Formula for minimal K-type, maximally compact case

Assumptions: \( G \) linear connected semisimple

\[ t = \Theta \text{-stable Cartan subalgebra of } \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{k}, \mathfrak{t} \subset \mathfrak{k}, \mathfrak{t} \subset \mathfrak{p}, \]
\[ \Delta = \text{roots of } (\mathfrak{g}^c, \mathfrak{t}^c), \Delta^- = \text{roots of } (\mathfrak{g}^c, \mathfrak{t}^c) \text{ vanishing on } \mathfrak{t}. \]

Maximally compact assumption: No member of \( \Delta \) vanishes on \( \mathfrak{k} \).

\( M \) and \( M \) constructed in the usual way from \( \mathfrak{k} \), so that \( \mathfrak{k} \) is a compact Cartan subalgebra of \( M \).

We can speak of compact and noncompact roots - roots of \( (\mathfrak{g}^c, \mathfrak{t}^c) \) that vanish on \( \mathfrak{t} \) and whose root vectors are in \( \mathfrak{k}^c \) or \( \mathfrak{p}^c \), respectively.

(They are also in \( M^c \).)

We work only with \( M^\# = M \times \mathbb{Z} \) since discrete series of \( M \) are induced from \( M^\# \). (Probably \( M^\# \) is connected under our assumptions.)

Let \( \sigma = \) discrete series of \( M^\# \)

\[ \lambda_0 = \text{a Weyl-Chernikov parameter of } \sigma \text{ relative to } (\mathfrak{m}^c, \mathfrak{b}^c) \]

\[ \lambda = \lambda_0 + \delta_m^- - \delta_e^- = \text{unique minimal } K_{N_\# M^\#} \text{-type of } \sigma, \text{ with positive } \operatorname{spin}(\Delta^-) \text{ close to make } \lambda_0 \text{ dominant for } \sigma. \]

Remark: \( \mathfrak{k} \) is a Cartan subalgebra of \( \mathfrak{g} \).

Proof. Otherwise extend \( \mathfrak{k} \) to a maximal abelian subalgebra of \( \mathfrak{k} \) and then to a Cartan subalgebra of \( \mathfrak{g} \), and end up with a more compact Cartan subalgebra of \( \mathfrak{g} \) than \( \mathfrak{k} \) is.

Theorem. \( \Lambda = \lambda \) is the unique minimal \( K \)-type of \( \operatorname{ind}_K^{K_{N_\# M^\#}} \sigma \).
Lemma. Restriction from $t^c$ to $b^c$ carries $\Delta - \Delta_-$ onto the set $\Delta_c$ of roots of $(k^c, b^c)$.

Proof. Let $\beta$ be in $\Delta$ with $E_{\beta}$ in $q^c$, and write $\beta = \beta_n + \beta_0$. Then $E_{\beta} + \theta E_{\beta}$ is in $P^c$. If $H$ is in $b^c$, then

$$[H, E_{\beta} + \theta E_{\beta}] = [H, E_{\beta}] + \theta [H, E_{\beta}] = \beta_n(H) E_{\beta} + \beta_0(H) \theta E_{\beta}$$

$$= \beta_n(H) (E_{\beta} + \theta E_{\beta})$$

Hence $\beta_n$ is in $\Delta_c$ or $E_{\beta} + \theta E_{\beta} = 0$. (We know that $\beta_0 \neq 0$, since no member of $\Delta$ remains anyway on $b^c$.)

In the latter case, $-E_{\beta} = \theta E_{\beta}$ is a root vector for $\theta \beta = \beta_n - \beta_0$, and so $\beta = \theta \beta$, $\beta_0 = 0$. Then $\beta$ is in $\Delta^-$. Since $E_{\beta}$ satisfies $\theta E_{\beta} = -E_{\beta}$, $E_{\beta}$ is in $p^c$. Hence $\beta$ is in $\Delta_c$. We conclude.

Restriction carries $\Delta - \Delta_-$ into $\Delta_c$.

We show the map is onto $\Delta_c$. Thus let $\beta$ be in $\Delta_c$ with $X_{\beta} \in k^c$ an associated root vector $\neq 0$. Write

$$X_{\beta} = \sum_{\beta \in \Delta} E_{\beta} + H_0$$

$H_0 \in h_c$.

Then $H$ in $b^c$ implies

$$\beta_1(H) X_{\beta_c} = \sum_{\beta \in \Delta} \beta_1(H) E_{\beta}$$
If \( H \) is in \( \Delta_0 \), then it follows that \( \beta(H) = 0 \) whenever \( \beta \neq 0 \). If \( H = \lambda \beta \), then it follows that
\[
\frac{\beta(H \beta)}{\lambda \beta^2} = 1
\]
whenever \( \beta \neq 0 \) and that
\[
\frac{H_0}{\lambda \beta^2} = 0.
\]
Consequently \( H_0 = 0 \) and \( \beta_1 \beta = \beta_1 \) for every \( \beta \) for which \( \beta \neq 0 \). Applying \( \Theta \) and averaging, we obtain
\[
X_\beta = \sum_{\beta \neq 0} (E_\beta + \Theta E_\beta).
\]
Choose \( \beta \) so that \( E_\beta + \Theta E_\beta \neq 0 \) in this expression; this is possible since \( X_\beta \neq 0 \).

Then \( \beta \) is met in \( \Delta_0 \) and \( \Theta \beta \neq 0 \), so that \( \beta_1 \beta = \beta_1 \). This proves the map is onto.

Remark. We can regard \( \Delta \) as \( \subseteq \Delta_0 \), via the restriction map of Lemma 1.

Positive system \( \Delta_+^c \):

Let
\[
\Delta_0^c = \{ \beta_1 \beta \mid \beta \in \Delta \text{ and } (\beta_0, \beta) = 0 \} \subseteq \Delta_c
\]
\[
\Delta_{1,0}^c = \{ \beta_1 \beta \mid \beta \in \Delta \text{ and } (\beta_0, \beta) > 0 \} \subseteq \Delta_c.
\]
These notions depend only on \( \beta_1 \beta \), not on all of \( \beta \), since \( \beta_0 \) vanishes on \( \Theta \beta \).

Choose a positive system \( \Delta_{0,0}^c \) for \( \Delta_0^c \), and define
\[
\Delta^c_+ = \Delta_{1,0}^c \cup \Delta_{0,0}^c.
\]
Then \( \Delta^c_+ \) is a positive system for \( \Delta_c \), and \( (\Delta_0^-)^+ \subseteq \Delta^c_+ \).

Define \( \Delta \) in the obvious way.

Thereon. Relative to the positive system \( \Delta^c_+ \) of roots of \( (k^c, b^c) \), \( \Lambda = \lambda \)
is the unique minimal \( K \)-type of \( \pi \).

\[ K^{\text{adm}} = \sigma. \]
Lemma 2. \( \Lambda = \Delta \) is integral for \( K \), i.e., \( \exp \Lambda \) is a well-defined character of the torus \( B \) of \( K \).

Proof. \( \exp \Delta \) is a well-defined character of \( B \) as a taut in \( M \), since \( \Delta \) is the Bottcher weight of \( \sigma \).

Lemma 3. Suppose \( \beta_1 \) is a simple root for the system \( \Delta^+_c \) and is not the restriction of a member of \( (\Delta^-_c)^+ \). Then

(a) \( s_{\beta_1} \beta \in \Delta^- \)
(b) \( s_{\beta_1} \Delta^-_c \in \Delta^-_c \)
(c) \( \langle \beta^- \beta_1 \rangle \geq 0 \)
(d) \( s_{\beta_1}(\Delta^-_c)^+ \in (\Delta^-_c)^+ \) and hence \( \langle \beta^-_c \beta_1 \rangle = 0 \).

Proof. (a) Let \( \beta = \beta_1 + \beta \) be an extension of \( \beta_1 \) to a member of \( \Delta - \Delta_m \), by Lemma 1. Since \( \beta \) is not in \( (\Delta^-_c)^+ \), by assumption, \( \beta \) is not in \( \Delta^- \), thus \( \beta = 0 \). Thus

\[
\frac{2 \langle \beta, \theta \beta \rangle}{|\beta|^2} = \frac{2 \langle \beta_1 + \beta, \theta \beta_1 - \theta \beta \rangle}{|\beta_1 + \beta|^2} = -1, 0, \text{ or } 1. \tag{4}
\]

It cannot be 1, since otherwise \( \beta = 0 \) would be in \( \Delta \), and there are no members of \( \Delta \) that vanish on \( \beta \).

Suppose \( \langle \theta \beta_1 \rangle = 0 \). Then it follows that \( s_{\beta_1} s_{\beta_1} = s_{\beta_1} s_{\beta_1} \).

Since \( s_{\beta_1} \) fixes \( \Delta^-_c \), \( s_{\beta_1} \) acts on \( s_{\beta_1} s_{\beta_1} \) on \( \Delta^-_c \) and must carry \( \Delta^-_c \) into \( \Delta \), hence into \( \Delta^- \).
Suppose (a) is 1. Then it follows that \( s_{\beta_1} \) is in \( \Delta \). Thus \( s_{\beta_1} \) carries \( \Delta^- \) into \( \Delta \), hence into \( \Delta^- \).

(b) Suppose (b) is 0. Then \( \beta, \delta \beta \), and their negatives generate a subalgebra of \( \mathfrak{g} \) isomorphic to \( \mathfrak{sl}(2, \mathbb{C}) \), and it follows that \( s_{\beta} s_{\delta \beta} \) has a representative \( w \) in \( K \). Thus \( s_{\beta_1} \) acts on \( \mathfrak{g}_+ \) in the same way as an element \( Ad(w) \) with \( w \) in \( K \) that normalizes \( \mathfrak{g}_+ \). The element \( w \) must then normalize \( M \). Hence \( s_{\beta_1} \) leaves \( \Delta^-_c \) and \( \Delta^-_m \) stable.

Suppose (c) is 0. Here \( \beta \) is not a root of a split \( B_2 \) factor, and \( \delta \beta \) is not useful. Then "dessin group of a unipotent parabolic" and essentially, \( 2\beta_1 \) is a root of \( \Delta^-_m \) such that \( \pm 2\beta_1 \) are orthogonal to all other roots of \( \Delta^- \). Then it is clear that \( s_{\beta_1}(\Delta^-_c) = \Delta^-_c \).

(c) In view of (a),

\[
\begin{align*}
    s_{\beta_1}(2\varphi^-) &= \sum_{\alpha \in \Delta^+} s_{\beta_1} \alpha + \sum_{\alpha \in \Delta^+} s_{\beta_1} \alpha \\
    &= \sum_{\beta \in \Delta^+} \beta - \sum_{\beta \in \Delta^+} \beta \\
    &= 2 \varphi^- - 2 \sum_{\beta \in \Delta^+} \beta
\end{align*}
\]

\[
\begin{align*}
    s_{\beta_1}(\varphi^-) &= \varphi^- - \sum_{\beta \in \Delta^+} \beta \\
    \beta_{\to 0} &< 0
\end{align*}
\]
So the sum on the right, we have \( \beta > 0 \) and \( \beta, \beta < 0 \). Since \( \beta \)

is nonsingular for \( \Delta - \),

\[
\langle \beta, \beta \rangle > 0 \quad \text{and} \quad \langle \beta, \beta \rangle < 0.
\]

Hence

\[
\frac{2 \langle \beta, \beta \rangle}{161^2} \langle \beta, \beta \rangle > 0.
\]

Since \( \beta \) is in \( \Delta^+ \), \( \langle \beta, \beta \rangle > 0 \). Thus \( \frac{2 \langle \beta, \beta \rangle}{161} > 0 \), and (c) follows.

(d) Regard \((\Delta^-)^+\) as in \( \Delta^+_c \). Since \( \beta_1 \) is simple for \( \Delta^+_c \) and is not in \( \Delta^-_c \), \( s_{\beta_1}(\Delta^-)^+ \subseteq \Delta^+_c \). Then (d) shows that

\( s_{\beta_1}(\Delta^-)^+ \subseteq (\Delta^-)^+ \), and it follows that \( \langle \beta^- c, \beta_1 \rangle = 0 \).
Lemma 5. \( \Lambda = 2 \) is dominant for \( \Delta_2^+ \).

Proof. Let \( \beta_1 \) be simple for \( \Delta_2^+ \). We have

\[
\frac{2 \langle \gamma, \beta_1 \rangle}{10,12} = \frac{2 \langle \gamma_0, \beta_1 \rangle}{10,12} + \frac{2 \langle \bar{g}^+, \beta_1 \rangle}{10,12} \cdot (\star)
\]

If \( \beta_1 \) is the restriction of a member of \( (\Delta_2^+) \), then \( \beta_1 \) is simple for \( (\Delta_2^+) \) since \( (\Delta_2^+) \leq \Delta_2^+ \). Hence

\[
\frac{2 \langle \bar{g}^-, \beta_1 \rangle}{10,12} \geq 1 \quad \text{and} \quad \frac{2 \langle \bar{g}_0^-, \beta_1 \rangle}{10,12} = 2.
\]

Since \( \gamma_0 \) is \( \Delta^- \)-non-singular, we conclude \( (\star) \) is \( > -1 \), \( \beta_1 \) being positive. Since the left side of \( (\star) \) is an integer, by Lemma 2, and must therefore be \( \geq 0 \).

Now suppose \( \beta_1 \) is not the restriction of a member of \( (\Delta_2^+) \). Then the first term on the right of \( (\star) \) is \( \geq 0 \) since \( \beta_1 \geq 0 \), the second term is \( \geq 0 \) by Lemma 3c, and the third term is \( 0 \) by Lemma 3d. Hence the left side of \( (\star) \) is \( \geq 0 \).
Lemma 6. For $\Lambda = \lambda$, $\tau^\lambda$ occurs in $\text{Ind}^K_{K_0 K_K} \sigma$.

Remark. We shall use that $M^*$ is connected here.

Proof. Let $\Phi^\lambda$ be a highest weight vector for $\tau^\lambda$. Then we have

\[ \tau^\lambda(H) \Phi^\lambda = \lambda(H) \Phi^\lambda \quad \text{for } H \in k^E \]

\[ \tau^\lambda(E_{\beta}) \Phi^\lambda = \tau^\lambda(-\lambda_0 s_{\beta} \varepsilon_{\beta}) \Phi^\lambda = 0 \quad \text{for } \beta \in (\Delta^-)^+ \cap \Delta_c^+ . \]

Also $M^*$ is connected. Thus $\text{span} \{\tau^\lambda(k \cdot \mu^*) \Phi^\lambda \}$ is an irreducible $K_0 K_K$ module of type $\lambda$. Since $\sigma^\lambda$ occurs in $\sigma$ and $\tau^\lambda|_{K_0 K_K}$ has been shown to contain $\sigma^\lambda$, we conclude $\tau^\lambda$ occurs in $\text{Ind}^K_{K_0 K_K} \sigma$. By Frobenius reciprocity.

Lemmas 7. \( \langle p_c - p_c^-, \gamma \rangle \geq 0 \) for $\gamma \in (\Delta^-)^+$.

Proof. First we observe $s_\gamma$ leaves $\Delta_c - \Delta_c^-$ stable. In fact if $\beta_1$ is obtained by restriction to be from $\beta = \beta_1 + \beta_0$ with $\beta_0 < 0$ (cf. Lemma 1), then $s_\gamma \beta_1$ is obtained from $s_\gamma \beta = s_\gamma \beta_1 + s_\gamma \beta_0$.

Then we write

\[ s_\gamma(p_c - p_c^-) = s_\gamma \left( \frac{1}{2} \sum \beta_1 \right) = \frac{1}{2} \sum s_\gamma \beta_1 + \frac{1}{2} \sum s_\gamma \beta_1 \]

\[ = \frac{1}{2} \sum \beta_1 - \frac{1}{2} \sum \beta_1 = \frac{1}{2} \sum \beta_1 - \sum \beta_1 \]

\[ = p_c - p_c^- - \sum \beta_1 \]

\[ \beta_1 \varepsilon_{\Delta^+} \]

\[ \beta_1 \varepsilon_{\Delta^-} \]

\[ s_\gamma \beta_1 < 0 \]

\[ s_\gamma \beta_1 < 0 \]
Expanding the left side, we obtain
\[
\frac{2 \langle p_c - p_c^-, \gamma \rangle}{|\gamma|^2} \gamma = \sum_{\beta_i \in \Delta_c^+} \beta_i \cdot \beta_i \neq \Delta_c^-
\]
\[\gamma \beta_i \leq 0
\]

Taking the inner product with \( \lambda_0 \) and using the inequality \( \langle \beta_i, \lambda_0 \rangle \geq 0 \), we find
\[
\frac{2 \langle p_c - p_c^-, \gamma \rangle}{|\gamma|^2} \langle \gamma, \lambda_0 \rangle = \sum \langle \beta_i, \lambda_0 \rangle \geq 0.
\]

Since \( \langle \gamma, \lambda_0 \rangle > 0 \) for \( \gamma \in (\Delta^-)^+ \), the lemma follows.

Proof of theorem: Let \( \tau_{\lambda_0} \) be a minimal \( K \)-type of \( \tau \). By

Inductive reciprocity, \( \tau_{\lambda_0} |_{K_{\check{H}_0}} \) contains some \( K_{\check{H}_0} \) type \( \tau_{\lambda'} \) of \( \tau \).

Then \( \lambda' \) is a weight of \( \tau_{\lambda_0} \), and we have
\[
|\lambda_0 + 2p_c|^2 \leq |\lambda + 2p_c|^2 \quad \text{by Lemma 6 and minimality}
\]
\[
|\lambda + 2p_c^-|^2 \leq |\lambda' + 2p_c^-|^2 \quad \text{by minimality}
\]
\[
|\lambda'|^2 \leq |\lambda_0|^2 \quad \text{since \( \lambda' \) is a weight of \( \tau_{\lambda_0} \)}
\]
\[
\lambda' = \lambda_0 - \sum m_i \beta_i \quad \text{since \( \lambda' \) is a weight of \( \tau_{\lambda_0} \)}
\]
\[
(\beta_i \in \Delta_c^+, m_i \geq 0)
\]

We write
\[1 \lambda^2 = 1 \lambda' + 2 \phi - 1^2 - 4 \langle\lambda', \phi_c^-\rangle - 4 \phi_c^-1^2\]
\[\leq 1 \lambda' + 2 \phi_c^-1^2 - 4 \langle\lambda', \phi_c^-\rangle - 4 \phi_c^-1^2\]
\[= 1 \lambda'1^2 + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]
\[\leq 1 \Lambda_01^2 + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]
\[= 1 \Lambda_0 + 2 \phi_c^-1^2 - 4 \langle\Lambda_0, \phi_c^-\rangle - 4 \phi_c^-1^2 + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]
\[\leq 1 \Lambda + 2 \phi_c^-1^2 - 4 \langle\Lambda_0, \phi_c^-\rangle - 4 \phi_c^-1^2 + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]
\[= 1 \Lambda1^2 + 4 \langle\Lambda - \Lambda_0, \phi_c^-\rangle + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]
\[= 1 \lambda^2 + 4 \langle\lambda - \lambda' - 2\epsilon, \phi_c^-\rangle + 4 \langle\lambda' - \lambda, \phi_c^-\rangle\]

Hence,
\[4 \langle\lambda' - \lambda, \phi_c^-\rangle \leq -4 \langle\Sigma i\beta, \phi_c^-\rangle \leq 0.\]

By Schmidt's theorem, \(\lambda' - \lambda\) is the sum of members of \((\Lambda^-)^+\). Then
\[4 \langle\lambda' - \lambda, \phi_c^-\rangle > 0\] by Lemma 7.

We conclude first that \(\langle\Sigma i\beta, \phi_c^-\rangle = 0\), from which it follows that \(\lambda' = \Lambda_0\), and second that \(1 \lambda + 2 \phi_c^-1^2 = 1 \lambda' + 2 \phi_c^-1^2\) in the chain of inequalities above, from which it follows that \(\lambda' = \lambda\). Then \(\Lambda_0 = \lambda' = \lambda = \Lambda\), and the theorem is proved.