SINGULAR INTEGRALS AND THE PRINCIPAL SERIES

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Abstract.—The L^2 theory of singular integral operators on nilpotent Lie groups is studied, extending known results for $\mathbb{R}^n$. The intertwining operators for the representations of the simple Lie groups of real rank one turn out to be of this type. As a result we determine which representations of the principal series of these groups are irreducible.

The Calderón-Zygmund Theorem[1] gives conditions under which a principal-value convolution operator on $\mathbb{R}^n$ is a bounded operator on L^p, $1 < p < \infty$. We shall give a generalization of the L^2 part of this theorem to operators on nilpotent Lie groups and apply the resulting theorem to a problem concerning irreducibility of unitary representations of semisimple Lie groups. It will turn out, in the case of semisimple groups of real rank one, that the intertwining operators for the principal series are singular integral operators of the form given below. Thus our work may also be viewed as a step in the construction of intertwining operators for arbitrary real semisimple groups.[2]

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Let X be a (connected) simply connected nilpotent Lie group. A continuous one-parameter group $\{\delta_r, 0 < r < \infty\}$ of automorphisms of X will be called a one-parameter group of *dilations* if the differentials $(\delta_r)_*$ at the identity 1 of X satisfy $(\delta_r)_* = r^D$ for a diagonable transformation D with all eigenvalues positive. In this case a *norm function* on X is a function $|x|$ from $X - \{1\}$ to the positive real numbers, of class C^∞, and with the three properties

(*i*) $|x^{-1}| = |x|$,

(*ii*) $|\delta_r x| = r^q|x|$ for a fixed number $q > 0$,

(*iii*) $\int_{1 \le |x| \le r} |x|^{-1}\, dx = C \log r$ for a fixed C, where dx is a Haar measure on X, or equivalently $|x|^{-1}dx$ is invariant under dilations.

The number q is completely determined by the eigenvalues of D.

THEOREM 1.[3] *Let X be a simply connected nilpotent Lie group, $\{\delta_r\}$ a one-parameter group of dilations, $|x|$ a norm function on X, and $\Omega(x)$ a complex-valued C^∞-function on X-$\{1\}$ such that $\Omega(\delta_r x) = \Omega(x)$ for all r and x. Let t be a fixed real number, and suppose that one or both of the following conditions hold:*

(α) $\int_{c \le |x| \le d} \Omega(x)dx = 0$ *for some (or equivalently all) c and d with $0 < c < d$.*

(β) $t \neq 0$.

Let $f \in L^2(X)$. If (α) holds, then the limit

$$Tf(x) = \lim_{\epsilon \to 0, M \to \infty} \int_{\epsilon \le |y| \le M} |y|^{-1+it}\Omega(y)f(yx)dy$$

exists in L^2, and $f \to Tf$ is a bounded operator on $L^2(X)$. If (α) fails but (β) holds, *then the same conclusion is valid, provided that the limit is taken over appropriate sequences of ϵ's and M's tending to* 0 *and* ∞, *respectively.*

The second theorem is a converse and shows the necessity of (α) when $t = 0$.

THEOREM 2. *Let X, $\{\delta_r\}$, $|x|$, and $\Omega(x)$ be as in Theorem 1, and suppose that (α) does not hold. Then there exists no distribution $d\mu$ defined on $C_c^\infty(X)$ satisfying the following two conditions:*

(γ) *Away from the identity, $d\mu$ is equal to the function $\Omega(x)/|x|$.*

(δ) *The mapping $f \rightarrow \int f(yx)d\mu(y)$ defined for f in C_c^∞ extends to a bounded operator mapping $L^2(X)$ into itself.*

The proofs of the theorems depend on two lemmas.

LEMMA 1. *Let $0 \leq r < 1$, $c_1 \geq 0$, $c_2 \geq 0$. If $T_1, \ldots, T_N$ are linear operators on a Hilbert space with $\|T_j\| \leq c_1$, $\|T_i^*T_j\| \leq c_2 r^{|i-j|}$, and $\|T_iT_j^*\| \leq c_2 r^{|i-j|}$ for all i and j, then $\|T_1 + \ldots + T_N\| \leq C$, where $C = C(c_1,c_2,r)$ is a constant independent of N.*

The statement of Lemma 1 was formulated by the authors some time ago, and this conjecture was communicated to M. Cotlar. A proof was later obtained independently by Cotlar and the authors. The lemma extends in a nontrivial way an earlier result of Cotlar[4] in which the T_j were assumed to be Hermitian and mutually commuting.

LEMMA 2. *If $|x|$ is a norm function on a simply connected nilpotent Lie group, then there exist constants C and v with $0 < v \leq 1$ such that*

$$\left|\frac{|xy|}{|x|} - 1\right| \leq C\left(\frac{|y|}{|x|}\right)^v$$

whenever $|y| \leq |x|$ and $x \neq 1$. (Here define $|1| = 0$.)

We turn to the application to semisimple groups. Let $G = KAN$ be an Iwasawa decomposition of a connected semisimple Lie group, let θ be the Cartan involution of G corresponding to K, let M be the centralizer of A in K, let M' be the normalizer of A in K, and let ρ be half the sum of the positive restricted roots. Then MAN is a closed subgroup whose finite-dimensional irreducible unitary representations are all of the form $man \rightarrow \lambda(a)\sigma(m)$, where $\lambda(a)$ is a unitary character of A and σ is an irreducible representation of M. The *principal series* of unitary representations of G is parametrized by (σ,λ) and is obtained by inducing these representations of MAN to G.

These representations may be viewed as operating on a space of functions on the simply connected nilpotent group θN as follows. If $g \in G$, then except for a closed subset of lower dimension, every x in θN has a decomposition $xg = man\bar{n}$ with $man \in MAN$ and $\bar{n} \in \theta N$. Write $xg = m(xg)a(xg)n\bar{n}(xg)$. If σ operates in the finite-dimensional space V, then the member of the principal series corresponding to (σ,λ) is unitarily equivalent with the unitary representation $U(g)$ on $L^2(\theta N) \otimes V$ given by

$$(U(g)f)(x) = e^{\rho \log a(xg)}\lambda(a(xg))\sigma(m(xg))f(\bar{n}(xg)).$$

In the application of Theorems 1 and 2, we shall determine for which (σ,λ) this representation is irreducible, *under the assumption that* $\dim A = 1$.

Assume now that $\dim A = 1$. Then the order of M'/M is 2; write $M' =$

$M \cup m'M$. Given σ and λ, we define $\sigma^{m'}(m) = \sigma(m'mm'^{-1})$. A theorem of Bruhat[5] implies that the induced representation corresponding to (σ,λ) is irreducible unless λ is trivial and σ and $\sigma^{m'}$ are equivalent. If we assume that $\lambda = 1$ and σ and $\sigma^{m'}$ are equivalent, then it is possible to define $\sigma(m')$ uniquely, except for a minus sign, in such a way that σ becomes a representation of M' on the same space V.

Under these assumptions, suppose that T is a bounded operator on $L^2(\theta N) \otimes V$ that intertwines $U(g)$, i.e., $TU(g) = U(g)T$. Then T is given on $C_c^\infty(\theta N) \otimes V$ by convolution with a matrix-valued distribution $d\mu(x)$ as $T(f)(x) = \int d\mu(y)f(yx)$. If $d\mu$ is supported at the identity of θN, it is easy to see that T is a multiple of the identity. In the contrary case, a computation shows that $d\mu(y)$ is given on functions supported away from the identity by integration against the matrix-valued function

$$ce^{\rho \log a(ym')}\sigma(m')\sigma^{-1}(m(ym')), \qquad (*)$$

where $c \neq 0$. (The matrix appears on the left in the integration.)

Now, in the notation of Theorems 1 and 2, put $X = \theta N$. The operations of conjugation by the members of A provide a one-parameter group of dilations, $e^{-\rho \log a(ym')}$ is a norm function, and $\sigma(m')\sigma^{-1}(m(ym'))$ has the same homogeneity property as Ω. The obvious vector-valued versions of Theorems 1 and 2 imply that convolution with the function (*) can be made into a bounded operator on $L^2(\theta N) \otimes V$ if and only if $\sigma(m')\sigma^{-1}(m(ym'))$ has integral 0 over shells defined relative to the norm function. The integral of this matrix-valued function over a shell is a scalar matrix, and the result is the following.

THEOREM 3. *Let* $\dim A = 1$. *Then the representation* $U(g)$ *corresponding to* $(\sigma,1)$ *is reducible if and only if*

(1) σ *is equivalent with* $\sigma^{m'}$, *and*

(2) $\int_{c \leq e^{\rho \log a(ym')} \leq d} Tr\{\sigma(m')\sigma^{-1}(m(ym'))\}dy = 0$ *whenever* $0 < c < d$.

Here Tr *denotes the trace.*

Examples: The simple Lie algebras for which $\dim A = 1$ fall into three infinite classes (the classical cases), and there is also one exception.[6] We shall examine condition (2) in the classical cases for the corresponding groups that have simply connected complexifications.[7]

(1) Let G be the twofold (universal) covering of $SO_e(p,1)$, $p \geq 3$. Then $M = \text{Spin}(p-1)$ is a double covering of $SO(p-1)$, and M' is a double covering of the orthogonal group $O(p-1)$. When σ and $\sigma^{m'}$ are equivalent, (2) becomes

$$Tr\,\sigma\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = 0,$$

where the notation refers to either preimage in M' of the diagonal orthogonal matrix. When p is even, this condition can be shown to mean that reducibility occurs if and only if σ is nontrivial on the two-element center of $\text{Spin}(p-1)$. When p is odd, the condition means that every member of the principal series is irreducible.

(2) Let $G = SU(p,1), p \geq 2$. Then M is the quotient of $U(1) \times U(p-1) = \{(e^{i\theta},\omega)\}$ by the relation $e^{2i\theta} \det \omega = 1$. The representation σ is of the form $\sigma(e^{i\theta},\omega) = e^{in\theta}\sigma_0(\omega)$, σ is always equivalent with $\sigma^{m'}$, and (2) becomes

$$\int_0^\pi e^{-in\theta} Tr\, \sigma_0 \begin{pmatrix} e^{2i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \sin^{p-2}\theta\, d\theta = 0.$$

Fix σ_0. If $|n|$ is sufficiently large, then reducibility occurs if and only if $n \equiv p \bmod 2$.

(3) Let $G = Sp(p,1), p \geq 2$. Then $M = SU(2) \times Sp(p-1)$. View $Sp(p-1)$ as the group of "unitary" matrices with quaternions as entries. The representation σ is an outer product of the form $\sigma = R^{(n)} \otimes \sigma_0$, where $R^{(n)}$ is the irreducible representation of $SU(2)$ of dimension $n+1$. For this group, σ is always equivalent with $\sigma^{m'}$, and (2) becomes

$$\int_0^{\pi/2} Tr\, R^{(n)}\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} Tr\, \sigma_0 \begin{pmatrix} e^{-2i\theta} & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cos^{2p-3}\theta \sin^2\theta\, d\theta = 0,$$

where $e^{-2i\theta}$ is the quaternion $\cos 2\theta - \mathbf{i} \sin 2\theta$. Fix σ_0. If n is sufficiently large, then reducibility occurs if and only if $n \equiv 1 \bmod 2$.

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[1] Dunford, N., and J. T. Schwartz, *Linear Operators, II* (New York: Interscience, 1963), p. 1072. See also Muckenhoupt, B., *Pacific J. Math.*, **10**, 239–261 (1960).

[2] For the connection between the rank-one case and the general case, see Kunze, R., and E. M. Stein, *Am. J. Math.*, **89**, 385–442 (1967), and Schiffmann, G., *Compt. Rend.*, **266**, 47–49 (1968).

[3] The assumptions can be considerably modified as to their form; in particular the smoothness requirement can be greatly relaxed, and the theorem has the usual vector-valued variants.

[4] Cotlar, M., *Rev. Math. Cuyana*, **1**, 41–55 (1955).

[5] Bruhat, F., *Bull. Soc. Math. France*, **84**, 97–205 (1956).

[6] See Helgason, S., *Differential Geometry and Symmetric Spaces* (New York: Academic Press, 1962), chap. 9, for the notation and classification.

[7] We omit $SL(2,R)$ in our discussion of the classical cases in order to simplify the notation. Nevertheless, that case can be treated by the same methods; the nontrivial intertwining operator is then the Hilbert transform.