Notes of a Course
Functional Analysis
Given by William Feller
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(Some lectures near the beginning were given by Christopher Anagnostakis.)

Personal Notes of A. W. Knapp
(Penciled comments in the margin for the first 15 pages were added probably by Benjamin Weiss.)
Functional analysis

Suggested reading:
A. E. Taylor, Functional analysis
Kelley, General topology
Functional analysis

Metric spaces

Definition:

A set $S$ of points $x$ and a function $p: S \times S \to \mathbb{R}$ is a metric space if

1) $p(x, y) = 0$ if and only if $x = y$
2) $p(x, y) = p(y, x)$
3) $p(x, z) \leq p(x, y) + p(y, z)$ \quad \text{triangle inequality}

Remark:

$p(x, y) > 0$ follows from $x = y$ in (3), the fact that $p(x, x) = 0$ and the use of symmetry.

Examples:

1) Let $f$ be any strictly increasing function $f: \mathbb{R} \to \mathbb{R}$ and define a metric on $\mathbb{R}$ by $p(x, y) = |f(x) - f(y)|$.
2) Discrete metric.
3) Hedgehog and the limiting case.

The straight line metric is used across the circle.

Remarks:

A subset of a metric space is a metric space under the induced metric. The sum of two metrics is a metric. Geodesics are paths where equality holds in the triangle inequality.

Definition:

Let $S$ and $T$ be metric spaces. Then $f: S \to T$ is continuous if for any $\epsilon > 0$ there exists a $\delta$ such that $p_T(f(x), f(a)) < \epsilon$ whenever $p_S(x, a) < \delta$.

Remark:

For a function of several variables in $S$, one $\delta$ is required for each variable for the definition to be used.
Definition:
An open ball (or simply a ball about \( p \)) is a set of the form \( \{ x \in S | \rho(x, p) < r \} \), where \( r > 0 \).

Theorem:
\( \rho(x, y) \) is a continuous function of two variables.
Proof:
Let \( \epsilon > 0 \) be given. If \( \rho(x, a) < \frac{\epsilon}{2} \) and \( \rho(y, b) < \frac{\epsilon}{2} \), then
\[
\rho(x, y) \leq \rho(x, a) + \rho(a, b) + \rho(b, y)
\]
or
\[
\rho(x, y) - \rho(a, b) < \epsilon.
\]
Similarly, \( \rho(y, b) - \rho(x, y) < \epsilon \).
Hence \( |\rho(x, y) - \rho(a, b)| < \epsilon \). Q.E.D.

Definition:
\( \{ x_n \} \) is a Cauchy sequence if for any \( \epsilon > 0 \) there is an \( N \) such that \( \rho(x_n, x_m) < \epsilon \) whenever \( n, m > N \). A metric space is complete if every Cauchy sequence has a limit.

The completion of a metric space is the set of all equivalence classes of Cauchy sequences with the natural metric and the canonical identification of \( \{ x_n \} \) with \( x \). The completion is complete.

Definition:
Let \( S \) be a metric space. Then \( A \subset S \) is dense in \( S \) if \( A = S \) or if every ball in \( S \) contains a point of \( A \).
\( A \) is dense in a subset \( T \) of \( S \) if \( \overline{A} \supset T \).
Remark: If \( A \) is dense in \( S \), then \( A \) is dense in every open subset of \( S \).

Definition: \( A \) is nowhere dense if \( A \) contains no open set. Otherwise, \( A \) is dense in \( S \).

Naive definition p. 59
Example:
Take the unit interval and remove an open interval of length $\frac{1}{n}$ from the middle. By induction remove at the $n$th step open intervals from each remaining interval in such a way that the points removed have length $\frac{1}{n^2}$. The limiting set is closed and has no interval of positive length; hence it is nowhere dense. Now, filling the blank spaces with replicas of the limiting set, we obtain inductively a set which is everywhere dense. It has measure 1 or 0 according as the limiting set had positive measure less than one or measure equal to zero. We shall prove that in any case not every point of the unit interval is in the resulting set.

Baire category theorem

Definition:
Let $S$ be a metric space. A subset $A$ of $S$ (possibly $S$ itself) is said to be of the first category in $S$ if $A$ is the denumerable union of nowhere dense sets of $S$. Otherwise, $A$ is said to be of the second category.

Note: Subsets of sets of first category are of first category, and supersets of sets of second category are of second category.

Baire theorem

Hence the complement of a set of first category is dense.

Theorem:
Every neighborhood of a complete metric space is of second category in the space.

Proof:
By the note it is sufficient to prove the result for
an open set \( G \) in \( S \). Suppose \( G \subseteq \bigcup \overline{A}_n \), where \( A_n \) is nowhere dense for each \( n \). \( \overline{A}_n \) contains no ball of \( S \) and hence no ball of \( G \). Find a ball \( S_1 \) in \( G - \overline{A}_n \) such that \( \delta(S_1) < \frac{1}{3} \). Since \( \overline{A}_n \) does not contain \( S_1 \), there exists an \( s, \overline{S}_1 \) not in \( \overline{A}_n \). Let \( S_2 \) be a ball about \( s \), such that \( \overline{S}_1 \subseteq S_2 \cap \overline{A}_n \) and \( \delta(S_2) < \frac{1}{3} \). Then \( S_2 \cap (\overline{A}_n \cup \overline{A}_m) = 0 \).

Proceeding inductively we obtain a decreasing sequence of sets \( S_1 \supseteq S_2 \supseteq \ldots \) and a Cauchy sequence of points \( \{s_m\} \). Since \( S_m \cap (\overline{A}_m \cup \overline{A}_m) = 0 \) and \( \delta(s_m) \cap (\overline{A}_m \cup \overline{A}_m) = 0 \).

But \( \overline{A}_m \supseteq G \) and \( G = \bigcap S_m \). Therefore \( \bigcap S_m = 0 \).

Let \( s \) be the limit of \( \{s_m\} \). If \( s \notin S_m \) for some \( m \), then \( s \notin \overline{S}_m \), and \( s \) is not a limit point of \( S_m \). Hence \( s \notin S_m \) for every \( m \), \( \bigcap S_m \neq 0 \), and the theorem is proved.

**Lemma:**

Let \( X \) be any topological space in which every neighborhood is of second category. Let \( A \subseteq X \).

If \( A \) is of second category in \( X \), then it is of second category in \( X \).

**Proof:**

Write \( A = U B_n \), and suppose that \( B_n \) is nowhere dense in \( X \), for every \( n \). Then if \( C_m = A \cap B_n \) \( A = U (C_m) \)

and \( C_m \) is nowhere dense in \( X \), for every \( n \). Since \( A \) is of second category in \( X \), for some \( m \), there is a neighborhood \( \text{in} X \) \( N \subseteq C_m \), where the closure is taken in \( X \). Now \( N \cap X = 0 \), since otherwise \( N \) is a (non-empty) neighborhood contained in the \( X \) closure of \( C_m \). Therefore, \( N \cap C_m = 0 \) because \( C_m \subseteq A \subset X \). But \( N \cap C_m \) so that
\[ C^N - C^n \]. In other words, every point of \( N \) is a limit point of \( C^n \) and no point of \( N \) is in \( C^n \). Therefore no point of \( N \) is an interior point (in \( X \)), and \( N \) cannot be a neighborhood.

**Corollary:**
If \( A \) is of first category in \( X \), then it also is of first category in \( X \).

**Corollary:**
If \( A \) is of second category in \( X \), then it is of second category in itself.

**Remark:**
We may then say that \( A \) is of second category without ambiguity.

**Examples:**
1. Let \( g(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{k}{n}, \text{ rational} \end{cases} \)

   Then \( g(x) \) is discontinuous at rational points and continuous at irrational points. It is the everywhere limit of continuous functions (constant spikes).

2. Let \( h(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases} \)

   Then
   \[
   h(x) = \lim_{n \to \infty} \left[ \cos \left( \pi n x \right) \right]^{1/n}
   \]

   We shall show shortly that \( h \) is not obtainable as a single limit of continuous functions since it is discontinuous everywhere.

**Remark:**
The denumerable union of sets of first category is again of first category. Any set which is the category (in a space in which every neighborhood is of second category) is a dense set.
Theorem:

Let \( X \) be any topological space in which every neighborhood of the space is of second category. Suppose \( f_n \) is a sequence of continuous functions with domain \( X \) and with the property that \( f_n \) converges to a function \( f \) pointwise. Then the set of discontinuities of \( f \) is of first category.

Remarks:

The oscillation of a function in a neighborhood is the difference of the supremum and the infimum. The oscillation of \( f \) at \( x \) is said to be greater than or equal to \( \varepsilon \) if \( x \) is greater than or equal to \( x \) in every neighborhood of \( x \). A function is discontinuous at a point if and only if it has positive oscillation at the point.

Proof of theorem:

Let \( \varepsilon > 0 \) be any positive real number and let \( N = X \) be any neighborhood of \( X \). As a function of \( \varepsilon \) and \( N \), let \( A_{n\varepsilon} = \{ x \mid x \in N, |f_m(x) - f(x)| \leq \varepsilon/\varepsilon \} \).

Since \( f_m - f_n \) is continuous, \( A_{n\varepsilon} \) is closed in \( N \). Put

\[ A_{\varepsilon} = \bigcap_{n=1}^{\infty} A_{n\varepsilon} \]

Then \( A_{\varepsilon} = \{ x \mid x \in N, |f_m(x) - f(x)| \leq \varepsilon/\varepsilon \} \) and \( A_{\varepsilon} \) is closed, being the intersection of closed sets. By the definition of a limit, \( \bigcup A_{\varepsilon} = N \). But \( N \) is of second category, so that not all of the \( A_{\varepsilon} \) are nowhere dense. Thus there exists an \( \varepsilon \) such that \( A_{\varepsilon} = N \), where \( N \) is a subneighborhood of \( N \). Since \( A_{\varepsilon} \) is closed in \( N \), \( A_{\varepsilon} = N \). Now for all \( x \in N \), we have

\[ |f_m(x) - f(x)| \leq \varepsilon/\varepsilon \]

Fix \( x = x_0 \). By continuity of \( f_{x_0} \) there exists a
subneighborhood $N_2$ of $N_1$, containing $x_0$ such that for all $y$ in $N_2$
$$|f_n(x_0) - f_n(y)| \leq \epsilon/6. $$
By the triangle inequality we have
$$|f(x_0) - f(y)| \leq \epsilon/2$$
for all $y$ in $N_2$. Again by the triangle inequality we find that for all $x, y \in N_2$
$$|f(x) - f(y)| \leq \epsilon. $$
Hence for any neighborhood $N$ in the space there is a subneighborhood $N_2$ in which the oscillation of $f$ is no greater than $\epsilon$. Alternatively the set of points in $X$ for which the oscillation of $f$ exceeds $\epsilon$ is nowhere dense. Since the set $P$ of points of discontinuity of $f$ satisfies
$$P = \bigcup_{n=1}^{\infty} P_n,$$
$P$ is of first category.

**Lemma:**

Let $X$ be a topological space in which every neighborhood is of second category, and let $E$ be a set of first category in $X$. Then every neighborhood of $\tilde{E}$ (with the induced topology) is of second category in $\tilde{E}$.

**Proof:**

Let $N'$ be a neighborhood in $\tilde{E}$. Then $N' = N - E$, where $N$ is a neighborhood in $X$. If $N'$ is of first category in $\tilde{E}$, then it is of first category in $X$ and $(N-E) \cup E$, being the union of two sets of first category in $X$, is of first category in $X$. Thus $N$ is of first category in $X$, which is impossible.  \[Q.E.D.\]
Definition:
If $A \subseteq X$ and $f$ is defined on $X$, $f/A$ is the restriction of $f$ to $A$.

Definition:
A function $f$ has the Baire property if there exists a set $E$ of first category such that $f/E$ is continuous on $E$.

Remark:
The limit of continuous functions has the Baire property. So does the characteristic function of the rationals. Let $B$ be the collection of all functions $f$ with the Baire property.

Theorem:
Let $X$ be a topological space in which every neighborhood is of second category. Suppose $f_n \in B$ and $f_n \to f$ pointwise. Then $f \in B$.

Proof:
If $f_n \in B$, then there exists a set $E_n$ of first category such that $f_n/E_n$ is continuous. Put $E = \bigcup E_n$. Then $E$ is of first category. Since $E = E_n$, $f_n/E$ is continuous for every $n$. By the lemma, every neighborhood of $E$ is of second category in $E$. Thus by the preceding theorem, the discontinuities of $f/E$ form a set of first category in $E$ (and hence in $X$). Call the set of discontinuities of $f/E$ $E_0$. Then $f/(E - E_0)$ is continuous, or equivalently $f/(E \cup E_0)$ is continuous. Since $E \cup E_0$ is the union of two sets of first category, it is of first category and $f$ is in $B$. Q.E.D.
Corollary:

B is closed under sums, differences, products, multiplication by scalars, and limits.

Definition:

Let \( A \Delta B \) denote \((A-B) \cup (B-A)\), the symmetric difference of A and B. A point is in \( A \Delta B \) if and only if it belongs to one of the component sets.

Note:

\[ A \Delta B = C \implies A \Delta C = B. \]

For

\[ A \Delta C = A \Delta A \Delta B = 0 \Delta B = B. \]

Definition:

A set \( A \) in a space \( X \) has the Baire property if there exists an open set \( \Omega \) such that \( A \Delta \Omega = B \) is of first category.

Remarks:

1. Equivalently, \( A \) has the Baire property if there is a set \( B \) of first category such that \( A \Delta B \) is open.
2. If \( A \) is open, take \( \Omega = A \). Then \( A \) has the Baire property.
3. If \( A \) is closed, then \( A \) minus its interior is nowhere dense and is thus of first category. Take \( \Omega \) to be the interior of \( A \). We thus see that every closed set has the Baire property.

Theorem:

Let \( B \) be the class of all Baire sets in a topological space \( X \) in which every neighborhood is of second category. If \( A \in B \), then \( \overline{A} \in B \). If \( A \subseteq B \), then \( \overline{A} \in B \). Thus \( B \) is a Boolean field of sets.
Proof:

Complements:

Let \( A \in B \) and let \( \Omega \) be an open set for which \( A \Delta \Omega \) is of first category. Let \( \Pi = (\overline{\Omega}) \). We shall show that \( \Pi - \tilde{A} \) and \( \tilde{A} - \Pi \) are of first category. First

\[
\Pi - \tilde{A} = \Pi \cap \tilde{A} < \overline{\Omega} \cap \tilde{A} = \tilde{A} - \Omega < A \Delta \Omega
\]

and \( A \Delta \Omega \) is of first category. Second

\[
\tilde{A} - \Pi = \tilde{A} \cap \overline{\Omega} = [\tilde{A} \cap (\overline{\Omega} - \Omega)] \cup [\tilde{A} \cap \Omega] = (\overline{\Omega} - \Omega) \cup (\Omega - \tilde{A}) = (\overline{\Omega} - \Omega) \cup (A \Delta \Omega)
\]

and each of these sets is of first category.

Unions:

Let \( A_m \in B \) and let \( \Omega_m \) be open such that \( A_m \Delta \Omega_m \) is of first category. Let \( \Pi_m = (\overline{\Omega}) \) and put \( \Pi = U \Pi_m \). Then

\[
\Pi - A < U (\Pi_m - A_m)
\]

and

\[
A - \Pi < U (A_m - \Pi_m)
\]

since \( \Pi_m - A_m \) and \( A_m - \Pi_m \) are both of first category, the result follows.

Q.E.D.

Remark:

It can be shown that \( f \in B \) if and only if \( \{x | f(x) > a\} \in B \) for every \( a \).
Comments on point set topology

Definition:
A topological space is a set $X$ with a class of open sets satisfying:
1) $\emptyset$ and $X$ are open.
2) $A \cap B$ is open whenever $A$ and $B$ are.
3) $\cup A_n$ is open whenever $A_n$ is open.

Definition:
A space is Hausdorff if any two distinct points can be separated by open sets.

Definition:
A family of open sets $B$ forms a base at $p$ if $p \in B$ for every $B \in B$ and if whenever $p \in U$ and $U$ is open, there is a $B \in B$ such that $B \subset U$. A base for a topological space is a family of open sets such that there is a subfamily depending on $p$ which is a base at $p$.

Remark:
A class of sets satisfying $\cup B = X$ is a base if and only if for any pair $B_1, B_2 \in B$ and for any $x \in B_1 \cap B_2$, there is a $B_0 \in B$ such that $x \in B_0 \subset B_1 \cap B_2$.

Definition:
A space is compact if every covering by open sets has a finite subcovering.

Proposition:
In a Hausdorff space, any pair of compact sets can be separated by open sets.

Proof:
For each fixed $p$ in $A$, separate $p$ from each point in $B$, refine the covering of $B$, take the union of those sets as a set covering $B$ and the intersection of the corresponding sets.
as a set covering \( p \). Doing so for every \( p \) gives a covering of \( A \) which can be refined. Form the union of these sets and the intersection of the corresponding sets covering \( B \). Then \( A \) and \( B \) are separated.

Equivalent definition of continuity of \( f: X \rightarrow Y \):

1) For any \( x \in X \) and for any open set \( V \) with \( f(x) \in V \) there is a \( U \subseteq X \) such that \( x \in U \) and \( f(U) \subseteq V \).
2) The inverse image of every open set is open.
3) The inverse image of every closed set is closed.

Remarks:
The continuous image of a compact set is compact. In a Hausdorff space a compact set is closed.

Definitions:

1) A space is sequentially compact if every sequence has a cluster point.
2) A space is sequentially separable if it has a countable dense set.
3) A space is separable if it has a countable base.
4) A space is locally compact if every point has a compact neighborhood.
5) A space in which every open covering has a countable subcovering is said to have the Lindelöf property.

Examples:

1) The hedgehog topology has a countable base at each point but not a countable base.
2) The topology on the real line which has intervals \([a, b]\) as a base is sequentially separable but not separable.
A space with the Lindelöf property is compact if and only if every denumerable set has a cluster point.

Proof:
Suppose $X$ is not compact. There exists an open covering $\{U_i\}$ which has no finite subcovering. We may assume $\phi \not\in \{U_i\}$ and that the sequence of sets is irredundant. Choose $x_i \in U_i$, and by induction $x_n \in U_n - (U_1 U_2 \ldots U_{n-1})$. Then $\{x_n\}$ has no cluster point. A similar argument establishes the converse.

Remark:
A space with a countable base is Lindelöf.

Theorem:
In a metric space, compact and sequentially compact are equivalent.

Proof:
We prove that a sequentially compact metric space has a countable base. By induction cover the space with as many spheres of radius $\frac{1}{3}$ as possible. Finitely many suffice by sequential separability. Expand the spheres to have radius 1; they then cover the space. Repeat the argument for radius $\frac{1}{3^m}$ for every $m$. This procedure gives a countable base.

Remarks:
Every locally compact space has a one-point compactification in which the neighborhoods of $\infty$ are the complements of the finite compact sets. Spaces can be compactified in other ways; for example, the real line has a natural two-point compactification.
Definition:
The Tychonoff topology for a Cartesian product of spaces is the topology with base consisting of all finite intersections of sets which are restricted only in one coordinate (and thus the restriction is to an open set).

Standard probability space:
Let $\Omega$ be the Cartesian product of the real line with itself under an indexing by the set $\{t \mid t > 0\}$. Points in the space are functions $w(t)$. The result is a replica of the real line for every $t > 0$. If intervals form a base for the topology of the real line, a base for the product topology consists of sets like

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where the functions $w(t)$ are restricted to pass through the finite number of slits but are otherwise unrestricted.
Banach space

Linear space (over reals or complex numbers), strictly a linear set
- Group under addition
- Scalar multi. \(Ax\)

Rules:
\[ A(x+y) = Ax + Ay \]
\[ Ax = xA \]
\[ (a+b)x = ax + bx \]
\[ 1x = x \]

Normed linear space

Toplogy to be formed will be preserved under \(x \rightarrow x + a\). Metric, when present, is to be preserved.

Definition:
\[ p(x+a, y+a) = p(x, y) \iff \|x\| = p(x, 0) \]

Normed linear space is a pre-Banach space

Banach space: Complete normed linear space

Example:
Continuous functions on a compact space
\[ \|f\| = \sup |f(x)| \]
\[ \|f_n - f_m\| \rightarrow 0 \Rightarrow \text{uniform convergence. Hence Banach space} \]

Example:
Sequences \(x = x_1, x_2, \ldots\) with \(y\)
\[ \|x\| = \sup |x_n| \]
Linear functionals - linear mapping of space into scalars (now the real)

Notation: $x^* x$

$$x^*(ax + by) = ax^* x + by^* y$$

$x^* + y^*$ and $cx^*$ defined in standard way

Norm on space:

$$\| x^* \| = \sup_{\|x\|=1} \frac{|x^* x|}{\|x\|}$$

Said pre-Banach space

Take $x_{x}^*$:

For each $x$, $x^*_x$ is linear and $\Rightarrow f(x)$

Check that limit is a functional

Call limit $x^*_x$

Prove $\| x^*_x \| = \lim \| x^*_x \|$ and space is Banach space

This space is called the dual space

Example: Finite-dimensional

If $\|x\| = \sup |x^*_x|$ then $\|x^*_x\| = \sum |x^*_x|$. Spaces are not isometric, different metrics

Dual space is space of measures, set functions

Typical functional on $\ell^p$:

$$\int f(x) dx^*$$

now is one

General form will be $\int f(x) dx^*$, fixed measure $\{\mu\}$ carries weight one, zero measure elsewhere.
Compact not separable: Cartesian product of reals with itself, function space
Take $A$ to be the set of functions which are 1 on all but a finite number of points, where they are zero. $F = 0$ is ace, pt., but no sequence converges to it.
Some idea is that points are not 0 at sets.

Adjoint Banach space

\[ \|x^*\| = \frac{\sup |x^* x|}{\|x\|} \]

Bounded linear functional has bounded norm

Existence of $x^*$ to be shown. This follows directly from

Hahn-Banach Theorem (compact)

If $L \subseteq X$ is a closed linear space, and if $l^*$ a bounded linear functional on $L$, \exists $x^*$ such that $\|x^*\| = \|l^*\|$ and

\[ x^* l = l^* l \text{ for } l \in L. \]

Corollary: with $\|x\| = 1$

For each $x \in X$, there is an $x^*$ such that $x^* x = 1$, $\|x^*\| = 1$.

Corollary: with $x \notin L$, $x^* L$

If $L \subseteq X$ is closed, there is a $y^* \in X^*$ such that $y^* L = 0$ and

\[ y^* x \neq 0. \]

Proof: Let $L_1 = \{x | z = l + \alpha x, \ l \in L, \ \alpha \in \mathbb{R}\}$. Define

\[ l^* x = \alpha. \] Here $L^*$ has the required properties. Extend it.

Codimension 1 means: $L_1 = X$ and functional is unique up to mult.

by construction.

\[ x^* x = 0 \] is of codimension 1. If $y^* a = \alpha$ and $y^* b = \beta$, then

\[ y^* (\beta a - \alpha b) = 0. \] So $\beta a - \alpha b \in L$.

Value of functional given linear manifold of codimension 1 and everything
in space is a translate of this manifold. Intersection of
these things is as in finite spaces.
Note: Closed linear manifold $A$ is dense if $y^* A = 0$ implies $y^* = 0$.

Remark: In Euclidean space, a normal vector to a hyperplane is really an element of the dual space. In an arbitrary Banach space no inner product exists, so functional interpretation is only possible one.

For any $L$ the set $\{ x^* | x^* L = 0 \}$ is the set of all "normals".

This set is called $L^\perp$.

When $X$ and $X^*$ are identifiable, a point in $X^*$ is normal. This is situation in Hilbert space.

Example:

$x = (x_1, \ldots, x_n)$

$x^* = (x_1^*, \ldots, x_n^*)$

$\|x\| = \sqrt{\sum |x_j|^2}$

$\|x^*\| = \sum |x_j^*|$ $x$ and $x^*$ not identifiable

More generally can take

$\sum |x_j^*|^\beta = \|x\|^\beta, \quad \beta > 1$

$\sum |x_j^*|^\beta = \|x^*\|^\beta, \quad \beta + \frac{1}{\beta} = 1$

$p = 2$ is Euclidean space

Case above is limit as $p \to \infty$

In above example can take infinite sequence, finitely many non-zero, complete space. In $\ell^p$ space get sublinear sequences. But in other cases get null sequences just by looking at metric. If space is extended, null sequences from closed linear manifold.

If $\sum |x_j^*| < \infty$, define $x^* x = \sum x_j^* x_j$ and $|x^* x| \leq \|x\| \sum |x_j^*|$ and $\|x^*\| \leq \sum |x_j^*|$. To get equality take point with coordinates proportional to $x_j^*$.
But these functionals are all defined on closed linear manifold.

Here \( F \) functional vanishing on \( L \) and \( \neq 0 \) somewhere else.

Another example is all convergent sequences.

We have \( B_0 \), null seq.
\[ C \text{ convergent seq.} \]
\[ B \text{ whole space} \]
\[ B_0 \subset C \subset B \]

Adjoint to \( B_0 \) is as above because its value is determined on basis
\[ (1, 0, \ldots) \]
\[ (0, 1, \ldots) \]
\[ B_0^* = \sum \| x_j \| = (x^*) \]

\( B_0 \) is space of codimension one in \( C \) obviously.

Adjoint to \( C \) is \( B_0^* + \{ \text{finite} \} \)
\[ \xrightarrow{\text{finite}} \]
\[ \xrightarrow{\text{measure}} \]
\[ \xrightarrow{\text{limit}} \]

Adjoint to \( B \) exists by Hahn-Banach theorem. No functional on whole space is known.

Adjoint to adjoint space \( X^{**} \)
\[ x^* \in X^* \] of \( x \) is fixed and \( x^* \) runs through \( X^* \), get
functional on \( X^* \)
\[ f(x, x^*) = x^* \cdot x \text{ linearity is obvious} \]
Every \( x \) can be interpreted as an \( x^{**} \) algebraically.

By norm
\[ |f(x, x^*)| \leq \| x^* \| \| x \| \]

Hence norm is preserved. \( x \subset X^{**} \)
In the space above \( X = X^{**} \), reflexive space. But with \( \ell_0^\infty \), \( B = B^{**} \). Show this. Sequence

\[ X \subset X^* \subset X^{**} \subset \ldots \]

is strictly increasing.
Nonsense linear space

\[ T \text{ additive: } X \to Y \text{ is continuous iff } T \text{ is continuous at } 0. \]

Proof:

\[ \Rightarrow \text{ trivial} \]

\[ \exists \varepsilon > 0 \text{ means } \exists \delta \text{ s.t. } \forall \|x\| < \delta \implies \|Tx\| < \varepsilon. \]

Let \( x_0 \in X \). If \( \|x_0 - x\| < \delta \), then \( \|T(x_0 - x)\| < \varepsilon \)

\[ \|T(x_0) - T_x\| < \varepsilon \]

so continuity at \( x_0 \).

This even gives uniform continuity.

**Lemma:**

\[ T(ax) = aTx \]

If \( T(x_0) = x_0 \), then \( T \) is continuous iff \( T \) is bounded.

(\( \exists A \text{ for which } \|Tx\| \leq A\|x\| \) \( A > 0 \).)

Proof:

\[ \text{If } T \text{ is bounded, take } \|x\| < \frac{\varepsilon}{A}; \text{ then } \|Tx\| < \varepsilon. \]

\( \text{or } A = 0 \)

Suppose \( T \) continuous at \( 0 \). If \( T \) not bounded, can find \( x_n \in X \) such

\[ \|Tx_n\| \geq m\|x_n\|. \]

Then

\[ \| T \frac{x_n}{\|x_n\|} \| = \| \frac{1}{m\|x_n\|} Tx_n \| \]

\[ = \frac{1}{m\|x_n\|} \|Tx_n\| \]

\[ > 1. \]

But \( \| \frac{x_n}{\|x_n\|} \| = \frac{1}{m} \). So \( T \) is not continuous at \( 0 \). \( \text{Thus: } T \text{ linear: } X \to Y \text{, then } T \text{ is continuous iff } T \text{ is bounded.} \)

**Def:** \( \|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \neq 0} \|Tx\| \) if \( \|T\| < \infty \)
Let $X$ be a linear space. Suppose two norms $\| \cdot \|_1$, $\| \cdot \|_2$ induce the same topology. This means identity map

$$i : X \to X \quad \| \cdot \|_1, \quad \| \cdot \|_2$$

is continuous.

Continuity and linearity means

$$\| x \|_2 \leq A \| x \|_1$$

Sufficiently

$$\| x \|_1 \leq B \| x \|_2$$

The result requires linear structure.

Dual spaces:

Dual in all bounded transformation of $X \to C$

$$X^{**} = X.$$ Linear map is one-one since $x^*(x) = 0 \forall x^* \Rightarrow x = 0 \text{ by H.B.T.}$

$$(\Phi x)(x^*) = x^*(x)$$

Determinate:

$$\| \Phi x \| = \| x \|$$

$$\| \Phi x^* \| = \| x^* \| = 1$$

Now $\exists \gamma^* \in X^*$ for which

$$\gamma^*(x) = \| x \| \text{ and } \| \gamma^* \| = 1 \text{ by H.B.T.}$$

$$\| \Phi x \| = \| \gamma^*(x) \| = \| x \|$$

Hence $\Phi$ is an isometry.

$\Phi X$ is a closed subspace of $X^{**}$

Suppose $\Phi x_0 \neq \Phi x, x_0 \in X^{+*}$

$$\| \Phi x_0 - \Phi x \| \to 0$$

Hence and linear $\| x_0 - x \|$

So $x_0$ is Cauchy and $x_0 \to y_0 \in X^*$

$\Phi$ is continuous to $\Phi x_0 \to \Phi y_0 \in \Phi X$. So $x_0 = \Phi y_0 \in \Phi X.$
Higher dual space

\[ X \subset X^{**} \subset \cdots \subset \overline{X}^{2m} \subset \overline{X}^{2m+2} \subset \cdots \]

Then:

\( \text{either } X = X^{**} \text{ and hence } X = \overline{X}^{2m} \quad \text{or } X \subsetneq X^{**} \text{ and then } X^{2m} \not\subseteq \overline{X}^{2m+1} \subset \overline{X}^{2m+3} \)

Proof:

First half is easy. Suppose \( X \neq X^{**} \). Claim \( X^{*} \neq X^{***} \).

Suppose \( \Phi \) is embedding of \( X \to X^{**} \); \( \Phi \) is embedding of \( X \to X^{***} \).

Consider on \( X^{**} \) such that \( a \Phi x = 0 \), \( a \in X^{***} \).

Show \( a \Phi x \). Suppose:

\[ \Phi_{i} b = a \quad \text{where } b \in X^{*} \]

\[ \Phi_{i} b = a \]

For \( x \in X \), \( a(\Phi x) = 0 \). Then

\[ 0 = a(\Phi x) = (\Phi_{i} b)(\Phi x) = (\Phi x)(b) = b(x) \]

So, \( b = 0 \).

Q.E.D.

Theorem of uniform boundedness:

\( T \) a family of continuous linear transformations \( X \to Y \), \( \alpha \in A \).

Suppose \( \| T_{\alpha} x \| \leq N_{x} < \infty \) for \( x \in X \). Then means

\[ \| T_{\alpha} x \| \leq \infty \text{ at any } x. \]

Then \( \| T_{\alpha} \| \leq N < \infty \) for all \( \alpha \in A \).
Proof:

Look at $\{ \| T_\alpha x \| \leq m \| x \| \}$, this is closed for fixed $x, m$.

Form intersection over $x \in A$
Form union over $m$.

Given $x \neq 0$, choose $m$ such that $m \| x \| > N_x$. Then $x \in B$ is an intersection. And $D \in D$

So $\bigcup_{x \in A} \{ \| T_\alpha x \| \leq m \| x \| \} = X$.

By Baire category then,

$\exists x \| x_0 - x \| \leq r_0 \}$

$s_0 \neq 0$

$\subseteq \bigcap_{x \in A} \{ \| T_\alpha x \| \leq m_0 \| x \| \}$.

This means $\| x_0 - x \| \leq r_0$ implies

$\| T_\alpha x \| \leq m_0 (\| x_0 \| + r_0)$ for all $x$

On some open $B$, transformation are uniformly bounded.

Now by $x, y \neq 0$, then

$T_\alpha y = \frac{\| y \|}{s_0} T_\alpha \frac{\| y \|}{s_0} \frac{\| y \|}{s_0} \quad \text{by linearity}$

$= \frac{\| y \|}{s_0} T_\alpha \left( x_0 + \frac{x_0}{\| y \|} s_0 \right) - \frac{\| y \|}{s_0} T_\alpha x_0$

satisfies

$\| x_0 - x \| \leq r_0$

$\| T_\alpha y \| \leq 2 \frac{\| y \|}{s_0} m_0 (\| x_0 \| + r_0)$

for $x_0$ close

$\| T_\alpha y \| \leq \| y \| \quad \text{const. for all } y \in X \text{ and } x$

and $\| T_\alpha x \| \text{ const}$
Filters

DEF 1: If $X$ is any set and $F \subseteq P(X)$, then $F$ is a filter if

1) $\emptyset \notin F$, $F$ is not empty.
2) If $A \in F$ and $A \supseteq B$, then $B \in F$.
3) If $A_i \in F$ for finitely many, then $\cap A_i \in F$.

Examples:
1) Neighborhoods $U_x$, where $x \in X$, a topological space.
2) All sets containing a fixed point.
3) $X$ itself.

DEF 2: If $F_1$ and $F_2$ are filters on $X$, then $F_1$ is finer than $F_2$ if

$F_1 \supseteq F_2$.

Remark: 1) The filters on $X$ are partially ordered.
2) If $F_a$ are filters, then $\bigcap a \in A$ is a filter and this $\bigcap a \in A$

if $F_a$ under partial ordering.

Lemma 4: $\mathcal{B} \subseteq P(X)$ is contained in a filter if and only if $\mathcal{B}$ has the
finite intersection property.

Proof: Necessity is trivial. If $\mathcal{B}$ has f.i.p., let $\mathcal{B}$ be all sets $E$ such
that $E$ contains a finite intersection of sets of $\mathcal{B}$; then $\mathcal{B}$ is obviously a filter.

Corollary 5: If $F$ is a filter on $X$ and $A \subseteq X$, then there exists a filter $F'$
containing $F \cup \{ \bigcap A \}$ if and only if $E \in F \Rightarrow \bigcap A \in \mathcal{B} = \emptyset$.

Corollary 6: If $F_a$ is a filter for $a \in A$, then there is a filter $F'$
containing all $F_a$, $a \in A$, if $E \in F_a \Rightarrow \cap E \notin \emptyset$.

Theorem 1: The class of filters on a set $X$ is inductive.

Proof: Let $F_a$, $a \in A$, be totally ordered. Apply Corollary 6.

Definition 2: If $F$ is a filter and $B \subseteq \mathcal{B}$, then $B$ is a basis of $F$ if

$F = \{ E \mid E = B \text{ for some } B \in \mathcal{B} \}$. 

Lemma 9: $B \subset \mathcal{P}X$ is a basis for some filter if and only if

1) $\emptyset \notin B$, $B \neq \emptyset$.
2) If $B_1, B_2 \in B$, then $\exists B_3 \in B : B_1 \cap B_2 = B_3$.

Proof: obvious

Notation: The filter generated by $B$ will be called $\mathcal{F}_B$.

Example 10:
1) Open neighborhoods of a point are a basis of $\mathcal{N}_X$.
2) Let $N = \mathbb{Z}, \mathbb{Z}, ..., \mathbb{Z}$
   The sets $\{m, m+1, ..., j\}$ form a filter, the complements of finite sets,
   called the Fréchet filter.
3) It is a basis of itself.

Lemma 11: If $\mathcal{F}_i$ is a filter with basis $B_i$, $i = 1, 2$, then $\mathcal{F}_1 \supseteq \mathcal{F}_2$
   if and only if every set $B_1 \in B_2$ then exists a $B' \in B_1$ such that $B' \subseteq B_2$.

Proof: Lemma

Definition 12: An ultrafilter on $X$ is a maximal filter.

Example: The class of sets containing some point $x \in X$.

Theorem 13: Every filter $\mathcal{F}$ is contained in an ultrafilter.

Proof: Let $C$ be the class of filters containing $\mathcal{F}$; this class is
   inductively ordered. By Zorn's lemma, there exists a maximal
   element: this maximal element is a maximal filter.

Lemma 14: If $\mathcal{F}$ is an ultrafilter and $A \cup B \in \mathcal{F}$, then $A \in \mathcal{F}$ or $B \in \mathcal{F}$.

Proof: Suppose $A, B \notin \mathcal{F}$. Let $S = \{E \mid \forall E \neq \mathcal{F} \}$. Then $S$ is a filter,
   $B \in S$, $A \in S$, $B \cap S$. Contradiction.
Lemma 15: If \( B \) is a basis for a filter \( \mathcal{F} \) over \( X \) and if \( A \in X \) implies \( A \notin B \) or \( \bar{A} \notin B \), then \( \mathcal{F} \) is an ultrafilter.

Proof:
Suppose \( \mathcal{F}' \) is a filter. If \( A \in \mathcal{F}' \), then \( \bar{A} \notin \mathcal{F}' \). Hence \( A \notin B \). So \( \mathcal{F}' = B \). Thus, \( \mathcal{F} \) is maximal.

Note: Image of filters need not be a filter.

Lemma 16: Let \( B \) be a filter basis on \( X \) and let \( f : X \rightarrow Y \). Then \( f(B) \) is a filter basis.

Proof:
1) \( \phi \notin f(B) \neq \phi \) obviously.
2) \( f(A) \), \( f(A_1) \) \& \( f(A_2) \)
   
   \( \forall A \in A_1 \cap A_2 \), \( f(A) \in f(A_1) \cap f(A_2) \).

Lemma 17: If \( B \) is a basis of an ultrafilter \( \mathcal{F} \) over \( X \) and if \( f : X \rightarrow Y \), then \( f(B) \) is a basis of an ultrafilter.

Proof: Let \( \mathcal{F}' \) be generated by \( f(B) \). If \( A \in \mathcal{F}' \), then \( f^{-1}(A) = \mathcal{F} \). So \( f^{-1}(A) = B_2 \in \mathcal{F} \). Hence \( A \in f(B) \). Then \( f(A) \in \mathcal{F}' \) and by 15, \( \mathcal{F}' \) is an ultrafilter.

Definition 19: Let \( X \) be a topological space. If \( \mathcal{F} \) is a filter on \( X \) and \( x \in X \), then \( \mathcal{F} \) converges to \( x \) if \( \mathcal{F} \supset U_x \).

Lemma 20: If \( B \) is a basis for \( \mathcal{F} \), then \( B \) converges to \( x \) if and only if \( U_x \cap U_x \neq \phi \), where \( x \in x \).

Proof: By 11.

Theorem 21: \( X \) is Hausdorff if and only if every filter converges to at most one point.
Proof: Let $X$ be Hausdorff. If $x \neq y$ and $x, y \in F_a$, then $U \cap U_y \subseteq V \neq \emptyset$. If $x \in U_y$, then $\emptyset \neq F$. Conversely, suppose every filter has at most one point to which it converges. If there is a non-Hausdorff pair, suppose $U \subseteq U_x$, $V \subseteq U_y$ and $U \cap V = \emptyset$. Then $U_x \cup U_y$ has infinite intersection property.

By 6, $F \cap (U_x \cup U_y)$ a contradiction.

**Definition 22:** Let $B$ be a filter a basis on $X$. If $x \in B$, then $x$ is an accumulation point of $B$ if and only if $x \in B \cap B$, $B \in B$.

**Lemma 23:** If $B$ is a basis for $F$, then $x$ is an accumulation point of $B$ if and only if $x$ is an accumulation point of $F$.

**Proof:** Suppose $x$ is an accumulation point of $B$. Then $x \in B$ for all $B \in B$. Hence $B \cap B = \emptyset$ for $x \in B$. Conversely, if $x$ is an accumulation point of $F$, then $\emptyset \neq F$ and $x$ is an accumulation point of $F$.

**Lemma 24:** $x$ is an accumulation point of a filter $F$ if and only if there is an $y \in F$ such that $F_y \subseteq B$.

**Proof:** If $y \in F$, suppose $F \subseteq F$. Now $F \subseteq B \cup B$. Hence $F \subseteq B$ for $x \in B$. Conversely, $F \subseteq B$ for $x \in B$ and $y \in F$. Apply 4 to $x \cup B$. Then $x \in B$ and $x \in B$.

**Corollary 25:** If $x$ is an accumulation point of an ultrafilter $F$, then $F$ converges to $x$.

**Definition 26:** Let $B$ be a filter on $X$ and suppose $x \in Y$. Then $L_x f = x$ (limit along $x$ of $f$) if $F(*)$ converges to $x$.

**Proposition 27:** Let $f : X \to Y$. Then $f$ is continuous at $x$ if and only if $L_x f = f(x)$.

**Proof:** Use 20 and 26.

**Corollary 28:** $f$ cont. at $x$ if and only if $F$ converges to $x$ and $(F \cap F)$ converges to $f(x)$. 


Theorem 29: Let $\prod_{\alpha \in A} X_{\alpha} = X$ be a topological space and $F_\alpha$ be a filter on $X$. Then $F_\alpha$ converges to $\{x_\alpha\}$ if and only if $\prod_{\alpha \in A} F_\alpha$ converges to $x_\alpha$ for $\alpha \in A$.

Proof: $\Rightarrow$ From Th. 28, since projections are continuous. Conversely, let $V = \prod_{\alpha \in A} V_{\alpha}, V_{\alpha} \in V_{x_\alpha}$. With $V_{x_\alpha} = X_{\alpha}$ for all but a finite number of $\alpha$. $V = \bigcap_{\alpha \in A} V_{x_\alpha}$, $V_{x_\alpha} \in F_\alpha \bigcap_{\alpha \in A} F_\alpha$. Thus $A_{\alpha_0} \in A$ such that $F_{\alpha_0} \in V_{\alpha_0}$, $V = \bigcap_{\alpha \in A} V_{x_\alpha}$.

Theorem 30: A topological space $X$ is compact if and only if every ultra filter on $X$ converges (or filter has one pt. or f.i. $F_{\alpha_0}$).

Proof: (a) $\subseteq$ (b) by 25, 13, 24.

Other by standard argument using 4.
Theorem 31: Let $f: X \to Y$ be continuous and $X$ be compact. Then $f(X)$ is compact.

Theorem 32 (Tychonoff): The product of topological spaces is compact if and only if all the factors are compact.

Proof: $X = \prod_{a \in A} X_a$. If $X$ is compact, use $X_a = \{x_a\}$ and Theorem 31.

Let $X_a$ be compact, $a \in A$. Let $U$ be an ultrafilter on $X$. $F(p_a U)$ is an ultrafilter on $X_a$ by 17. Let $F(p_a U)$ converge to $x_a \in X_a$.

By 29 $U$ converges to $(x_a) = x$.

Theorem 33: Let $X = \prod_{a \in A} X_a$ be a product space and suppose infinitely many $X_a$ are non-compact. Then any compact set $B$ in $X$ has empty interior.

Proof:

Suppose $B^0 = \emptyset$ and $B$ is compact. Find a basic open set in $B^0$ such that $B^0 = \cap_{i=1}^{m} B_{x_i}$. Now $B = B \cap B^0 = \cap_{i=1}^{m} B_{x_i}$ $= X_a$ for $a \neq a_1, \ldots, a_m$, which is impossible.

Notes

Definition: A partially ordered set $D$ is directed ($\preceq$) if

$a, b \in D, a \preceq b \Rightarrow \exists \gamma \in D \text{ such that } \gamma \preceq a, \gamma \preceq b$.

Definition: A net is a mapping $S$ of a directed set $D$ into a set $X$.

Notation: $S(x) = S_x$

Suppose $X$ is a topological space.

Definition: A net $S: D \to X$ converges to $x \in X$ if for each $U \in \mathcal{V}_x$, there is an $a \in D$ for which $S_a \in U$ if $a \succeq a_0$. 
A point \( a \) is an accumulation point of a set \( S : D \to X \) if for each \( U \in \mathcal{U}_a \), \( x \in D \), there exists a \( B \supseteq a \) such that \( S_B \in U \).

**Definition:** \( C \subseteq D \) is cofinal if for each \( x \in D \) there is a \( B \) in \( C \) with \( x \in B \).

(A pt is an acc. pt. if to each \( U \in \mathcal{U}_a \), there is a cofinal set mapped into \( U \))

**Filter and nets:**

If \( D \) is directed, then the class \( \{ D_x \mid x \in D \} \) is a filter base.

If \( D_{x_1} \to D_{x_2} \), then \( x_1 \supseteq x_2 \).

\[ D_{x} = \{ B \in D \mid x \in B \} \]

\( S^F(S(BD)) \) is the filter associated with a net.

\( S \) is the class of supersets of any set of the form \( \{ x \mid x \supseteq y \} \).

\[ FS = \{ S^F(B) \} \]

**Lemma 35:** Let \( S : D \to X \) be topological and \( a \in X \). Then \( S_x \to a \) (\( S_x \) converges to \( a \)) if and only if \( FS \) converges to \( a \). Furthermore \( a \) is an accumulation point of \( S \) if and only if \( a \) is an accumulation point of \( FS \).

**Proof:** Let \( S_x \to a \).

Let \( U \in \mathcal{U}_a \). Then for some \( x \), \( SD_x \subseteq U \). Hence \( U \in \mathcal{U}_a \) and \( x \in FS \). Conversely let \( FS \to a \). Let \( U \in \mathcal{U}_a \). Since \( U \in FS \), find a set of which it is a superset. For the second half, let a be an
accumulation point of $S$. Let $E \subseteq S$, $E \subseteq SD_a$ for some $a \in D$.
Take $U \subseteq U_a$ and let $B \supseteq a$ be such that $S_B \subseteq U$. But then $E \subseteq U \neq \emptyset$, since $U$ is arbitrary, $a \in E$. Conversely, suppose $a$ is an accumulation point of $FS$. Let $a \in D$ and $U \subseteq U_a$.
Now $SD_a \subseteq FS$, so that $U \cap SD_a \neq \emptyset$ because $a \in E$ for $E \subseteq FS$.
Pick $B$ such that $S_B \subseteq U \cap SD_a$.

Let $F$ be a filter on $X$; it is directed by $3': E, F \subseteq F \Rightarrow E \cap F \subseteq F$.
Any mesh $S = E \subseteq F \Rightarrow X = F \subseteq X$ is a net. Assume also that $X \subseteq E$. Let $D_F = \{F \mid F \subseteq F \text{ and } F \cap E \}$, form $SD_F$. If $F \subseteq D_F$, then $S_F \subseteq F \cap E$. Hence $SD_F \subseteq E$. Therefore $FS \supset F$ is a refinement of $F$.

Lemma 35: If $F$ converges to $a$, then $S$ converges to $a$.

Proof: $FS \supset F \subseteq U_a$ and by 35.

Lemma 36: Let $A \subseteq X$. Then $a \in A$ if and only if there exists some filter $F$ converging to $a$ and $A \subseteq F$.

Proof: $a \in A \iff U \subseteq U_a \Rightarrow U \cap A \neq \emptyset$. By 5, this is equivalent to the existence of a filter $F$ such that $F \subseteq U_a$ and $F \ni a$.

Q.E.D.
Lemma 37: Let $A \subseteq X$, $a \in X$. Then $a \not\in \overline{A}$ if and only if there exists a net $S: D \to X$ such that $a$ is a limit of $S$ and $SDCA$ ($S_x \subseteq A$ if $x \in D$).

Proof:
From SDCA, $S_u \to a$. Let $U = U_a$. For each $x \in D$, let $V_x = U_a$. Then $SDCA \subseteq C_A$, $SDCA \subseteq C_{A^C}$, and $A^C \cup \emptyset = \emptyset$.

Conversely, let $a \not\in \overline{A}$. Let $D = U_a$. Now $U \supseteq D \to S_x \subseteq U \cup A$, where $S_x$ is any point in $U \cup A$, which is non-empty. Now $U \to a$.

By 36, $S_x \to a$.

Lemma 38: Let $f: X \to Y$ be continuous and $S: D \to X$ be a net with $S_x \to a \in X$. Then the composition map $f \circ S$ is a net and $(f \circ S)_x \to f(a)$.

Proof:
Let $V \ni f(a)$. Let $U = U_{f(a)}$ be such that $f(U) \subseteq V$. By convergence, $\exists x$ such that $SDCA \subseteq C_U$. Only $f$

$(f \circ S) \subseteq f(S \subseteq C_U) \subseteq f(U) \subseteq V$

Lemma 39: In a Hausdorff space, a net can converge to at most one point.

Proof: Immediate from directedness.

Examples:
1) Sequences
2) $\mathbb{R}^\mathbb{R}$, all functions from $\mathbb{R}$ to $\mathbb{R}$

$A = \{ f \in \mathbb{R}^\mathbb{R} \mid f(x) = 1 \text{ except for finitely many } x \in \mathbb{R}, \text{ where } f = 0 \}$

Let $f_1 \leq f_2$ if and only if $f_1(x) \leq f_2(x)$ for $x \in \mathbb{R}$. Directed under $\leq$. Let $S$ be the identity map. Then $S \cdot f \to 0$. 
To see this we note
\[ u = \{ f \mid f(x), \ldots, f(x_m) < \epsilon, \epsilon > 0, 1 \leq m, x_i \in \mathbb{R} \} \]

Let \( g(x) = \cdots = g(x_m) = 0 \) and \( g \subseteq A \). Then \( g' \subseteq g \) implies and not converges.

Banach spaces

Theorem: Let \( X \) be a normed linear space, \( X^* \) its dual. Let
\[ B^* = \{ x^* \in X^* \mid \| x^* \| \leq 1 \} \]
Then \( B^* \) is compact in the weak-star topology.

Remark:
\( X \), dual \( X^* \)
The smallest (convergent) topology on \( X^* \) which makes every mapping \( X^* \rightarrow y^* \times \) (field) \( x^* \), continuous is the weak-star topology. Note continuity in norm topology.

Topology on \( X^* \) is topology induced by product topology on \( \mathbf{field} \times \)

Proof:
Let
\[ C = \prod_{x \in X} \left[ -\| x \|, \| x \| \right] \subset \mathbf{field}^X \]
convex
\[ \prod_{x \in X} \mathbf{field} = \mathbf{field}^X \]

Now \( C \) is compact and \( C \supset B^* \). Let \( x^* \in B^* \). Then
\[ \left| x^* \right|_{\mathbb{Z} \times X} = \left| x^* \right|_{\mathbb{Z} \times X} \leq \| x^* \| \leq \| x \| \].

Thus
\[ y^* \times \left[ -\| y \|, \| y \| \right] \]
for every \( y \).

Hence \( B^* \subseteq C \). We shall show \( B^* \) is closed. Suppose \( x^* \notin B^* \).

Let \( x^*_k \) be a net in \( B^* \) which converges to \( x^*_k \). Now
\[ x^* (\alpha x + b) = \frac{1}{\alpha} x^* + \frac{b}{\alpha} x^* \]

(two limits, but product of Hausdorff spaces is Hausdorff)

So \( x^* \) is linear.

We now from \( \|x^*\| \leq 1 \).

Let \( \|x\| = 1 \). \( \|x^*\| = \sup |x^* x| \) with \( \|x\| \leq 1 \).

\[ |x^* x| = |\frac{1}{\alpha} x^*| \leq |\frac{1}{\alpha} x^* x| \]

\[ = |x^* x| \leq \|x^*\| \|x\| \leq 1 \text{ because } \|x^*\| \leq 1 \text{ in } \mathbb{R}^n. \]

Hence \( \|x^*\| \leq 1 \).

Thus \( B^* \) is closed in the weak-star topology, hence it is compact.

Notation:
- \( K = \mathbb{R} \) or \( \mathbb{C} \).

Definition: Let \( X \) be a vector space over \( K \). A function \( p : X \rightarrow \mathbb{R} \) is called a seminorm if:

1. \( p(\alpha x) = |\alpha| p(x) \)
2. \( p(x+y) \leq p(x) + p(y) \)

Remarks:
- \( p(0) = 0 \), \( p \geq 0 \).
- It is not necessarily true that \( p(x) = 0 \) implies \( x = 0 \).
Theorem: Let $X$ be a vector space over $\mathbb{R}$, $\| \cdot \|$ a semi-norm on $X$, $L$ a linear subspace of $X$, $\varphi$ a linear functional $f : L \to \mathbb{R}$, and $x_0 \in X - L$.

If $|\varphi| \leq 1$ on $L$, then there exists a linear $F : L + Rx_0 \to \mathbb{R}$ such that $|F| \leq 1$ on $L + Rx_0$, $F|L = \varphi$.

Proof:
A functional is determined by its value on $L$ and $x_0$.

\[
F|L = \varphi, \\
F(x_0) = a_0 \in \mathbb{R}
\]

Then $F$ is uniquely defined on $L + Rx_0$ by linearity.

Must have $|F(l + ax_0)| \leq \|l + ax_0\|$ for $l \in L$, $a \in \mathbb{R}$.

\[
|l + ax_0| = l + ax_0 - \varphi(l + ax_0) \leq \|l + ax_0\| \leq \|l + ax_0\| - \varphi(l + ax_0) = \|l + ax_0\| - Fx_0.
\]

If $a > 0$, $-\frac{1}{a}(\varphi(l + ax_0) - \varphi(l)) \leq Fx_0 \leq \varphi(l + ax_0) - \varphi(l)$.

If $a < 0$, $-\frac{1}{a}(\varphi(l + ax_0) - \varphi(l)) \geq Fx_0 \geq \varphi(l + ax_0) - \varphi(l)$.

In either case, $-\varphi(l + ax_0) - \varphi(l) \leq Fx_0 \leq \varphi(l + ax_0) - F(l)$.

This is necessary and sufficient. It remains to produce an $Fx_0$.

If $f(x - y) \leq f(x) - f(y)$ for $x, y \in L$ and $f(x + y) = f(x) + f(y)$.

So $f(x + y) \leq f(x) + f(y)$.

$\varphi(x) - \varphi(x_0) \leq f(x + x_0 - x) + f(x - x_0)$.

Take supremum on left:

$S = \sup_{x \in L} \left( f(x) - f(x - x_0) \right)$

$\varphi = \inf (f + f(x - x_0))$, both are finite.
Hence $3 \leq i$

Let $s \leq f(x_0) \leq i$. (Let $f(x_0) = s$, for example)

Then $f(x) - f(x - x_0) \leq f(x) \leq f(x_0) + f(x - x_0)$ for $x, y \in L$.

And the condition is satisfied.

Q.E.D.

Theorem: (Hahn-Banach)

Let $X$ be a vector space over $\mathbb{R}$, $f$ a semi-norm on $X$, $L$ a subspace of $X$, $f: L \to \mathbb{R}$ linear, $|f| \leq \rho$ on $L$. Then there exists an $\mathbb{F}$-linear $F: X \to \mathbb{R}$ such that $|F| \leq \rho$ on $X$ and $F|L = f$.

Proof:

Let $C$ be the class of all linear extension of $f$ on linear subspaces of $X$ satisfying the norm condition. Order by $(L', \rho') > (M, \rho)$ if and only if $L' \supset M$, $\rho'|M = \rho$.

This is inductive. Find maximal element. Apply lemma.

Q.E.D.

For normed linear spaces

If $L \subseteq X$ and $|f(x)| \leq M \|x\|$ on $L$, then

$p(x) = M \|x\|$ is a semi-norm on $X$.

with $|F(x)| \leq |f(x)| = M \|x\|$.

Complete the result.

Complex spaces

Lemma: Let $X$ be a vector space over $\mathbb{C}$. Suppose $f$ is $\mathbb{R}$-linear and $f: X \to \mathbb{R}$. Then $F(x) = f(x) - if(ix)$ is $\mathbb{C}$-linear and $\text{Real } F = f$.

Proof: as exercise.
Lemma. Let $X$ be a vector space over $\mathbb{C}$, $f : X \to \mathbb{C}$ be linear.

Let $f_1 = \frac{f + \overline{f}}{2} = \text{Real } f$. Then $f_1 : \mathbb{R} \to \mathbb{R}$ linear: $X \to \mathbb{R}$

and $f_1(x) = f(x) - i f_1(ix)$

For $H \in \text{Im } f$. Take $f_1$ from $f_1$, extend, apply first lemma.
Theorem: Let $X$ be a vector space over $\mathbb{C}$, $Y$ a subspace of $X$, $f$ linear from $Y$ to $\mathbb{C}$, $\varphi$ a seminorm on $X$, $|f| \leq \varphi$ on $Y$, then there exists an $F$ linear from $X$ to $\mathbb{C}$ and $|F| \leq \varphi$ on $X$ and $F/Y = f$.

Proof:

Let $f = f_1 - i f_1(i \cdot)$, where $f_1 = \text{Re} f = \frac{f + \bar{f}}{2}$.

$f_1$ is $\mathbb{R}$-linear and $|f_1| \leq |f| \leq \varphi$ on $Y$.

Extend $f_1$, $F_1 : X \to \mathbb{R}$ with $|F_1| \leq \varphi$, b. Take

$F = F_1 - i F_1(i \cdot)$

$F$ is $\mathbb{C}$-linear from $X$ to $\mathbb{C}$;

$F/Y = f$ since $F/Y = (F_1 - i F_1(i \cdot))(Y) = f_1 - i f_1(i \cdot) = f$.

$|F(x)| = e^{i \varphi} F(x) = F(e^{i \varphi} x) = F_1(e^{i \varphi} x)$, since $F(e^{i \varphi} x)$ is real

$\leq \varphi(e^{i \varphi} x)$

$= |e^{i \varphi} | \varphi(x)$

$= \varphi(x)$

On a normed linear space $X$,

$Y \subset X$, norm $\| \|$.

If $f, Y \to K$ is continuous

$|f(x)| \leq \|x\| \|f\|$ known if $f$ on $Y$.

Set $|f(x)| = \|x\| \|f\|$

Extend $f$ to $F$:

$F : X \to K$.

$|F(x)| \leq \|F\| \|x\|$

and so $\|F\| = \|f\|$.
Corollary: Let $X$ be a normed linear over $K$ ($\mathbb{R}$ or $\mathbb{C}$); let $x_0 \in X$. Then there exists a continuous linear functional $\phi^* \in X^*$ on $X$ such that $\|\phi^*\| = 1$ and $\phi^*(x) = \|x\|$

Proof:

John $Y = Kx_0$. This is a closed linear subspace of $X$. Let $f(x) = x_0 \|x\|$ for $x \in K$.

$\|f\|_Y = 1$ because $|f(x)| = \|x\| \|x\|$. If $\|x\| = 1$, then $|f(x)| = 1$. Extend $f$ to $g^* \in X^*$ with $\|g^*\| = 1$.

Corollary: Let $Y$ be a closed linear proper subspace of $X$ and $x_0 \in \overline{Y}$. (Then $d = \inf \|x_0 - y\| > 0$). Then there exists a $y^* \in X^*$ such that $y^*|Y = 0$, $y^*(x_0) = 1$, and $\|y^*\| = 1/d$.

Proof:

Consider $Y + Kx_0$. The sum is direct: $Y + Kx_0$. Define $l(y + ax_0) = x$, well-defined because sum is direct. Now

$$\sup_{y + ax_0 \neq 0} \frac{|l(y + ax_0)|}{\|y + ax_0\|} = \sup_{\frac{1}{\|x\|}a : \frac{1}{\|x\|}a \neq 0} \frac{|l(\frac{1}{\|x\|}a + x_0)|}{\|\frac{1}{\|x\|}a + x_0\|} = \sup_{\frac{1}{\|x\|}a \neq 0} \frac{1}{\|\frac{1}{\|x\|}a + x_0\|} = \frac{1}{d}$$

Extend $l$ by Hahn-Banach Theorem.
Lemma: Let \( X \) be a normed linear space. If \( L \) is a closed maximal proper subspace of \( X \), then there exists a linear function \( \ell^* \in X^* \) such that \( \ker \ell^* = L \), and conversely.

Proof:

If \( L \) is closed and proper, take any \( x_0 \in X \setminus L \) and apply the preceding corollary to \( L + Kx_0 \). Conversely, let \( \ell^* \in X^* \), \( \ell^* \neq 0 \). Let \( L = \ker \ell^* \). \( L \) is closed because \( \ell^* \) is continuous and \( L = L^{-1}(0) \).

\( L \) is proper because \( \ell^* \neq 0 \). Maximal: \( \alpha \notin L \), to show \( X = L + \alpha \).

Let \( x \in X \). Then \( \ell^*(\alpha \frac{x}{\ell^*(x)} + L) = \alpha \). If \( \ell^*(x) = \alpha \), then

\[
\ell^*(x - \alpha \frac{x}{\ell^*(x)}) = 0
\]

So \( x - \alpha \frac{x}{\ell^*(x)} \in L \)

and \( x = \alpha \frac{x}{\ell^*(x)} + L \).

\( L^{-1}(\alpha) = \alpha \frac{x}{\ell^*(x)} + L \)

\( X = U L^{-1}(\alpha) = U \left( \alpha \frac{x}{\ell^*(x)} + L \right) = \left( \frac{\alpha}{\ell^*(x)} + L \right) \).

Hence \( L \) is maximal.

Theorem: Let \( X \) be a normed linear space. If \( \lambda^* \) is a separable norm on \( X \), then \( X^* \) is separable if and only if \( X \) is separable.

Proof:

Let \( \{ x^*_\mu \mid 1 \leq \mu \} \) be dense in \( X^* \).

\[
\| x^*_\mu \| = \sup_{\| x \| = 1} | x^*_\mu(x) | \iff \| x^*_\mu \| = \frac{\| x^* \|_1}{\| x \|_1} \| x^*_\mu \|_1 = 1
\]

To each \( \mu \geq 1 \), take two \( x^*_\mu \) such that \( x^*_\mu(x^*_\mu) > \frac{1}{\mu} \| x^*_\mu \|_1 \). Take the linear span \( \mathcal{F}[x_\mu] \); we shall show that \( \mathcal{F} = X \), and this is sufficient (take closed finite linear combinations of \( x_\mu \)). Suppose otherwise. Then there exists \( x^* \in X^* \) such that \( x^* \neq 0 \) and \( x^* x_\mu = 0 \).
\[ \text{Let } x_n^* \to x^*. \text{ Then } 0 \leq \| x^* - x_n^* \| \]
\[ = \| x^* - x_n^* - x_n^* + x_n^* \| \]
\[ \geq \| x_n^* - x_n^* \| \| x_n^* \|
\]
\[ = \| x_n^* - x_n^* \| \]
\[ = \| x_n^* \| / 2 \]

So \( \| x_n^* \| \to 0 \) and \( x_n^* \to 0 \). So \( x^* = 0 \), contradiction.
Theorem: (Open Mapping Principle)

Let $T$ be continuous and linear from $X$ to $Y$, $X$ and $Y$ both Banach spaces. If $T$ is onto, then $T$ is open.

Proofs:

1) To show if $U \subset Y_0$ open at 0, then $TU \subset Y_0$.

Let $W$ be an open neighborhood of 0 in $X$ such that $W - W \subset U$.

Write $X = \bigcup_{n=0}^{\infty} nW$

$Y = TX = UTn(W) = U_nTW = U_n\overline{TW}$

Let $B_2$ Basic category there exists $n_0$ such that $n_0\overline{TW} = n_0\overline{TW} \supset W'$ open and non-empty (homeomorphism)

So $n_0\overline{TW} \supset \frac{1}{2} W'$

$\overline{TU} \supset \overline{T(W-W)} = \overline{TW-TW} \supset \overline{TW} - \overline{TW} \supset \frac{1}{2} W' - \frac{1}{2} W' \in U_0$, in fact it is open.

2) If $\varepsilon_0 > 0$, then there exists an $\eta_0$ such that $TB(0,2\varepsilon_0) \supset B(0,\eta_0)$.

Take $\varepsilon_0, \eta_0, \ldots, \eta_0 < \varepsilon_0$. Choose $\eta_n$ for $0 \leq n$ such that $TB(0,\varepsilon_n) \supset B(0,\eta_n)$ for $n > 0$ and such that $\eta_n \to 0$.

Take $x_n \in B(0,\varepsilon_n)$ such that $||x_n|| < \eta_n$. But $x_n \in B(0,\eta_n) \subset TB(0,\varepsilon_n)$. Let $x_n \in B(0,\varepsilon_n)$ such that $||Tx_n - y|| < \eta_n$, but $Tx_n - y \in B(0,\eta_n) \subset TB(0,\varepsilon_n)$.

Thus $x_n \in TB(0,\varepsilon_n)$ and continue $||x_n|| < \eta_n$ and $||Tx_n + (Tx_n - y)|| < \eta_n$.

Now $||\sum x_i|| < \sum \varepsilon_i$, and the latter is Cauchy. Let

$x = \lim_{n \to \infty} \sum_{i=0}^{m} x_i$

By continuity of norm

$||x|| \leq \sum_{i=0}^{m} \varepsilon_i < 2\varepsilon_0$ and
\[ T_2 = 2 \frac{2}{3} T_2 \text{ by continuity of } T \]
\[ = y. \quad \text{This proves (2)} \]

Here if \( U \) is any neighborhood of \( 0 \in X \), then \( T(U) \) is a neighborhood of \( 0 \in Y \).

3) \( T(U) \) is any open set. Let \( y \in T(U), y = T(x), x \in U \).

Find \( U \in U_X \) such that \( x + U \subseteq O \). Then
\[ T(x + U) = T(x) + T(U) \subseteq T(U) \]

\( T(U) \) is a subset of \( O \) in \( X \) and \( T(x + U) \subseteq T(U) \) is open of \( T(x) \) in \( Y \). Hence \( T(U) \) is open.

Corollary:

Let \( T \) be continuous linear one-one and onto such that \( X \rightarrow Y \), \( X \) and \( Y \) are Banach spaces. Then \( T^{-1} Y \rightarrow X \) is continuous and linear.

Proof: \( T^{-1} \) exists and is linear. To show converse image of open set is open, which is the theorem.

Corollary (Closed graph theorem):

Remark: If \( T : X \rightarrow Y \), then the graph of \( T, \text{Graph}(T), \) is
\[ \{x, Tx \mid x \in X \times Y \}. \]

Statement: If \( X \) and \( Y \) are Banach spaces and \( T \) linear \( X \rightarrow Y \), then
\[ T^{-1} \text{ closed implies } T \text{ continuous}. \]

Proof: Projection to \( X \) is continuous is linear one-one onto. So inverse is continuous. Second projection is continuous and composition \( T \circ \text{cont} \)
(Closed subset of \( X \times Y \) is Banach with \( ||x|| = ||x|| + ||y|| \) )
Corollary:

If \( X \) is a Banach space under \( \| \cdot \|_1 \) and \( \| \cdot \|_2 \), and if
\[
\| \|_1 \leq \alpha \| \|_2
\]
(the identity from \( Z \) to \( Y \) is continuous), then there exists a \( \beta \) such that
\[
\| x_2 \|_2 \leq \beta \| x_1 \|_1 \quad \text{for} \quad x \in X.
\]

Proof:

Identity is continuous, linear, one-one, onto. Be universe in continuous.

Examples:

1) Let \( X \) be an infinite dimensional normed linear space over \( K \). Let \( B \) be a basis of \( X \) over \( K \). Let basis be such that \( \| b \| = 1 \) for \( b \in B \). We shall define a linear functional on \( B \) and extend by linearity. Take countable subset \( \{ e_n \} \). Define \( f(e_n) = n \) for the rest. Then
\[
f \| \| = \sup_{\| x \| = 1} | f(x) | = n.
\]

So linear functional is unbounded and therefore not continuous.

Graph is a subspace and is not closed. This is a non-closed subspace.

2) Separable space with non-separable dual.

Let \( X = l_1 = \{ x_1, x_2, \ldots \} \sum |x_n| < \infty \), \( x_i \in K \).
\[
\| x \| = \sum |x_n|
\]

\( X \) is a separable Banach space.

Fundamental set is countable \( (0, \ldots, 1, 0, \ldots) \): dense as that in \( X \).

Space with countable fundamental set is separable.

\( X^* \) is the space of all bounded sequences.

To see the limit at effect of \( f \) on fundamental sequence. Then define a linear functional to go the other way. Can show name is unique.
Known: A normed linear space is locally compact if and only if it is finite-dimensional.

**Proof:**

In a Banach space, $B(0, 1) = \{ \| x \| \leq 1 \}$. We shall show $\{ x : \| x \| = 1 \}$ is not compact. Define inductively

$\| x_n \| = 1$ for $1 \leq n \leq m$ and $\| x_n - x_m \| > \frac{1}{2}$ for $n \neq m$.

Take $x$ in $\text{span} \{ x_1, x_2, \ldots, x_m \}$, so $x = \sum x_i$. Then

$d(x, V_m) > 0$.

Make $d(y, x) < 2d(x, V_m)$.

$\| x_n - y \|$, $y \in V_m$.

Take $x_{m+1} = \frac{x_n - y}{\| x_n - y \|} \notin V_m$, $\| x_{m+1} \| = 1$.

Now let $x \in V_m$ and then

$\| x_{m+1} - x \| = \frac{1}{\| x_n - y \|} \frac{1}{2}$.

$\| x_{m+1} - x \| > \frac{1}{2}$.

This sequence has no limit point.

**Q.E.D.**

Every space which is compact but not sequentially compact. Take

$X = \mathbb{R}_o$, the product topology.

$x_n$ is a net of functions of the set of countable sequences into $0, 1$.

$f^n(s) = s_m$, $s_m = m$ the coordinate; $f^n$ does not converge.
Lema:
Let $X$ be a vector space over a field $F$ and suppose $X = A \oplus B$. Then there exist linear maps $\phi : X \to A$ and $\psi : X \to B$ such that $\phi y + \psi y = y$ for all $y \in X$. ($\phi + \psi = 1$)

Proof:
Write $y = y_1 + y_2$ uniquely. Set $\phi y = y_1$, $\psi y = y_2$, etc.

Remark:
Suppose $X = A \oplus C$. If we define $\phi'$ and $\psi$, then $\phi \neq \phi'$ in general.

Proposition:
Let $X$ be a Banach space.
Suppose $P : X \to X$ is a bounded linear map of $X$ into $X$, and suppose $P^2 = P$. Then $\text{image} \ P$ is a closed linear subspace of $X$ and $(I - P)^2 = I - P \oplus \text{image} \ (I - P)$.

Proof:
Image $P$ is linear. Assume $\text{image} \ P = \ker (I - P)$, which is closed. Let

$y = P \cdot y - x = y_1$. Hence $y - P \cdot y = 0$, $(I - P) y = 0$, and $y \in \ker (I - P)$

And conversely. Also $(I - P)^2 = I - P - P + P^2 = I - P$ by expanding.

Finally, for $x = \text{image} \ P \oplus \text{image} \ (I - P)$. If $x = 0$ in intersection,

$P \cdot x = 0$ and $P x = 0$. Hence $x = 0$. In spanning, $x = P x + (I - P) x = P x + (I - P) x$.

Q.E.D.

Proposition:
Let $X$ be a Banach space.
Suppose $X = A \oplus B$, $A$ and $B$ closed. Then there exist a $P : X$ such that $P^2 = P$ and \text{image} $P = A$, \text{image} $(I - P) = B$.

Proof: Set $P = \phi$ in Lemma. Use closed graph theorem.
Let graph of $P: X \to A$ be $\Gamma$. Let $(x_y, Px_y) \in \Gamma \to (y, z) \in X \times A$.

Then $x_y \to y$, $x_y = Px_y + (I-P)x_y$.

$\downarrow$  $\downarrow$  $\downarrow$ so that $y = z + y_P$.

Now $Px_y \in A$ and $(I-P)x_y \in B$. Since $A$ and $B$ are closed, $z \in A$ and $y \in B$. By uniqueness in lemma $z = P_{x_y}$. Hence graph is closed and $P$ is continuous.

Let $T$ be bounded linear $X \to Y$ over $K$. Then $T$ induces $T^* = f \circ T \in X^*$ with $f \in Y^*$.

Hence $T^* : Y^* \to X^*$.

$T^*$ is linear and is bounded.

$\|T^*f\| = \|foT\| = \sup_{\|x\| = 1} |(f \circ T)x| = \sup_{\|x\| = 1} |f(Tx)| \\
\leq \sup_{\|x\| = 1} \|f\| \|Tx\| = \|f\| \|T\|$

$= \|f\| \|T\|$

Hence $\|T^*\| \leq \|T\|$

Proposition:

$\|T^*\| = \|T\|$

Proof:

Note that $T^{**} : X^{**} \to Y^{**}$ and $\|T^{**}\| \leq \|T^*\| \leq \|T\|$. We will show $\|T\| \leq \|T^{**}\|$. 
We show the diagram commutes; this is sufficient.

\[(T^{++} \cdot \varphi_x) \varphi = ?\]

\[\varphi \in Y^*\]

\[Y^* \to X^* \to K\]

\[(T^{*+} \cdot \varphi_x) \varphi = (\varphi_x \cdot T^*) \varphi\]

\[= \varphi_x (T^* \varphi)\]

\[= (T^{*+} \varphi) \varphi \]

\[= (5 \circ T) \varphi \]

\[= 3 \left( T^* \varphi \right) \]

\[= \left[ g \left( T^* \varphi \right) \right] \varphi \]

\[T^{*+} \varphi \varphi = 4 (T^* \varphi) \]

\[T^{*+} \varphi = 4 \circ T \]

Now \[\| T^* \varphi \| = \| 4 (T^* \varphi) \| = \| T^{*+} (\varphi_x) \| \leq ||T^{++}|| \| \varphi_x \|\]

\[-||T^{++}|| \| \varphi \|. \quad \text{Q.E.D.}\]

**Proposition:**

Let \( X \) be normed linear over \( K \) and \( L \) a closed subspace of \( X \).

In \( X/L \) set \[\| x + L \| = \inf_{z \in L} \| x + z \|.\] Then \( X/L \) is a normed linear space over \( K \). \( X \) complete implies \( X/L \) complete.

**Proof:**

Norm is \[= \inf_{z \in L} \| x + z \| \]  

1) \[\| x + L \| = \| x + (x + z) \| = \inf_{z \in L} \| x + z \| , \] which for \( x \neq 0 \)
\[
\begin{align*}
&= \frac{\text{Erf}}{2^L} \|x + \alpha z\|L \\
&= |\alpha|^L \|x + z\|L = |\alpha| \|x + LI\|
\end{align*}
\]

For \( \alpha = 0 \), norm is zero.

2) Triangle inequality is okay.

3) \( \delta_0 = k(x + LI) \), take \( x + \delta_0 \to 0 \). Then,

\[L^\nu - \delta_0 \to 0\] and result \( x \in L \) comes from closure.
\[ x / L, \quad \| x / L \| = \inf_{g \in L} \| x - g \| \]

1) \( \phi : X \to x / L \) is continuous

2) \( \| \phi \| \leq 1 \)

3) If \( X \) is complete, \( x / L \) is complete

Suppose \( \| x_n + L - (x_m + L) \| \to 0 \)

Then \( \| x_n - x_m \| \leq \frac{1}{2^n} \)

Choose a subsequence \( x_{n_k} \) such that \( x_{n_k} \to x \) and

\( \| x_{n_k} - x_{n_{k+1}} \| < \frac{1}{2^n} \)

So \( x_{n_k} \to x \), \( x_{n_k} + L \to x + L \)

Subsequence converges; so whole sequence converges.

If \( X \) is normed linear and \( A \subseteq X \), then \( A^\perp = \{ f | f \neq x^* \text{ and } f(A) = 0 \} \).

\( A^\perp \) is a closed linear subspace of \( X^* \)

Proposition:

If \( X \) is a normed linear space and \( L \) closed in \( X \), then \( (X/L)^* \) is isometric to \( L^\perp \).

Proof:

Let \( \phi : X \to X / L \) and \( f \in (X / L)^* \).

Now \( f \to f \circ \phi \in L^\perp \)

since \( f \circ \phi = f(\phi(x)) = f(x) = 0 \) \( \forall \phi \in L^\perp \).

\( \phi \) is linear. \( \phi \) is one-to-one. Let

\( O = \forall f = \forall \phi \text{ and } f = 0 \text{ since } \phi \text{ is onto.} \)

Here \( \phi \) is one-to-one. For onto let \( l \in L^\perp \).

Then \( l : X \to X / L \) with kernel \( A \).

Factor map: \( X \xrightarrow{\phi} \mathbb{R}^n, \quad l = l' \circ \phi \text{ since } L \subseteq \ker \phi. \)
$l'$ is continuous:

Let $x_n + L \to 0$. Then $y_n \to x_n \to L$ with $\|y_n\| \to 0$.

Then $0 \leq l'(y_n) = l'(x_n + L)$

and $l'(x/L^*)$.

Hence $X$ is onto.

Let $f_2(x/L^*)$. Then $\|f_2\| = \sup_{x/L} \frac{|f(x+L)|}{\|x+L\|}$

$= \sup_{x/L} \frac{|f(x)+L|}{\|x+L\|}$

$f(x+L) = f(x+L)$

$= \sup_{x/L} \frac{1}{\|x+L\|} \frac{|f(x)|}{\|x\|}$

$= \sup_{x/L} \frac{|f(x)|}{\|x\|}$

$= \sup_{x/L} \frac{|f(x)|}{\|x\|}$

$= \|f\|$

$= \|f\|$

$L^* \text{ is isomorphic to } X^*/L^*$:

$f : X^* \rightarrow L^* \in L^*$

$X^* \rightarrow L^* \rightarrow 0 \text{ in exact by Hahn–Banach.}$

Hence $L^* \cong X^*/\ker \rho = X^*/L^*$
Bounded linear transformation on $X$ from Banach space. If $X$ is a normed ring, we call it an algebra, called a Banach algebra, in which
\[ \|AB\| \leq \|A\| \cdot \|B\| \]
over $C$.

**Lemma:** Let $X$ be a normed linear space in which
\[ \|x + y\|^2 + \|x - y\|^2 = 2 \left[ \|x\|^2 + \|y\|^2 \right]. \]

Let \( (x, y) = \sum_{\sigma = 0}^{\infty} \frac{i^\sigma}{\sqrt{4}} \|x + i^\sigma y\|^2 \). Then $(x, y)$ is a sesquilinear form (linear in first, conjugate linear in second).

$(x, y^\dagger) = (x, y)$, and $(x, x) = \|x\|^2$.

**Proof:**
\[ \alpha = 2(\|x + y + z\|^2 + \|y - z\|^2) = \|x + 2y\|^2 + \|x + 2z\|^2 \]

If $z = 0$,
\[ 2(\|x + y\|^2 + \|y\|^2) = \|x + 2y\|^2 + \|x\|^2 \]
and similarly for $y = 0$.

Substitute
\[ \alpha = 2 \left( \|x + y\|^2 + \|y\|^2 \right) - \|x\|^2 + 2 \left( \|x + z\|^2 + \|z\|^2 \right) - \|x\|^2 \]
\[ \|x + y + z\|^2 = \|x + y\|^2 + \|x + z\|^2 + \|y + z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2 \]
\[ 2(\|x + y\|^2 + \|y\|^2) = \|x + 2y\|^2 + \|x\|^2 \]

For $(x, y) = (\bar{y}, x)$, just write things down:
\[ (x, y) = \sum_{\sigma = 0}^{\infty} \frac{i^\sigma}{\sqrt{4}} \|y + i^\sigma x\|^2 = \frac{1}{\sqrt{4}} \sum_{\sigma = 0}^{\infty} \|\frac{1}{i^\sigma} y + x\|^2 \]
\[ = \sum_{\sigma = 0}^{\infty} \|\frac{1}{i^\sigma} y + x\|^2 \]
\[ = (x, y) \quad \text{Easy for } \|x, x\| = \|x\|^2 \]
Prove that \(-1, i\) can be factored out of \(\langle x, y\rangle\).

By additivity, \(m(x, y) = \langle x, y\rangle\) for integers.

Then \(\langle \frac{1}{n} x, y \rangle = \frac{1}{n} \langle x, y \rangle\)
and gets for all reals, then for all Gaussian reals.

\[
0 \leq \|x - r\|_1 = \langle x - r, x - r \rangle
= \|x\|_1^2 - \langle r, x \rangle - \langle x, r \rangle + \langle r, r \rangle
\]

through reals.

Let \(\rho \to \langle x, y \rangle / \|y\|_1^2\) through reals, and

\[
\|\langle x, y \rangle\| = \|x\|_1 \|y\|_1^2
\]

Of \(r_n \to x\)

\[
\|r_n - x\|_1 = \|r_n - x + \|y\|_1^2 \|y\|_1^2
\]

\[
= \|r_n - x\|_1 \|x\|_1 \|y\|_1^2 \|y\|_1^2
\]

Hence constant multiples by

Schwarz's inequality.

Converse is also known.
Definition. A Hilbert space over $\mathbb{C}$ is a Banach space over $\mathbb{C}$ with a positive symmetric sesquilinear form such that $\|x\|^2 = (x, x)$.

Schwarz inequality: $|\langle x, y \rangle| \leq \|x\| \|y\|$

Theorem. Let $A$ be a non-empty closed convex subset of $H$.

Then there is a unique $\bar{x}_0 \in A$ such that $
abla \|x_0\| \leq \|x\|$ for $x \in A$.

Remark. Need $x, y \in A \Rightarrow \frac{x+y}{2} \in A$.

Proof.

$$\inf_{x \in A} \|x-\bar{x}\| = d > 0$$

Take $\bar{x}_0 \in A$ such that $\|\bar{x}_0\| \leq d$.

$$\|\bar{x}_0 - \bar{x}\|^2 = 2 (\|\bar{x}_0\|^2 + \|\bar{x}\|^2) - 2 \|\bar{x}_0 + \bar{x}\|^2 = 2 (\|\bar{x}_0\|^2 + \|\bar{x}\|^2) - 2 \|\frac{\bar{x}_0 + \bar{x}}{2}\|^2 \leq 2 (\|\bar{x}_0\|^2 + \|\bar{x}\|^2 - 2d^2) \Rightarrow 2(d^2 + d^2 - 2d^2) = 0.$$ 

So $\{\bar{x}_0, \bar{x}\}$ is Cauchy with limit $\bar{x}_0$, which is in $A$ since $A$ is closed. $\|\bar{x}_0\| = d$ by continuity.

Suppose $\|\bar{x}_0 + \bar{x}\| \leq \|\bar{x}\|$ for $x \in A$.

$$\|\bar{x}_0 + \bar{x}\| \leq \|\bar{x}\|$$

$$\|\bar{x}_0 - \bar{x}\|^2 + 2(\bar{x}_0 + \bar{x}) = 2 \|\bar{x}_0\|^2 + 2 \|\bar{x}\|^2 \leq 4 \|\frac{\bar{x}_0 + \bar{x}}{2}\|^2 = \|\bar{x}_0 + \bar{x}\|^2$$
Corollary:

Let \( A \neq \emptyset \) be closed and convex in \( H \). Then there exists exactly one \( w_x \in A \) such that \( \| x - w_x \| = \| x - w \| \) for \( w \in A \).

Proof:

Just translate to zero.

Perpendicularity:

Definition:

\[ x \perp Y \text{ if } (x, y) = 0 \]

Definition is symmetric.

Proposition:

If \( M \) is a closed subspace of \( H \) and \( x \notin M \), then there is a unique \( u \in M \) such that \( x - u \perp M \) and \( \| x - u \| \leq \| x - w \| \) for \( w \in M \).

Remark:

If \( A \subset H \), then \( A^\perp = \{ x : x \perp A \text{ for all } y \in A \} \).

Proof:

\( M \) is convex. Let \( u \in M \) with \( \| x - u \| \leq \| x - w \| \) for \( w \in M \) and \( u \) is unique. If \( \perp \), find \( y \in M \) with \( (x - u, y) = 0 \).

Make \( \| y \| = 1 \). Project \( x - u \) onto \( y \) and look at

\[
\frac{x - (x - u, y)}{\| y \|^2} 
\]

\[
\| x - u - (x - u, y) y \|^2 = \| x - u \|^2 - (x - u, y)(y, x - u) \\
- (x - u, y)(x - u, y) \]
Lemma: 1) If $A \subseteq B$, then $A^\perp \supseteq B^\perp$.
2) $A^\perp = A$
3) $A^\perp = A$ if and only if $A$ is a closed subspace.
4) $A^\perp$ is the smallest closed subspace containing $A$.

Proof:
3) One direction trivial. Suppose conversely that $A$ is closed and $z \in A^\perp$, $z \notin A$. Take $x \in A$ such that $z - x \in A^\perp$. Since $(z, z - x) = 0$ and $(x, z - x) = 0$ and $||z - x||^2 = 0$. Hence $z = x$.
4) $A \subseteq \text{lin span } A \subseteq \text{lin span } B = B$. $A^\perp \supseteq B^\perp$. Then $A^\perp = B^\perp$ and $A^\perp = B^\perp = B$.

Lemma:
Let $M$ and $N$ be closed linear subspaces with $M \cap N$ (or $M \subseteq N^\perp$ or $N \subseteq M^\perp$). Then $M + N$ is a closed linear subspace and $M + N = M \oplus N$.

Proof:
If $x \in M \cap N$, then $x \perp x$ and $x = 0$. Take sequence in $M \oplus N$, $x_n + y_n \to z$. It is Cauchy.
$||x_n + y_n - (x_n + y_n)||^2 \to 0$. Reorder and
distribute.

\[ ||x_v + y_v - (x_n + y_n)||^2 = ||x_v - x_n||^2 + ||y_v - y_n||^2 \]

\[ x_v \rightarrow x_0 \]
\[ y_v \rightarrow y_0 \]

Hence \( x_0 + y_0 \in M + N \) since \( M \) and \( N \) are each closed. QED.

**Lemma:**

Suppose \( x \in H \) and \( \hat{x} = y \in H' \rightarrow (\gamma, x) \). Then \( \hat{x} \in H^* \).

**Proof:**

\( \hat{x} \) is linear and bounded because

\[ |\langle \hat{x}, y \rangle| = |(\gamma, x)| = ||y|| ||x|| \]

And \( ||\hat{x}|| = ||x|| \). But

\[ ||x||^2 = \langle x, x \rangle = \langle \hat{x}, x \rangle \leq ||\hat{x}|| ||x|| \]

Every linear functional arises in this manner, to be proved.

**Theorem (Riesz):**

Let \( l \in H^* \). Then there exists a unique \( x_l \in H \)

such that \( l(x) = \langle x, x_l \rangle \) for \( x \in H \).

**Proof:**

**Uniqueness:**

Suppose \( \langle x, x_l \rangle = \langle x, x' \rangle \) for all \( x \). Then \( (x, x_l - x') = 0 \).

Hence \( x_l - x' = 0 \).

**Existence:**

If \( l = 0 \), take \( x_l = 0 \). If \( l \neq 0 \), then \( b = l(x_l) \) is a closed maximal proper subspace of \( H \). Take vector
not in kernel and take perpendicular to it. Make it have norm 1. Thus find $x \in H$ with

$$x \perp \ker \ell \quad \text{and} \quad \|x\| = 1.$$  

Then

$$H = \ker \ell + \mathbb{C}x.$$

Let $x_0 = \ell(x) x$.

Take $z \in H$, $z = y + \alpha x$. Then

$$\ell(z) = \ell(y + \alpha x) = \alpha \ell(x) = \alpha \ell(x_0) \ell(x) = \alpha \ell(x) \ell(x) (x, x) = \alpha \ell(x) x, \ell(x) x) = (\alpha x_0, x_0) = (z, x_0) \quad \text{Q.E.D.}$$

Make $x \rightarrow \hat{x}$ is conjugate linear, is onto by (3), and is norm preserving. Hence $H$ is homeomorphic to $H^*$. Weak topologies are also homeomorphic.
$L^* \cong X^*/L^\perp$ is an isometry

$f + L^\perp \mapsto f|L^\perp$, where $f \in X^*$.

Norm preservation:

$\|f + L^\perp\| = \inf_{g \in L^\perp} \|f + g\|$

$(f + g)|L^\perp = f|L^\perp$, $\|f + g\| \geq \|f|L^\perp\|$

$= \|f|L^\perp\|$.

If $f \in L^+$, by (11B) there is an $f \in X^*$ such that

$f|L^\perp = f$ and $\|f\| = \|f|L^\perp\|$.

Then $f \in P^{-1}[E_0^+]$. Here equality holds and

$\|f + L^\perp\| = \|f|L^\perp\|$.

**Theorem:** If $M$ is a closed subspace of a Hilbert space $H$, then

$M \oplus M^\perp = H$.

**Proof:**

We know $N = M \oplus M^\perp$ is closed. $N \subseteq M$, $N \subseteq M^\perp$

$N^\perp \subseteq M$

$N^\perp \subseteq M \cap M^\perp = \{0\}$

$N = N^\perp = H$.

**Example:**

$l_2 = \{x = (x_1, x_2, \ldots) | x_n \in C, \Sigma |x_n|^2 < \infty\}$

$(\alpha, \beta) = \Sigma \overline{x_\alpha} \beta_n$

**Definition:** A subset $\{e_n | x \in A\}$ of $H$ is called **orthonormal** if and only if $(e_n, e_m) = \delta_{nm}$. A basis of $H$ is a maximal orthonormal subset.
Theorem: Any orthonormal set is contained in some basis of \( H \).

Proof: By von Neumann's theorem.

Corollary: Every Hilbert space has a basis.

Theorem: Let \( \{e_1, e_2, \ldots, e_n\} \) be a basis of \( H \). Then for \( x \in H \), at most a countable number of \( (x, e_n) \)'s are not zero.

Let \( (x_1, x_2, x_3, \ldots) \) be such that \( (x, e_n) = 0 \) for \( n \neq \{x_1, x_2, \ldots\} \)

Then \( x = \sum \frac{(x, e_n)}{|(x, e_n)|^2} e_n \), and

\[
\|x\|^2 = \sum |(x, e_n)|^2, \quad \text{Parseval}
\]

Proof:

Let \( F \) be finite in \( A \). Then

\[
0 \leq \|x - \sum_{\alpha \in F} (x, e_\alpha) e_\alpha \|^2
= (x, x) - \sum_{\alpha \in F} (x, e_\alpha)(x, e_\alpha) - \sum (x, e_\alpha)(e_\alpha, x) + \sum |(x, e_\alpha)|^2
= \|x\|^2 - \sum |(x, e_\alpha)|^2
\]

Hence \( \sum |(x, e_\alpha)|^2 \leq \|x\|^2 \)

This means that at most a countable number of \( (x, e_n) \)'s are non-zero, but then be contained in \( (x, e_\alpha) \) for \( 1 \leq \alpha \leq n \).

Then \( \sum_{1 \leq \alpha \leq n} |(x, e_\alpha)|^2 \leq \|x\|^2 \)

Hence \( \sum_{1 \leq \alpha \leq n} |(x, e_\alpha)|^2 \leq \|x\|^2 \), Bessel's inequality

which holds for any sequence of orthonormal elements.
Lemma: If \( \{ e_\nu \} \) is orthonormal in \( H \) and \( \sum |x_\nu|^2 < \infty \),
then \( \sum x_\nu e_\nu \) exists and \( \| \sum x_\nu e_\nu \| = \sum |x_\nu|^2 \).

Proof:

Look at \( \| \sum x_\nu e_\nu - \sum x_\nu e_\nu \|_2 \)
\[ = \| \sum (x_\nu - x_\nu) e_\nu \|_2 \]
\[ = \sum |x_\nu - x_\nu| \rightarrow 0. \]

Hence \( \sum x_\nu e_\nu \) converges, say to \( \sum x_\nu e_\nu \). (Completion)

Now \( \| \sum x_\nu e_\nu \| = \sum |x_\nu|^2 \)

By continuity norm of left converges to norm of right.

End of Lemma.

So form \( x' = \sum (x, e_\nu) e_\nu \) and suppose \( x' \neq x \).

\[ (x' - x, e_\nu) = \begin{cases} 0 & \text{for } \nu \neq \nu_0 \\ (x'_\nu e_\nu) - (x_\nu e_\nu) = 0 & \text{for } \nu = \nu_0. \end{cases} \]

\[ (x', e_\nu) = (x, e_\nu) \]

from sum by continuity of inner product.

Now \( \sum \frac{x' - x}{\| x' - x \|} \) is orthonormal.

By maximality of \( B \), \( x' = x \).

Penrose follows from relation for \( x' \). Q.E.D.
Let $H$ be a Banach algebra.

For $x \in H$:

$$||x|| = \sup_{||y||=1} |(x, y)|$$

By Riesz theorem together with HB.

If $A$ is an operator,

$$||A|| = \sup_{||x||=||z||} |(Ax, y)|$$

(shown easily)

Theorem: If $A \in H$, there is exactly one $A^* \in \mathcal{L}(H^*)$ and

$$(A^*x, y) = (x, A^*y)$$

for all $x, y \in H$.

Proof:

For fixed $y$ let $x \in H \rightarrow (Ax, y)$.

This is linear and bounded:

$$|(Ax, y)| \leq ||A|| ||x|| ||y||$$

By Riesz, map is $x \rightarrow (x, A^*y)$, uniquely.

For each $y$ we have

$$(Ax, y) = (x, A^*y)$$

for all $x$.

Now

$$(x, A^*(y^* + z)) = (Ax, y^* + z)$$

$$= \bar{x} (Ax, y) + (Ax, z)$$

$$= (x, A^*y) + (x, A^*z)$$

$$= (x, A^*y) + (x, A^*z)$$

$$= (x, A^*y + A^*z)$$

for all $x$.

So

$$A^*(y + z) = (A^*y, z)$$

Hence linearity.

For boundedness,

$$||A^*|| = \sup_{||y||=1} |(A^*y, z)|$$

from above.

$$= \sup_{||z||=1} |(Az, A^*y)|$$

$$= \sup_{||y||=1} |(A^*y, z)| \leq ||A|| ||x|| ||y|| = ||A|| ||x||$$
Thus $\|A^*\| \leq \|A\|$

Now $A^* = A$ trivially. Hence $\|A\| = \|A^*\|$. Q.E.D.

$A^*$ is called the adjoint of $A$.

Identities

1) $(\alpha A + \beta B)^* = \overline{\alpha} A^* + \overline{\beta} B^*$

2) $(AB)^* = B^* A^*$

A is called self-adjoint if and only if $A = A^*$

Note: For any $A$, $A = B + iC$ with $B$ and $C$ self-adjoint

$c = \frac{A - A^*}{2i}$ and $b = \frac{A + A^*}{2}$ are self-adjoint.

$U$ is unitary if $UU^* = 1 = U^* U$
Discussion of adjoint spaces

Consider space of continuous functions on compact space with norm \( \sup |f(x)| \).

Problem reduces to discussion of measures.

Valuation - additive set function defined on a kind of class of sets to be specified. Characteristic functions of these sets give positive linear functionals \( \chi_A(x) = 1 \text{ or } 0 \) for simple functions.

Extend by \( \tilde{\mu} \) if \( \tilde{\mu} \), \( \overline{\epsilon} \text{ and } \epsilon \), \( E(\overline{\epsilon} - \tilde{\epsilon}) < \epsilon \), then \( E(f) \) is defined uniquely.

We call this \( E(f) = \int f(x) \mu(dx) \).

Given a functional.

Take \( f \) and let \( \mathcal{A} = \{ x \mid f(x) > 0 \} \).

\( f \rightarrow \chi_{\mathcal{A}} \).

Also \( E(f_{\mathcal{A}}) \geq \chi_{\mathcal{A}} \) and \( \chi_{\mathcal{A}} \) will be a functional. Then integrate.

Measures lead to functionals.

Let space be locally compact.

The compact space \( E(1) \) is finite, so that \( 0 \leq f \leq 1 \| f \|_1 \) and \( E(f) \leq \| f \|_1 E(1) \).

If postulate that compact sets have finite measure, take a sequence \( \tilde{\mu} \), increasing to the whole space, and problem is easy. We get all \( \sigma \)-finite measures this way.

Another example:

Bounded harmonic functions in the disk. \( \text{NASC} \) is \( u_p = \text{Average } u \).

Under sup norm, they form a Banach space.

Completeness: Uniform limit is continuous and

\[ u = \lim u_n = \text{lim a.e. } u_n = \text{a.e. } u \]

Uniform limit of harmonic functions is bounded harmonic.
Subspace of bounded cont. funs on space. We do not get an algebra; product of harmonic functions is harmonic. So no $f^m$ and $X$ does not have to be a monostone limit.

For cont. funs, $f \leq g$ and $f + g$ are in. (locally max and min).

Cont. funs with $v$ and $w$ form a "Priy space" (linear lattice).

Is every two functions there is a unique "maximum"?

For harmonic funs $f \leq g$ is the unique harmonic fun greater than $f$ and $g$. Similarly for $f + g$. We get a linear lattice. Ordering

$$f \leq g$$ if $f + g = g$.

Measure theory (including passing to limit) can be done from lattice point of view. Get Daniell integral.

Properties indicate that some lattices are monomorph to continuous funs on a compact space. (This done is actually a proper measure theory on the unknown compact space).

Poisson integral = positive cont. fun on boundary $\leftrightarrow$ unique bounded harmonic (fun in disk) for example.

Probability problem in reverse: Given harmonic funs and obtain boundary. Fuc inside corresponds to max $(f, g)$ on boundary. Problem obtaining proper boundary.
Denumerable space

Natural topology

Function is a column vector infinite in both directions.

Attach to m two positive numbers \( p + q = 1 \). Neighboring are \( m+1 \)
and \( m-1 \).

Average operator is \( y_n = p x_{n+1} + q x_{n-1} \), or \( y = T x \).

Look at functions \( x \) such that \( x = T x \); these are harmonic functions.

The function 1 is a solution. There are no more than two solutions
since values at two points lead to a recursion equation.

We consider only bounded solutions, which we may assume non-negative.

Random walk

Path space is a function space

We know how to assign probabilities to intervals: product of \( p \)'s and \( q \)'s.

One of the problems is to ask for probabilities on a larger space.

Situations

1. \( p_n < q_n \) for \( n > 0 \), drift from integers toward zero

2. \( p_n > q_n \) also for \( n > 0 \), drift toward origin. If drift is strong
   none never, then is no boundary and 1 is only solution

3. Drift in one direction

4. Drift outward, probabilities associated with going to + and = \( \infty \).

Let \( i \) be any point \( > 0 \).

What is probability that process never reaches origin. Call it \( y_i \).

\[ y_i \text{ defined for } i > 0 \]

\[ y_i = p_i y_{i+1} + q_i y_{i-1} \text{ for } i > 1 \]

\[ y_1 = p_1 y_2 \]

Does there a solution to the system? If there is no solution \( y \neq 0 \)

then there is no drift. If solution exists it is unique \( y \)

\( y \)'s are monotonically increasing. If non-trivial solution make

limit one provided it is bounded.
\[
\frac{\Delta x_{i+1} - \Delta x_i}{\Delta x_i - \Delta x_{i-1}} = \frac{q_i}{p_i}
\]

By induction
\[
\Delta x_{i+1} - \Delta x_i = \Delta x_i \frac{q_i}{p_i} \frac{p_i}{p_2} \ldots \frac{p_i}{p_i}
\]

For boundedness let
\[
P_0 = \frac{q_0}{p_0}, \quad \text{question is whether}
\]
\[
\sum_{k=1}^{\infty} p_k = P_k \quad \text{converges.}
\]

NASC

Product with \( q_i \) should be 1 if converges.

Shifting indices gives \( P_0 \) never reach \(-1\). Pass to limit to get

\(-\infty\) value. Set \( P_0 = \frac{P_n}{P_k} \) and get symmetric condition

Let
\[
\sigma_k = P_k + P_k P_{k+1} + P_k P_{k+1} P_{k+2} + \ldots
\]

\[
S_{n+1} = 1 + \sigma_k + P_{k-1} \sigma_{k-1} + \sigma_{k-1} \sigma_{k-2} + \ldots
\]

If both series converge let
\[
\frac{S_k}{S_k + \sigma_k} = \frac{S_k}{S_k + \sigma_k}
\]

Set
\[
S_{k+1} + \sigma_{k+1} = \frac{S_k (S_k + \sigma_k)}{S_k + \sigma_k}
\]

and this is a solution.

Probability of drift to right is harmonic, \( 1 - \gamma \) it is the probability in the other direction.

Lattice is linear combinations of \( y \) and \( 1 - y \). Attach weights to right and left points, like Poisson integral; harmonic function come from boundary values.

Another example: three lines
\[
\begin{align*}
\ldots & \quad \ldots \\
& \quad \ldots \\
\ldots & \quad \ldots
\end{align*}
\]
\[
\begin{align*}
\ldots & \quad \ldots \\
& \quad \ldots \\
\ldots & \quad \ldots
\end{align*}
\]
If $\exists q_n < \infty$, then only finitely many steps backward and hence process stays on top or bottom line from some point on. And conversely. Boundary will be one or two points. But this is not a compactification if there is oscillation (then one pt.)

Harmonic functions in disk

Defined by integral over largest circle divided by area. Uniform probability in disk, single out functions which on 1 or on one are, 0 elsewhere; associate are with function Build up boundary from it.

Continuous functions on a space $\cap U$, positivc minimum and maximum

Properties (abstract lattice):

1) $x \land x = x$ idempotent
2) $x \land y = y \land x$ commutative
3) $x \land (y \lor z) = (x \land y) \lor z$ associative
4) $x \land (x \lor y) = x$ absorption law

Partial ordering properties (1)

$x \leq y$ means $x \land y = x$ and $x \lor y = y$

Riesz space - a linear lattice (over the reals)

Compatibility conditions:

$x > 0, y > 0 \Rightarrow xy > 0$

$x \leq y \Rightarrow x + z \geq y + z$

Properties derived:

1) $(x + z) \land (y + z) = (x \land y) + z$
2) $xy \leq xz \Rightarrow (x \lor y)$ and $\lor$ for $x > 0$
3) $x \geq y$ implies $-x \leq -y$
4) $(x) \lor (-y) = -(x \land y)$
IV. \[ x + y = x \cup y + x \wedge y \]
\[ x \wedge (y \cup z) = (x \wedge y) \cup (x \wedge z) \]
\[ x \cup (y \wedge z) = (x \cup y) \wedge (x \cup z) \]

**Proof:**

1) \[ x + (y - x \wedge y) \geq y \]
\[ x + (x - x \wedge y) \geq y \]
and hence the maximum \[ x \cup y \]
\[ x - (x \cup y - y) \leq x \]
and hence the minimum \[ x \wedge y \]

Here (1) holds.

2) and (3) are dual
\[ (x \wedge y) \cup (x \wedge z) \leq x \]
\[ y \wedge z \]
Here \( (x \wedge y) \cup (x \wedge z) \leq x \wedge (y \wedge z) \)
\[ (x \wedge y) \cup (x \wedge z) = (x + y - x \cup y) \cup (x + z - x \cup z) \]
\[ \geq (x + y - x \cup y \wedge z) \cup (x + z - x \cup y \wedge z) \]
\[ = y \wedge z + x - x \cup y \wedge z \]
\[ = x \wedge (y \wedge z) \]

Here (2)

In a Reed space define \[ x^+ = x \cup 0, \quad x^- = -x \wedge 0 \]
\[ |x| = x^+ + x^- \]

**Properties**

1) \[ x = x^+ - x^- \]
\[ x = x + 0 = x \cup 0 + x \wedge 0 = x^+ - x^- \]

2) \[ x^+ \wedge x^- = 0 \]
\[ x^+ \wedge x^- = (x + x^-) \wedge x^- = (x \wedge 0) + x^- = -x^- + x^- = 0 \]

3) \[ x \cup x^+ = x^+ \]
\[ x \wedge x^+ = x \]

4) \[ x \geq 0 \text{ implies } x = x^+ = |x|, \quad |x + y| \leq |x| + |y| \]
5) $x^+ - x^+ = |x|$

Proof of inequality:

$|x| + |y| \geq (x^+ + y^+) \cup (x^- + y^-) = |x+y|$

Positive elements form a cone $C$

We have shown $X = C \cup (-C)$

A Riesz space is a Banach lattice if it is normed and such that

1) if $0 \leq x \leq y$, then $\|x\| \leq \|y\|$

2) $\|1_{x}\| = \|x\|$.

Let $X$ be a Riesz space. A linear functional is a real-valued $f$ such that $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$. $f$ is positive if $x \geq 0$ implies $f(x) \geq 0$.

A functional is lattice-bounded if $\sup_{0 \leq x \leq 1} |f(x)| < M_x$.

Proof: if $f$ is lattice-bounded, then $f = f^+ - f^-$, where $f^+ \geq 0$, $f^- \geq 0$. Conversely, if such a decomposition exists, $f$ is lattice bounded.

Proof: if $f \geq 0$, $f$ is bounded. $y \leq x$ implies $x = y + z$ and $f(x) = f(y)$. If decomposition exists, $f$ is lattice bounded. Conversely, define $f^- (-x) = \sup_{0 \leq a \leq x} f(ay)$ and similarly for $f^+$.

$f^+(x_1 + x_2) = \sup_{0 \leq a \leq x_1 + x_2} f(a) \geq \sup_{0 \leq a \leq x_1} f(a + x_2) = f^+(x_1 + x_2)$. 

$f^-(x_1 + x_2) = \sup_{0 \leq a \leq x_1 + x_2} f(-a) \leq \sup_{0 \leq a \leq x_1} f(-a + x_2) = f^-(x_1 + x_2)$.
\[ f^+(x_1) + f^+(x_2) \]

In reverse inequality, for \( 0 \leq y \leq x \) define

\[ y_1 = y \cap x_1 \]
\[ y_2 = y - y_1 \]
\[ y_2 = y - (y \cap x_1) = x_1 \cup y - x_1 \]

since \( x_1 + y = x_1 \cup y + x_1 \cap y \)

\[ x_1 u (x_1 + x_2) - x_1 \]
\[ = x_2 \quad \text{and} \quad y_2 \leq x_2 \]

Now \( f^+(x_1 + x_2) = \sup_{0 \leq y \leq x_1 + x_2} (f(y_1) + f(y_2)) \)
\[ \leq \sup_{y_1 \leq x_1} (f(y_1) + f(y_2)), \text{ etc.} \]
\[ \sup_{y_2 \leq x_2} \]

Hence linearity. Etc.

This definition gives the minimal decomposition

lattice-bounded functionals therefore themselves form a thing
space, the adjoint space.
Now on adjoint space
If \( x \geq 0 \) and \( x^* \geq 0 \) and \( 0 \leq y \leq x \), then \( x^* y \leq x^* x \) for lattice bounded functionals

**Lemma:**
If \( x_n \to x \) in the norm of the Banach space, then \( x_n e \to x \to a \)

**Corollary:**
If \( x_n \geq 0 \) and \( x_n \to x \), then \( x \geq 0 \).

**Proof of Lemma:**

\[
|x-y| = x y - x y
\]

\[
1 x y = (x-y) u 0 - (x-y) n 0
\]

\[
= (x y) - y - (x y) y
\]

by lifting property

Then \( |x y - y u a| + |x n a - y n a| = |x-y| \) by distributive law

Hence \( |x-y| \geq |x n a - y n a| \)

So \( |x_{n a} - x n a| \leq |x_{n} - x| \)

\[
\| (1 x_{n a} - x n a) \| \leq \| (1 x_{n} - x) \|
\]

\[
= \| x_{n} - x \|
\]

\[
\to 0
\]

**Other norm condition on \( x^* = f \)**

\[
\| f \| = \| (1 f) x \|
\]

\[
w = f^+ + f^-
\]

Want \( \| f \| = \| w \| \)

\[
|f(x)| \leq f^+ (1 x) + f^- (1 x) = w (1 x) \leq \| w \| \| (1 x) \|
\]

\[
\to \| f \| \leq \| w \|
\]

**Other direction:**
Take \( \| x \| = 1 \) and \( x \) such that \( w (x) \geq \| w \| - \epsilon \).

Assume \( x > 0 \) without loss of generality. Then there is a \( y \)

\( 0 < y < x \) such that \( f(y) > f^+(x) - \epsilon \) by def. of \( f^+ \)
Write \( f = f^+ - f^- \), so that
\[
\|f(x-y)\| \leq f^-(x) + \epsilon \\
f(2y-x) \geq m(x) - 2\epsilon
\]

Also \((2y-x)^+ \leq x\)
and \((2y-x)^- \geq x\)
so that \(\|2y-x\| \leq x\)
\[
\|f(2y-x)\| \leq \|f\| \|2y-x\| \leq \|f\| \|x\| = 1
\]
\[
f(2y-x) \leq \|f\| \|2y-x\| \leq \|f\| \|x\|
\]
Hence \(\|f\| \geq m(x) - 2\epsilon \geq \|m\| - 3\epsilon\) Q.E.D.

Continuous functions on a compact space \( X \), want functionals on set.
Let \( u \) be cont. on \( X \). Want to prove exists measure such that
\[
x^*(u) = \int u(x) \, d\mu(x)
\]
for \( x^* \geq 0 \)

Will extend class of functions in domain of \( x^* \) and show there are
many characteristic functions in sets. Then we define \( \mu(A) = x^*(X_A) \).

For simple functions \( x^*(\sum c_i X_{A_i}) = \sum c_i \mu(A_i) \).

Extension will work for functions \( u \) such that
\[
\bar{c} < u < \underline{c}
\]

We shall start with a family of functions \( B \) and consider smallest family
closed under +, x, limits.

But we may want to enlarge class for certain fixed \( x^* \)'s so that if
\( 0 \leq f \leq g \) and \( x^*(g) > 0 \), then \( x^* f = 0 \).

Cantor function has property that any uncountable set is the image of a set of
measure zero.
A family of subsets of a set $X$ is an algebra if it is closed under finite unions and complements and if $X \in \mathcal{A}$, $\emptyset \in \mathcal{A}$.

$\mathcal{A}$ is a monotone class if together with any monotone (ascending) sequence of sets it contains the limit.

If $\mathcal{A}$ is an algebra and $B_n \in \mathcal{A}$ implies $\bigcup B_n \in \mathcal{A}$, then $\mathcal{A}$ is a $\sigma$-algebra.

$\lim B_n = \text{set of points in } B_n \text{ infinitely often} = \bigcap_{n=1}^{\infty} B_n$

If $R_n = B_n \cup B_{n+1} \cup \cdots$, then $R_n \uparrow \lim B_n$

$\lim B_n = D_n = B_n \cup B_{n+1} \cup \cdots$. Then $\bigcup D_n \uparrow \lim B_n$

$B_n \to B$ if $\lim B_n = B$

If $B$ is a family of functions, $B$ is a monotone class if it is closed under pointwise limits. $B_0$ is the smallest $\sigma$-class containing $B$.

$\mathcal{F}_{\infty}$: For $A \in \mathcal{F}_{\infty}$, $\mu(A) > 0$. Whenever $A_i : n \in A = \emptyset$ for $i \in \mathbb{K}$,

$\sum_{i=1}^{\infty}\mu(A_i) = \mu(\bigcup_{i=1}^{\infty}A_i)$.

Example: discrete space and measure

Monotone condition is equivalent to complete additivity

Completion: Adjoin all sets $B \Delta \Gamma$, where $\Gamma \in A$, $A,B \in \mathcal{F}_{\infty}$, $\mu(A) = 0$.

The set forms a $\sigma$-ring. $\mathcal{F}_{\infty}$ is a $\sigma$-algebra $(X, \mathcal{F}_{\infty}, \mu)$

Consider the product space $\mathbb{R}^{[0,1]}$

All basic sets containing $0$ are assigned measure 1; others are assigned measure 0.

The $\mathcal{F}_{\infty}$-$\sigma$-algebra consists of those sets describable by countably many coordinates.

The set $\{0\}$ is not in the $\sigma$-algebra, even after completion. It is not a $G_\delta$ and Baire measures are insufficient.

The measure assigning weight one to $\{0\}$ and extended is a Borel measure.
For a finite algebra $B$, if $f$ and $g$ are elements of $B$, then $\mu(f) + \mu(g) = \mu(f + g)$.

Every measure on an algebra can be extended to the smallest $\sigma$-algebra containing the algebra.

$$(X, \mathcal{F}, \mu).$$

For $A \in \mathcal{F}$, set

$$\mu(\Omega) = \sup_{A \subseteq \Omega} \mu(A)$$

$$\mu(\Omega) = \inf_{A \subseteq \Omega} \mu(A)$$

The sets in $\mathcal{F}$ are those for which $\mu(A) = \mu$.

If $A_1, A_2, \ldots$ define a simple function to be one satisfying $f = \sum_{i=1}^{N} c_i \chi_{A_i}$ and set

$$E(f) = \sum_{i} c_i \mu(A_i)$$

Functions which can be approximated by simple functions are those which, for all $\epsilon > 0$, $\exists \delta > 0$ such that for all $\delta > 0$ there exists a simple function $f$ such that $|f(x) - f(y)| < \delta$.

For $f \geq 0$ let

$$A_{\epsilon} = \{x \mid (a_{\epsilon}) \leq f(x) \leq \epsilon \leq c\}.$$ Then $\forall A_{\epsilon} \neq \emptyset$ and $f$ is bounded.

If only finitely many $A_{\epsilon}$ are non-empty, let

$$\pm = \sum_{\epsilon \in \mathbb{N}} c \chi_{A_{\epsilon}}$$

$$E = \sum_{\epsilon \in \mathbb{N}} c \chi_{A_{\epsilon}}$$

Class of these functions is monotone containing the indicator functions

$$C = \{f \mid f_n(x) \geq 0 \} \in \mathcal{F}_0$$

$$C \to C$$

It is the smallest such class.

Let $L$ be the set of positive linear functionals on $C$. For $x \in X$ and $f \in L$, note that $E \in \mathcal{F}$.

Let $f(x) = 1$, and that if $f_n \to 0$, then $E(f_n) \to 0$.

With the multiplication in $L$, $f_n$ is defined or can be approximated if $f \in \mathcal{F}$. Let

$$L = \{x \mid f(x) > 0 \}$$

For some $f \in \mathcal{F}$

$$f \to \text{indicator} \to 1,$$ so that measure can be defined since the class is monotone.
Consider the class of continuous functions on a compact space $X$.

$E(\mathcal{F}) \rightarrow \text{measure of } \{ x \in \mathcal{F}(x) > 0 \}$

But some open sets are not representable in the form $\{ x \in \mathcal{F}(x) > 0 \}$, as we have seen.
$X$ compact. Additive functional $E$ on cont. func., assumed non-negative.
$E(1) = 1.$

Closed sets:
$\{ x \mid f(x) > 0, \text{ some } f \text{ with } 0 \leq f \leq 1 \}$

Union of denumerably many
$A_n$ defined by $f^*_{A_n}$
$\bigcup A_n$ defined by $\sum \frac{1}{2^n} f_{A_n}$

Every one of them is open, but the class is not closed under non-denumerable unions.
In a countable-base space, one can talk just about open sets.

Basic sets: smallest $\sigma$-algebra generated by open sets.
Borel sets:

Define non-negative set function
$\mu(A) = \sup \{ E(f) \mid f \text{ conlin } A_n, 0 \leq f \leq 1; f(x) = 0 \text{ if } x \notin A \}$

(f vanishes outside $A$)

If $f > 0$ on $A$,
$\mu(A) = \lim_{n \to \infty} E(f_n)$

Lemma: If $f_n \uparrow \phi$, $g_m \downarrow \psi$, $f_n$ and $g_m$ continuous, then $\lim E(f_n) = \infty$ and $\lim E(f_n) = \lim E(g_m)$

Remark: This shows the remark above.

Proof: $g_m$ fixed, $f_n \uparrow \phi$, $g_m \downarrow \psi \Rightarrow f_n \uparrow \phi \wedge g_m = \phi \wedge g_m$.
So automatically uniform convergence. Hence $E(f_n) \to E(\phi \wedge g_m)$ and
$\lim E(f_n) = E(\phi \wedge g_m)$ for every $m$.
Then
$\lim E(f_n) = \lim E(\phi \wedge g_m)$

Lemma: If $A = \bigcup A_n$, $A_n$ a countable sets, then $\mu(A) \leq \sum \mu(A_n)$

Equality holds if sets are disjoint.
Proof:
To a fix \( f_n \leq 1 \) outside \( A_n \). Define \( f = \sum \frac{1}{m} f_n \).

If \( \sum \frac{1}{m} f_n \) is finite, we are done. Otherwise,
and \( \frac{a^m + b^m}{2} \geq (a+b)^{1/m} \) so that

\[ P^{1/m} \leq \sum \left( \frac{1}{m} f_n \right)^{1/m} \rightarrow \sum \mu(A_n) \]

where finite partial sums.

For non-overlapping sets, the reverse inequality is easy.

Null set: a set which can be covered by a countable set of measure
less than \( \varepsilon \) for any \( \varepsilon \).

Union of denumerably many null sets is null.

Egoroff's theorem:

Let \( f_n \) be a sequence of cont. func. \( f_n \geq 0 \) converging to \( f \) finite.
The convergence is uniform except on a set contained in a

Uniform set of measure \( \varepsilon \).

Introduce \( \| f \|_1 = E(1f) \) and make continuous functions a normed space.

L^1 norm. Take Cauchy sequence in L^1 norm, call it \( f_n \). We
want the abstract limit to be interpreted as function. There
exists a subsequence \( f_{n_k} \rightarrow f \) converging uniformly except
on a set of measure less than \( \varepsilon \). If \( \| f \|_1 \rightarrow 0 \), then

\( f = 0 \) on that set. (Convergence in measure implies subsequence
converging pointwise on set of measure \( 1 - \varepsilon \))

Remark: All representatives \( f \) agree except on null set.

Identify limit with the class. Conversely, if convergence
is as above, \( E(\text{limit}) \) is uniquely defined.
Example:

\[ \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \ldots \]

Converges in norm, not pointwise.

Proof:

Consider \( A = \{ x \mid f(x) > a \} \); this is a contour set. If associated \( g \leq 1 \), then \( g \leq \frac{a}{f} \) and hence

\[ m(A) \leq \frac{1}{a} E(f). \]

Fix a \( N \).

Pick \( \gamma_i \) such that \( ||f_{m_i} - f_{n_i}|| < \frac{1}{N^i} \), \( m, n \geq \gamma_i, \gamma_i \) arbitrary.

(Shows convergence geometric).

Look at

\[ f_{m_0} + (f_{m_1} - f_{m_0}) + \ldots + (f_{m_i} - f_{m_{i-1}}) = f_{m_i} \]

The

\[ A_n = \left\{ x \mid |f_{m_n}(x) - f_{m_{n-1}}(x)| > \frac{1}{M^n} \right\} \]

\[ m(A_n) \leq N^n E\left( ||f_{m_n} - f_{m_{n-1}}|| \right) \]

\[ \leq \frac{M^n}{N^{n-1}} \]

Outside \( \bigcup_{n} A_n \), the series \( \sum_{n=1}^{\infty} (f_{m_n} - f_{m_{n-1}}) \) converges uniformly by M-test. Difference bounded by \( \sum \frac{1}{N^n} \) except for finitely many terms. And this completes the proof; there exists \( \phi \) such that \( \phi = 0 \) on the big set.
\[ \|p\|_1 = E(\|p\|_1), \text{ for } p \text{ Cauchy in norm.} \] Then for any \( \varepsilon > 0 \) and \( A \) with \( \mu(A) < \varepsilon \) and \( f_n \to f \) uniformly.

Take \( \varepsilon > 0 \). Find \( A_n \) and use subsequences and diagonals to get a subsequence that converges a.e. pointwise. But uniform convergence is stronger.

Now plays only a small role. We need only "convergence in measure". Look at \( \{x : |f_n(x) - f_m(x)| > \varepsilon\} \). If \( \mu \) of this \( \to 0 \), then \( f_n \) forms a Cauchy sequence in measure. Integral norm convergence \( L^1 \) enforces this convergence. We have convergence in measure implying subsequences converge a.e. pointwise.

Any two subsequences converge to equivalent \( P \)'s by corollary.

Similarly if \( f_n \) and \( g_m \) are equivalent.

So every Cauchy sequence has an equivalence class of \( P \)'s with it.

Example:

Take \([0, 1]\). Write \( x = \xi_1 \xi_2 \cdots \) with \( \xi_n \equiv 0 \text{ or } 1 \). Have sequence of \( f_n \) representing number of ones \(-n/2\). Call them \( f_n \). Look at \( f_n \).

Pick an \( N \). For \( \varepsilon \) small, the set \( \{x : |f_n(x)| > \varepsilon \} \) is small. Strong law of large numbers gives a.e. convergence. From CLT

\[ \frac{f_n}{n} \to 0 \quad \text{in measure of } A_n \to 0 \]

\[ \Pr \left( \frac{f_n}{n} > \varepsilon \right) \text{ is small. But } \lim_{n \to \infty} \frac{f_n}{n} \text{ pointwise oscillates, and convergence in pointwise almost nowhere. } \lim_{n \to \infty} \frac{f_n}{n} = \frac{1}{2} \text{ at most points.} \]
We have a large class of forms and we can associate $E(f) = \lim E(f_n)$. If $f_n$ is Cauchy, then $f^+_n$ and $f^-_n$ are Cauchy since

$$|f(x) - g(x)| \geq |f^+(x) - g^+(x)| \quad \text{in all cases}$$

Suppose $f_n \uparrow f$. If $E(f_n)$ is bounded, then $f_n$ is Cauchy. If $f_n \downarrow f$, then $E(f_n)$ is Cauchy.

Let $f_n$ arbitrary $\to 0$ converge to $f(\mathbf{x})$. If $\forall n, f_n f_{n+1} \cdots f_m$ exists as $m \to \infty$, then $\lim_{m \to \infty} f_m(x) = f(\mathbf{x})$.

As $n \to \infty$, this decreases (and $f_n f_{n+1}$ is in class when $f_n$ and $g$ are). Hence $f_m \cdots f_n$ exist and is in class. Set

$$t_n = f_{n-1} \cdots f_1$$

Then $t_n \uparrow$ and \( \lim_{n \to \infty} t_n(x) = \lim_{n \to \infty} f_n(x) = f(\mathbf{x}) \), which is in class if all $E_n$ are bounded. Suppose that $f_n \leq F$ with $E(F) < \infty$.

Then $T_n$ is in class and

$$E(t_n) \leq E(f_n) \leq E(T_n)$$

and

$$\lim E(t_n) \leq \inf E(f_n) \leq \sup E(f_n) \leq \inf E(T_n) = \inf E(f_n)$$

Here if $f_n \to f$ and $f_n \leq F$, then $E(f_n) \to E(f)$

(Dominated Convergence)

Always $E(f_n) \to E(f)$ (Fatou's)

$$\lim E(f_n)$$
13, the class we have constructed, is a conditionally monotone class (bounded limits are in class).

Egoroff:

\( f_n \rightarrow f \) finite.

If \( f_n \in E \) and suppose \( f_n \rightarrow f \) pointwise, a contour set of measure \( \epsilon \) and the convergence is uniform. If \( \epsilon \) is bounded, \( f_n \) is Cauchy and can be replaced by constant. If \( \epsilon \) is not bounded, take \( f_n \rightarrow f \).

For every \( \alpha \), apply results to the truncation.

\( E \) is conditionally monotone closed under \( \cap \) and \( \cup \).

Definition of measurable sets, those such that \( X_A \in E \):

\[ X_A \cap X_B = X_{A \cap B} \]

Limits of functions \( X \), \( X \) are in \( E \) so that family contains contour sets and is closed under countable unions and intersections. Measurable sets form \( \sigma \)-algebra. Define \( \mu(X) = E(X_A) \). Every simple function is in \( E \), so every uniform limit of them is in \( E \).

If \( f \in E \), then \( \exists X(f(x) > a) \) is measurable since it is limit of \( \exists X(f(x) > a) \) for cont. \( X \). Take sets where

\[ \exists \frac{m-1}{m} \leq f(x) < \frac{m+1}{m} \Rightarrow X = A_m \]

\[ \frac{m-1}{m} = \sum \epsilon_X X_{A_m} \]

\[ \frac{m+1}{m} = \sum \mu_X X_{A_m} \]

And \( f \) is limit of these:

\[ E(f) = \int_X f(x) \mu(dx) \]

This is the Riesz theorem.
Class of functions is complete.

If $f \geq 0$, $E(f) = 0$ then $E(g) = 0$.

Restrict $B$ to $B_0$, the smallest monotone $\sigma$-field containing contour functions. These are Baire functions.

Look at $u(f, g)$, $f, g \in B_0$, $u$ defined in a region of the plane, $u$ taken as a Baire function.

Theorem: $u(f, g) \in B_0$. (Expectation need not be finite, of course)

Proof:
The smallest $\sigma$-field containing contour sets. $f \in B_0$ if

$\exists \Omega, f(x) > a \iff \exists \Omega, f(x) > a$.

To prove result suppose $u$ and $f$ are continuous. Take $\Omega_f$ family of all $g$ for which there is true.

Claim for cont. $g$, monotone limits are in class. We can use $u(f(x), g(x)) > a$. So statement is true for every Baire function. Fix $u$ cont. $g$ arbitrary, let $f$ be anything.

Repeat, etc. for $u$.

Can extend any field $E$ to class $B_0$. Can complete class for any field $E$.

For any $A$ with $X_A \in B_0$, there is $\Omega$ contour $A \in \Omega$ with $\mu(\Omega_A) < \mu(A) + \epsilon_n$. Then $\mu(\cap \Omega_n) = \mu(A)$. Every set has a $G_\delta$ bigger than it, or $F_\sigma$ smaller than it, both with same measure.
Start with cont. func.

Basic family $B_0 = \text{smallest } \sigma \text{-algebra containing cont. func.}

For $A \subseteq \mathbb{R}$, enlarge cont. func. to $B_0$ by throwing in Cauchy sequences and limits $B = B_0$

$B$ is the completion of $B_0$ with respect to $E$.

Define

$\Sigma_0 = \text{smallest } \sigma \text{-algebra containing cont. func.}$

$\Sigma = \text{sets represented by indicators in } B$

$\Sigma = \Sigma_0$

Continuous function give cont. func.

Take monotone limits once up and down to get everything in $\Sigma$ (except $\text{negligible measure}$)

For every $x \in B$ there is $\mu(B_n) \leq \mu A + \epsilon$.

If $B_n \uparrow B_0$ then $\mu(B) = \mu(A)$. To prove first statement we use original theorem.

An exceptional set $E$ for a seq of cont. func. can be substituted as

$E \in \mathcal{F}_A$ uniformly outside $E$. So suppose

$|f(x) - X_n(x)| < \epsilon \frac{1}{2}$ on $E$.

Define $B = \{ x \in E | f(x) > \frac{1}{2} \}$. Is cont. func.

Measure $\mu(B) \leq \mu(A) + \epsilon$

because $1 \leq f(x) > 1 - \epsilon$ for $x \in A \cap E'$

$0 \leq f(x) < \epsilon$ for $x \in A' \cap E'$

$E(f) \leq 1 \mu(E) + 1 \mu(A \cap E') + \epsilon \mu(A' \cap E')$

$\leq 2 \epsilon + \mu(A)$

$E_1 = \{ x | f(x) > 1/2 \} \leq \mu(A) + \epsilon$ nowhere.

Conclude.

From $E_1 \cup E$, cont. func. containing $A$. Measure is as close to $A$ as wanted. So limit of monotone sequence has same measure of $A$. 
Start with $C = \omega$.

From $C_0$ and $C_5$, motion limits up and down.

Note $C_0 = C_5$.

From $C_0$ and continue.

$C_5$

"Young classification." This process does not end but exhausts $B_0$.

$C_0$ corresponds to count set, $C_5$ completes.

To any $f$ in $B_0$, there is one $f_r$ in $C_0$ and one in $C_5$ such that integrals are same as $f$ and $f$ is squeezed between the two $f_r$s.

Banach classification

$C_1$ -- set $f$ in first class

$C_2$ -- end limits in second class

Each by transfinite induction

One limit is two monotone limits; $C_0$ and $C_5$ are in $C_2$.

We defined measure for count set, null set, finite Cauchy sequences.

Borel class is $B_0$ with class; other were Banach sets.

Start with open set $\Omega$.

$\Omega$ contains many count sets.

Define $\mu(\Omega) = \sup_{\Lambda \in \Omega} \mu(\Lambda)$. Must be $\geq \mu$.

Earthly reason to demand equality unless $\Omega$ is a count set. Equality means regular measure.
We have examples with non-Baire measurable operators.

Some definition is:

\[
\mu(D) = \sup_{0 \leq f \leq 1} E(f),
\]

Everything is the same except we get more null sets.

There are more Borel functions than Baire functions. Measure agrees on Borel sets.

As suspected before, it follows that now from an associated

Borel merely adds null sets.

Baire sets - useful in measure theory.

Borel sets - topology.

Proof of Egoroff:

Given \( \sigma \)-algebra of functions and measure. Suppose \( f_n \rightarrow f \) a.e. does not matter. Claim \( \int \eta \) not of measure \( \leq \epsilon \) outside of which convergence is uniform.

Cell \( A_{n}(\epsilon) = \{ x \mid | f_{n_{1}}(x) - f_{n_{2}}(x) | > \epsilon, \text{ some } f_{1} \text{ and } f_{2} \} \)

\( A_{n}(\epsilon) \) measure = union of measurable sets.

\( \text{Fixed } \epsilon = A_{n}(\epsilon) \downarrow \phi \) by everywhere convergence.

Hence \( \mu(A_{n}(\epsilon)) \rightarrow 0 \) (finite case used)

Take \( \epsilon_{k} \downarrow 0 \). Fixed \( \epsilon_{k} \) such that

\[
\mu(A_{n_{k}}(\epsilon_{k})) < \frac{\epsilon}{17^{k}}
\]

Set \( A = \bigcup A_{n_{k}}(\epsilon_{k}) \)
Claim convergence is uniform outside $A$. Let $\varepsilon > 0$.

$$|f_{n+p}(x) - f_{n+q}(x)| < \varepsilon \quad \text{outside } A, \quad \forall m, p, q$$

for $m$ sufficiently large. Take $m$, $p$, and $q$ such that

$$A = A_{\varepsilon_0}(\varepsilon_0)$$

Hence uniform Cauchy criterion. \(\Box\)
Radon-Nikodym idea

\( \mu \) and \( \nu \) two finite measures

\( \mu - \nu \) is signed measure

Claim: \( \mu - \nu = \alpha^+ - \alpha^- \), carried by two different sets

If \( 0 \leq \mu \leq 1 \)

\( 0 \leq \nu \leq 1 \)

Then can decompose \( \mu - \nu \) into two different sets

Do with \( X = X_1 \cup X_2 \)

on \( X_1 \) \( \mu \geq \nu \)

on \( X_2 \) \( \mu \leq \nu \)

Set all 2-norms on both sides.

A: For each \( B \subseteq A \),

\( \alpha^+ \nu(B) \geq \mu(B) \geq \alpha^- \nu(B) \)

Make ratio lie between \( \mu \) and \( \nu \).

Split space into layers

\[ 17 \leq A \leq 18 \]

\[ 16 \leq \nu(B) \leq \mu(B) \leq 17 \leq \nu(B) \]

\[ \alpha^- \leq \nu(B) \leq \mu(B) \leq \alpha^+ \nu(B) \]

Can affix \( \mu \) by \( \nu \) on that set.

On sets take \( \Sigma A \nu(A) \), integral is bigger.

Squeeze \( \mu \) between integral of two stop flows. Essentially unique.

So

\[ \int_A \sigma(x) \nu(dx) \leq \mu(A) \leq \int_A \bar{\sigma}(x) \nu(dx) \]

and

\[ \mu(A) = \int_A \sigma(x) \nu(dx) \]

true for all nets that correspond to layers. All net has to be done in look at layers for ratios 0 and \( \infty \).

0 gives set of measure zero. Look at top sets
B \subset A_m \quad \mu(B) \geq m \nu(B)

As \( n \to \infty \), either \( \mu(A_n) \to 0 \), which is fine, or
\[ \mu(A_n) \to \alpha > 0 \] while \( A_n \uparrow \Gamma \), \( \nu(\Gamma) = 0 \).

This gives singular component. We must therefore exclude \( A \cap T \). This is precisely absolute condition.

Looking at difference notion is key idea. Problem are in sets of measure zero and in checking the singular part.

Heleos deals with \( \mu \) rather than \( \mu/\nu \), to keep things

ounded.

Differentiation problem: \( \int f(x) \nu(dx) \) generalized.

If in a space we can take an atomic grid (Euclidean sense, etc.), if to every \( x \in A \uparrow \nu X \), then if \( \mu(A_n) \) has lim > \( \alpha \)

for all \( x \in B \), then \( \mu(B) \geq \alpha \nu(B) \). Same for lower limit (cover \( B \) by such sets). Take \( m \) and \( n \) rational

\( m \geq n \). If \( \lim n \) and \( \lim l \leq n \), then that

observation gives contradiction unless both \( \mu \) and \( \nu \) are

gzero on \( m \). Take all pairs of rationals, throw out

this null set. So except for fixed \( E \), \( \nu(E) = 0 \),
then for \( x \in E \), \( \lim \frac{\mu(A_n)}{\nu(A_n)} \) converges, say to \( f(x) \).

Then \( \mu(A) = \int f(x) \nu(dx) \) with respect to one fixed

grid. This grid give \( f \) like a.e. But want arbitrary

der sets converging to null. This is impossible hard in \( \dim > 1 \).
For discs \( \frac{1}{\pi a^2} \int \int_{D \in \mathbb{C}} \rho(x,y) \, dx \, dy \) and "functions that remain uniformly thick" (ellipses, squares, nice rectangles) can be defined. But can't for arbitrary open sets.

Gubini:

Given \( X \) and \( Y \), form \((X,Y)\)

For \( \Sigma_x, \Sigma_y, \mu_x \) and \( \nu_y \), form rectangles

\[ A \times B, \quad A \in \Sigma_x, \quad B \in \Sigma_y \]

Set

\[ \mu(A_x) = \mu(A) \times \nu(B) \]

Extend to \( \sigma \)-algebra containing rectangles. Set in product

is good if for every \( x \), section

at \( x \) is in \( \Sigma_y \) and

symmetrically. Good sets form a monotone class contain \( \sigma \)-algebra. Take

\[ \nu(A_x) = \text{fin of } x \]

This is measurable now.

\[ \int \nu(A_x) \, \mu(dx) = \text{not for } \omega \text{ over } A = \text{measure} \]

\[ = \mu(A) \nu(B) \text{ for rectangles} \]

So extends to smallest \( \sigma \)-algebra. If

\[ \omega(A) = \int \nu(A_x) \, \mu(dx) \]

Then

\[ \int \phi(p) \, \omega(dp) = \quad \text{Gubini} \]