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Mathematics

Classification of irreducible tempered representations of semisimple Lie groups

(unitary representations/limits of discrete series/character identities/intertwining operators)

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ABSTRACT For each connected real semisimple matrix group, one obtains a constructive list of the irreducible tempered unitary representations and their characters. These irreducible representations all turn out to be instances of a more general kind of representation, here called basic. The result completes Langlands's classification of all irreducible admissible representations for such groups. Since not all basic representations are irreducible, a study is made of character identities relating different basic representations and of the commuting algebra for each basic representation.

For each connected linear real semisimple Lie group G, we shall give a constructive list of the irreducible tempered representations of G and their characters. "Tempered" here means that the character, as a distribution on the smooth functions of compact support on G, extends continuously to Harish-Chandra's Schwartz space (1) on G.

The significance of such a list results from a theorem of Langlands (2) that provides a classification of all irreducible admissible representations of G once one knows the irreducible tempered representations. All irreducible unitary representations are admissible, but it remains unknown which irreducible admissible representations can be made unitary. However, all irreducible tempered representations can be made unitary.

1. Irreducible tempered representations

The classification problem quickly simplifies to the problem of exhibiting the complete reduction of certain standard representations if we apply the following known results.

(*a*) Every irreducible tempered character is a constituent of some induced character $\mathrm{ind}_P{}^G\Theta$, where $P = MAN$ is a cuspidal parabolic subgroup and Θ is a discrete series character on M, unitary character on A, trivial on N. See refs. 2 and 3. [This result is also implicit in the work of Harish-Chandra (4), though not explicitly stated there.]

(*b*) With Θ and Θ' as in (*a*), either $\mathrm{ind}_P{}^G\Theta$ and $\mathrm{ind}_{P'}{}^G\Theta'$ have no constituents in common, or they are equal. See ref. 2.

(*c*) $\mathrm{ind}_P{}^G\Theta = \mathrm{ind}_{P'}{}^G\Theta'$ only if MA is conjugate by G to $M'A'$. If $MA = M'A'$, then $\mathrm{ind}_P{}^G\Theta = \mathrm{ind}_{P'}{}^G\Theta'$ if and only if Θ and Θ' on MA are conjugate under the Weyl group $W(A{:}\,G)$ of the parabolic. See ref. 4.

(*d*) The irreducible constituents of $\mathrm{ind}_P{}^G\Theta$ occur with multiplicity one. See ref. 5.

When these facts are combined, we obtain the required classification as soon as we can exhibit each $\mathrm{ind}_P{}^G\Theta$ in (*a*) as a sum of irreducible characters. (It will appear that all the irreducible characters in the decomposition of $\mathrm{ind}_P{}^G\Theta$ are tempered; see also ref 6, p. 71.)

We begin by describing in more detail the characters that we shall consider and the finite group R that leads to the decomposition. Let $P = MAN$ be the Langlands decomposition (4) of a cuspidal parabolic subgroup of G, and let T^- be a compact Cartan subgroup of M. Then $T = T^-A$ is a Cartan subgroup of G. Lie algebras will be denoted by corresponding lower-case German letters.

Each discrete series representation of M is determined by a nonsingular linear form λ on $i\mathfrak{t}^-$ and a character η on the center Z_M of M. The form λ satisfies the integrality condition that $\lambda - \rho$ (with ρ equal to half the sum of the positive roots of M in some order) lifts to a character $e^{\lambda-\rho}$ on $\exp \mathfrak{t}^-$, and λ and η satisfy the compatibility condition that $e^{\lambda-\rho}$ and η agree on their common domain. The discrete series character is denoted $\Theta^M(\lambda, C, \eta)$, where C is the unique Weyl chamber of $i\mathfrak{t}^-$ with respect to which λ is dominant. Two such characters $\Theta^M(\lambda, C, \eta)$ and $\Theta^M(\lambda', C', \eta')$ are equal if and only if $\eta = \eta'$ and there is some w in the Weyl group $W(T^-{:}M)$ with $w\lambda = \lambda'$ and $wC = C'$.

For representations that are limits of discrete series (see ref. 7)‡ on M, the characters are of the same form $\Theta^M(\lambda, C, \eta)$, except that λ is allowed to be singular and C is not uniquely determined by λ. These characters are irreducible or zero‡, and the criterion for equality of two such characters is the same as for discrete series.

If ν is a real-valued linear functional on $\mathfrak{a}$, we let

$$\Theta^{MA}(\lambda, C, \eta, \nu) = \Theta^M(\lambda, C, \eta) \otimes \exp i\nu$$

which is a character of MA. If we extend trivially on N and induce, we obtain characters

$$\mathrm{ind}_P{}^G\Theta^{MA}(\lambda, C, \eta, \nu) \qquad [1]$$

which we call *basic characters* if Θ^M is a limit of discrete series and we call *induced discrete series characters* if Θ^M is a discrete series character.

Fix an induced discrete series character [1]. Let W be the stability group of Θ^{MA}, namely,

$$W = \{w \in W(A{:}G) \mid (\Theta^{MA})^w = \Theta^{MA}\} \qquad [2]$$

The group W corresponds to standard self-intertwining operators of the induced representation (8). We shall recall shortly from ref. 8 a semidirect product decomposition $W = W(\Delta')R$, where Δ' is a root system whose Weyl group $W(\Delta')$ is normal in W. The group R has the following properties.

(*a*) The dimension of the commuting algebra for the induced representation is exactly the order $|R|$ of R. [In terms of the operators of ref. 8, $W(\Delta')$ corresponds to trivial operators, and R corresponds to independent operators. The operators for W span the commuting algebra by Harish-Chandra's completeness theorem (See ref. 9, Theorem 38.1).]

(*b*) $R = \Sigma \mathbf{Z}_2$. See ref. 5.

(*c*) There exists a set of positive orthogonal real roots $\mathcal{H} = \{\alpha_1,$

‡ G. Zuckerman, "Tensor products of infinite-dimensional and finite-dimensional representations of semisimple Lie groups, in preparation."

$\ldots, \alpha_q\}$ on t, i.e., roots vanishing on t^-, such that (*i*) the only roots in the span of $\mathcal{H}$ are the $\pm\alpha_j$; (*ii*) each r in R is of the form $p_{\alpha_{j1}}, \ldots, p_{\alpha_{jn}}$; and (*iii*) each α_j occurs in the decomposition of some r in R.

The system Δ' is defined as

$$\Delta' = \{\alpha \text{ useful root of } \mathfrak{a} | \mu_{\Theta^M,\alpha}(i\nu) = 0\} \quad [3]$$

where $\mu_{\Theta^M,\alpha}$ is the Plancherel factor described in [8]§. Here Δ' is a root system, $W(\Delta')$ is contained in W, and W leaves Δ' stable. Then

$$R = \{w \in W | w\alpha > 0 \text{ for } \alpha > 0 \text{ in } \Delta'\}$$

Associated to each of the real roots $\alpha_1, \ldots, \alpha_q$ is a Cayley transform that corresponds within the root's $SL(2,\mathbf{R})$ subgroup to conjugation by

$$2^{-1/2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

and carries the root to a new Cartan subgroup on which it is imaginary. Let *Cayley* denote the commuting product of the q individual Cayley transforms. With obvious notation, we can define the ingredients of new basic characters as follows:

$$t^{-*} = t^- \oplus i\mathbf{R}\, Cayley\{\alpha_1, \ldots, \alpha_q\}$$
$$\mathfrak{a}^* = \mathfrak{a} \ominus \mathbf{R}\{\alpha_1, \ldots, \alpha_q\}$$
$$t^* = t^{-*} \oplus \mathfrak{a}^*$$
T^*, M^*, A^*, P^* constructed as in (4)
$$\lambda^* = \lambda \text{ extended by } 0$$
$$\eta^* = \eta|_{Z_{M^*}}$$
$$\nu^* = \nu|_{\mathfrak{a}^*}$$
C^* chosen so that $\overline{C^*} \supseteq C$.

There are exactly 2^q choices for C^*. A preliminary form of the identity we seek is

$$\text{ind}_P{}^G\Theta^{MA}(\lambda, C, \eta, \nu) = \sum_{j=1}^{|R|} \text{ind}_{P^*}{}^G\Theta^{M^*A^*}(\lambda^*, C_j^*, \eta^*, \nu^*)$$

and the only problem is to describe the relevant chambers C_j^*.

Let $W_{\mathcal{H}}$ be the Weyl group generated by the reflections in the roots *Cayley* (α_j). The order of $W_{\mathcal{H}}$ is 2^q, and $W_{\mathcal{H}}$ permutes the 2^q choices for C^*. Also, by means of the Cayley transform, we can regard R as contained in $W_{\mathcal{H}}$. It is natural to conjecture that $C_j^* = r_j C_0^*$ with r_j in R, but this is not so, as one sees already in $SL(4,\mathbf{R})$. The actual parametrization is by $\hat{R}$, the dual group of R.

Namely, let

$$E_{\mathcal{H}} = \{w \in W_{\mathcal{H}} | w \in W(T^{-*}:M^*)\}$$

If w is in $E_{\mathcal{H}}$, then C^* and wC^* lead to the same character. One proves that

$$E_{\mathcal{H}} = \left\{ w \in W_{\mathcal{H}} \middle| \begin{array}{l}\text{for each } r \text{ in } R, w \text{ and } r \text{ have an} \\ \text{even number of factors } p_{\alpha_j} \text{ in common}\end{array} \right\} \quad [4]$$

We can recast [4] in terms of the dual $\hat{R}$. Let $\varphi: W_{\mathcal{H}} \to \hat{W}_{\mathcal{H}}$ be the isomorphism given by mapping the basis $p_{\alpha_1}, \ldots, p_{\alpha_q}$ to the dual basis. Then

$$\varphi(E_{\mathcal{H}}) = \text{annihilator of } R \text{ in } \hat{W}_{\mathcal{H}} \quad [5]$$

Consequently $W_{\mathcal{H}}/E_{\mathcal{H}}$ is canonically isomorphic with $\hat{R}$.

§ For the definition of "useful," see ref. 10.

THEOREM 1. *Let* $ind_P{}^G\Theta^{MA}(\lambda, C, \eta, \nu)$ *be an induced discrete series. Then*

$$\text{ind}_P{}^G\Theta^{MA}(\lambda, C, \eta, \nu) = \sum_{w \in W_{\mathcal{H}}/E_{\mathcal{H}} \cong \hat{R}} \text{ind}_{P^*}{}^G\Theta^{M^*A^*}(\lambda^*, wC^*, \eta^*, \nu^*)$$

for any choice of the chamber C^* *extending* C. *The characters on the right are nonzero and irreducible.*

COROLLARY. *Every irreducible tempered character is basic.*

2. Generalized Schmid identities

If we proceed on a more abstract level than in Section 1, we can obtain some information about all basic characters. We begin with one of Schmid's character identities (11). Let M^* be connected and reductive, having a compact Cartan subgroup T^{-*} with M^*A^* contained in a connected complexification. For a Weyl chamber C, suppose MA is constructed in the standard way by applying a Cayley transform to a noncompact C-simple root α with $\langle\lambda^*, \alpha\rangle = 0$. Let γ_α be the element corresponding to minus the identity in the $SL(2,\mathbf{R})$ subgroup for α. Schmid's identity is

$$\Theta^{M^*}(\lambda^*, C, \eta^*) + \Theta^{M^*}(\lambda^*, p_\alpha C, \eta^*) = \text{ind}_P{}^{M^*}\Theta^{MA}(\lambda^*|_{\alpha^\perp}, C^{p_\alpha}, \eta^* \otimes \zeta, 0) \quad [6]$$

where ζ is the character of $\{1, \gamma_\alpha\}$ given by

$$\zeta(\gamma_\alpha) = (-1)^{2\langle\rho_\alpha,\alpha\rangle/\langle\alpha,\alpha\rangle}$$

with ρ_α equal to half the sum of the roots whose restriction to $\mathfrak{a}$ is $c\alpha$ with $c > 0$.

Now let us drop the assumption that M^* is connected, retaining the other hypotheses. Then generalizing [6] splits into two cases, depending on whether the reflection p_α is in the Weyl group $W(T^{-*}:M^*)$.

THEOREM 2. (*i*) *If* p_α *is not in* $W(T^{-*}:M^*)$, *then* $Z_M = \{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$ *and*

$$\Theta^{M^*A^*}(\lambda^*, C, \eta^*, \nu^*) + \Theta^{M^*A^*}(\lambda^*, p_\alpha C, \eta^*, \nu^*) = \text{ind}_P{}^{M^*}\Theta^{MA}(\lambda^*|_{\alpha^\perp}, C^{p_\alpha}, \eta^* \otimes \zeta, \nu^* \oplus 0) \quad [7a]$$

with the characters on the left distinct unless both are zero.

(*ii*) *If* p_α *is in* $W(T^{-*}:M^*)$, *then* $|Z_M/\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}| = 2$. *If* $(\eta^*\otimes\zeta)^+$ *and* $(\eta^*\otimes\zeta)^-$ *denote the two extensions of* $\eta^* \otimes \zeta$ *from* $\{1, \gamma_\alpha\}Z_{M_0}Z_{M^*}$ *to* Z_M, *then*

$$\Theta^{M^*A^*}(\lambda^*, C, \eta^*, \nu^*) = \text{ind}_P{}^{M^*}\Theta^{MA}(\lambda^*|_{\alpha^\perp}, C^{p_\alpha}, (\eta^* \otimes \zeta)^+, \nu^* \oplus 0)$$
$$= \text{ind}_P{}^{M^*}\Theta^{MA}(\lambda^*|_{\alpha^\perp}, C^{p_\alpha}, (\eta^* \otimes \zeta)^-, \nu^* \oplus 0) \quad [7b]$$

COROLLARY. *Every basic character is contained in an induced discrete series character.*

We call [7a] and [7b] *generalized Schmid identities*. We now invert them.

THEOREM 3. *Let* MA *be given, let* α *be a real root, and construct* M^*A^* *from* MA *and* α. *Then* $ind_P{}^{M^*}\Theta^{MA}(\lambda, C, \eta, \nu)$ *is the right side of a generalized Schmid identity if and only if* $\langle\nu, \alpha\rangle = 0$ *and* $\eta(\gamma_\alpha) = (-1)^{2\langle\rho_\alpha,\alpha\rangle/\langle\alpha,\alpha\rangle}$.

Now by iterating *Theorem 3* and applying induction in stages, we can start with a basic character and successively decompose it into basic characters obtained from Cartan subgroups that are more compact. Call a basic character *final* if there are no real roots α meeting the conditions of *Theorem 3*. *Theorem 1* and an argument with tensor products‡ yield *Theorem 4*.

THEOREM 4. *Every final basic character is irreducible.*

3. Intertwining operators

Fix a basic character $\Theta = \mathrm{ind}_P{}^G\Theta^{MA}(\lambda, C, \eta, \nu)$. The Plancherel factor corresponding to Θ^M and an $\mathfrak{a}$-root α is given as follows. If α is even in the sense of ref. 10 and β denotes a root for the full Cartan subalgebra, let

$$\mu_{\Theta^M,\alpha}(i\nu) = \prod_{\beta|_{\mathfrak{a}}=\alpha} \langle \lambda + i\nu, \beta\rangle \qquad [\mathbf{8a}]$$

If α is odd and 2α is not an $\mathfrak{a}$-root, let

$$\mu_{\Theta^M,\alpha/2}(i\nu) = \mu_{\Theta^M,\alpha}(i\nu) = \Bigl(\prod_{\beta|_{\mathfrak{a}}=c\alpha, c>0} \langle \lambda + i\nu, \beta\rangle\Bigr) f_{\Theta^M,\alpha}(i\nu) \qquad [\mathbf{8b}]$$

where $f_{\Theta^M,\alpha}(i\nu) = \tanh(\pi\langle\nu,\alpha\rangle/|\alpha|^2)$ or $\coth(\pi\langle\nu,\alpha\rangle/|\alpha|^2)$ according as $\eta(\gamma_\alpha) = -(-1)^{2\langle\rho_\alpha,\alpha\rangle/|\alpha|^2}$ or $= +(-1)^{2\langle\rho_\alpha,\alpha\rangle/|\alpha|^2}$ (see ref. 9, Part III). By means of refs. 8 and 13, one can relate these Plancherel factors to the normalizing factors of the standard intertwining operators developed in ref. 8.

Also we define modified Plancherel factors $\mu'_{\Theta^M,\alpha}$ as follows: If α is even, $\mu' = \mu$. If α is odd and 2α is not an $\mathfrak{a}$-root, we take $\mu'_{\Theta^M,\alpha/2} = \mu'_{\Theta^M,\alpha}$ to be the same as $\mu_{\Theta^M,\alpha}$, except that the product in [**8b**] is extended only over those roots β with $|\beta| \leq |\alpha|$.

Now let W be the stability group of Θ^{MA}, as in [**2**], and construct the standard self-intertwining operators for Θ as in ref 8. These operators are parametrized by W. Let

$$\Delta' = \{\alpha \text{ useful root of } \mathfrak{a} | \mu'_{\Theta^M,\alpha}(i\nu) = 0\}$$

One shows that Δ' is a root system whose definition is consistent with ref. 8 if Θ is an induced discrete series character. Moreover, $W(\Delta')$ is contained in W, and W leaves Δ' stable. Let

$$R = \{w \in W | w\alpha > 0 \text{ for } \alpha > 0 \text{ in } \Delta'\}$$

The corollary to *Theorem 4* and the Hecht-Schmid character identity (ref. 4, p. 139) combine to yield *Theorem 5* below. Harish-Chandra's completeness theorem in the induced discrete series case and an argument with tensor products[‡] yield *Theorem 6*.

THEOREM 5. *The standard self-intertwining operators corresponding to members of* $W(\Delta')$ *are all scalar.*

THEOREM 6. *The standard self-intertwining operators corresponding to the stability group* W *span the commuting algebra of* Θ.

COROLLARY. *The dimension of the commuting algebra of* Θ *does not exceed* $|R|$.

4. Induced nondegenerate characters

Let $\Theta^{MA}(\lambda, C, \eta, \nu)$ be a limit of discrete series on M and a unitary character on A. We introduce a subclass of such characters that includes all cases in which Θ^M is a discrete series and that is given in terms of nonsingularity of λ with respect to certain roots of M. Specifically we say that Θ^{MA} is *nondegenerate* if, for each imaginary root α, $\langle\lambda, \alpha\rangle = 0$ implies that p_α is not in $W(T^-{:}M)$.

Equivalently Θ^{MA} is nondegenerate if and only if it is not equal to the left side of some generalized Schmid identity [**7**]. (In the case of [**7a**], this condition means that Θ^{MA} cannot be taken as one of the terms in such a way that the other term equals 0.)

For an induced nondegenerate character, a Plancherel factor vanishes only when the corresponding modified Plancherel factor vanishes. This fact allows us to connect Δ', which is defined in terms of μ', with the normalizing factors of the standard intertwining operators, which are given in terms of μ. The corollary to Theorem 6 above and the lemma in ref. 8 yield *Theorem 7*.

THEOREM 7. *For an induced nondegenerate character, the dimension of the commuting algebra equals the order* $|R|$ *of the* R*-group.*

It follows that $R = \Sigma\mathbf{Z}_2$ for any induced nondegenerate character. With little change the theory of Section 1 then applies in this wider context. Moreover, the irreducible constituents obtained from the theory are again nondegenerate.

If we use the corollary to *Theorem 3* to imbed an induced nondegenerate character in an induced discrete series character and then apply *Theorem 1* to the latter, we find that the complete decompositions of the two characters are compatible in an obvious sense.

Thus we obtain the following picture of the reducibility of a general basic character: by a succession of steps involving generalized Schmid identities, we can rewrite the basic character as an induced nondegenerate character corresponding to a more noncompact Cartan subgroup. At this stage matters become canonical and are specified by the R-group. The given induced nondegenerate character is completely reduced by a formula compatible with the reduction of the induced discrete series representation in which it imbeds.

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