_\alpha}((\lambda + \lambda_M)|\mathfrak{a}_\alpha),$$

where the product is over all positive $\mathfrak{a}_p$-roots α such that $w\alpha$ is negative and $\alpha/2$ is not an $\mathfrak{a}_p$-root. Here the factor for α on the right is the Plancherel measure for the real-rank-one group constructed from the $\mathfrak{a}_p$-root α.

COROLLARY 2. *Let β be a positive $\mathfrak{a}$-root such that $t\beta$ is not an $\mathfrak{a}$-root for $0 < t < 1$, and let ξ be imbedded at (σ,λ_M). Then there exists a constant $c_{\xi,\beta}$ depending on ξ,β, and the imbedding such that*

$$\mu_{\xi,\beta}(\lambda)^{-1} = c_{\xi,\beta}\gamma(P_\beta:\bar{P}_\beta:\xi:\lambda|_{\mathfrak{a}_\beta})\gamma(\bar{P}_\beta:P_\beta:\xi:\lambda|_{\mathfrak{a}_\beta}).$$

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