for all $f \in C_{\text{com}}(G)$. If K is a compact set in G, we can apply (8.22) to all f that are $\geq$ the characteristic function of K. Taking the infimum shows that $d\mu_l(L_{g^{-1}}K) = d\mu_l(K)$. Since G has a countable base, the measure $d\mu_l$ is automatically regular, and hence $d\mu_l(L_{g^{-1}}E) = d\mu_l(E)$ for all Borel sets E.

A nonzero Borel measure on G invariant under left translation is called a **left Haar measure** on G. Theorem 8.21 thus says that a left Haar measure exists.

In the construction of the left-invariant m form ω before Theorem 8.21, a different basis of G would have produced a multiple of ω, hence a multiple of the left Haar measure in Theorem 8.21. If the second basis is $Y_1, \dots, Y_m$ and if $Y_j = \sum_{i=1}^m a_{ij} X_i$, then the multiple is $\det(a_{ij})^{-1}$. When the determinant is positive, we are led to orient G in the same way, otherwise oppositely. The new left Haar measure is $|\det(a_{ij})|^{-1}$ times the old. The next result strengthens this assertion of uniqueness of Haar measure.

Theorem 8.23. If G is a Lie group, then any two left Haar measures on G are proportional.

PROOF. Let $d\mu_1$ and $d\mu_2$ be left Haar measures. Then the sum $d\mu = d\mu_1 + d\mu_2$ is a left Haar measure, and $d\mu(E) = 0$ implies $d\mu_1(E) = 0$. By the Radon-Nikodym Theorem there exists a Borel function $h_1 \geq 0$ such that $d\mu_1 = h_1\, d\mu$. Fix g in G. By the left invariance of $d\mu_1$ and $d\mu$, we have

$$\begin{aligned}\int_G f(x)h_1(g^{-1}x)\,d\mu(x) &= \int_G f(gx)h_1(x)\,d\mu(x) = \int_G f(gx)\,d\mu_1(x)\\ &= \int_G f(x)\,d\mu_1(x) = \int_G f(x)h_1(x)\,d\mu(x)\end{aligned}$$

for every Borel function $f \geq 0$. Therefore the measures $h_1(g^{-1}x)\,d\mu(x)$ and $h_1(x)\,d\mu(x)$ are equal, and $h_1(g^{-1}x) = h_1(x)$ for almost every $x \in G$ (with respect to $d\mu$). We can regard $h_1(g^{-1}x)$ and $h_1(x)$ as functions of $(g, x) \in G \times G$, and these are Borel functions since the group operations are continuous. For each g, they are equal for almost every x. By Fubini's Theorem they are equal for almost every pair (g, x) (with respect to the product measure), and then for almost every x they are equal for almost every g. Pick such an x, say x_0. Then it follows that $h_1(x) = h_1(x_0)$ for almost every x. Thus $d\mu_1 = h_1(x_0)\,d\mu$. So $d\mu_1$ is a multiple of $d\mu$, and so is $d\mu_2$.

$$= \int_{N^- \times MAN} F(\bar{n}) e^{-2\rho_A \log a} \varphi(\bar{n}man)\, d\bar{n}\, d_r(man) \qquad \text{by (8.39)}$$

$$= \int_{N^- \times MAN} F(\bar{n}man) \varphi(\bar{n}man)\, d\bar{n}\, d_r(man) \qquad \text{by (8.47)}$$

$$= \int_G F(x)\varphi(x)\, dx \qquad \text{by Proposition 8.45.}$$

The proposition follows.

For an illustration of the use of Proposition 8.46, we shall prove a theorem of Helgason that has important applications in the harmonic analysis of G/K. We suppose that the reductive group G is semisimple and has a complexification $G^{\mathbb{C}}$. We fix an Iwasawa decomposition $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$. Let $\mathfrak{t}_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{m}_{\mathfrak{p}}$, so that $\mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}}$ is a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{g}$. Representations of G yield representations of $\mathfrak{g}$, hence complex-linear representations of $\mathfrak{g}^{\mathbb{C}}$. Then the theory of Chapter V is applicable, and we use the complexification of $\mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}}$ as Cartan subalgebra for that purpose. Let Δ and Σ be the sets of roots and restricted roots, respectively, and let Σ^+ be the set of positive restricted roots relative to $\mathfrak{n}_{\mathfrak{p}}$.

Roots and weights are real on $i\mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}}$, and we introduce an ordering such that the nonzero restriction to $\mathfrak{a}_{\mathfrak{p}}$ of a member of Δ^+ is a member of Σ^+. By a **restricted weight** of a finite-dimensional representation, we mean the restriction to $\mathfrak{a}_{\mathfrak{p}}$ of a weight. We introduce in an obvious fashion the notions of **restricted-weight spaces** and **restricted-weight vectors**. Because of our choice of ordering, the restriction to $\mathfrak{a}_{\mathfrak{p}}$ of the highest weight of a finite-dimensional representation is the highest restricted weight.

Lemma 8.48. Let the reductive Lie group G be semisimple. If π is an irreducible complex-linear representation of $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{m}_{\mathfrak{p}}$ acts in each restricted weight space of π, and the action by $\mathfrak{m}_{\mathfrak{p}}$ is irreducible in the highest restricted-weight space.

PROOF. The first conclusion follows at once since $\mathfrak{m}_{\mathfrak{p}}$ commutes with $\mathfrak{a}_{\mathfrak{p}}$. Let $v \neq 0$ be a highest restricted-weight vector, say with weight ν. Let V be the space for π, and let V_ν be the restricted-weight space corresponding to ν. We write $\mathfrak{g} = \theta\mathfrak{n}_{\mathfrak{p}} \oplus \mathfrak{m}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}} \oplus \mathfrak{n}_{\mathfrak{p}}$, express members of $U(\mathfrak{g}^{\mathbb{C}})$ in the corresponding basis given by the Poincaré-Birkhoff-Witt Theorem, and apply an element to v. Since $\mathfrak{n}_{\mathfrak{p}}$ pushes restricted weights up and $\mathfrak{a}_{\mathfrak{p}}$ acts by scalars in V_ν and $\theta\mathfrak{n}_{\mathfrak{p}}$ pushes weights down, we see from the irreducibility of π on V that $U(\mathfrak{m}_{\mathfrak{p}}^{\mathbb{C}})v = V_\nu$. Since v is an arbitrary nonzero member of V_ν, $\mathfrak{m}_{\mathfrak{p}}$ acts irreducibly on V_ν.

Theorem 8.49 (Helgason). Let the reductive Lie group G be semisimple and have a complexification $G^{\mathbb{C}}$. For an irreducible finite-dimensional representation π of G, the following statements are equivalent:

(a) π has a nonzero K fixed vector
(b) $M_{\mathfrak{p}}$ acts by the 1-dimensional trivial representation in the highest restricted-weight space of π
(c) the highest weight $\tilde{\nu}$ of π vanishes on $\mathfrak{t}_{\mathfrak{p}}$, and the restriction ν of $\tilde{\nu}$ to $\mathfrak{a}_{\mathfrak{p}}$ is such that $\langle \nu, \beta\rangle / |\beta|^2$ is an integer for every restricted root β.

Conversely any dominant $\nu \in \mathfrak{a}_{\mathfrak{p}}^*$ such that $\langle \nu, \beta\rangle / |\beta|^2$ is an integer for every restricted root β is the highest restricted weight of some irreducible finite-dimensional π with a nonzero K fixed vector.

PROOF. For the proofs that (a) through (c) are equivalent, there is no loss in generality in assuming that $G^{\mathbb{C}}$ is simply connected, as we may otherwise take a simply connected cover of $G^{\mathbb{C}}$ and replace G by the analytic subgroup of this cover with Lie algebra $\mathfrak{g}$. With $G^{\mathbb{C}}$ simply connected, the representation π of G yields a representation of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then of $\mathfrak{g}^{\mathbb{C}}$, and then of the compact form $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$. Since $G^{\mathbb{C}}$ is simply connected, so is the analytic subgroup U with Lie algebra $\mathfrak{u}$ (Theorem 6.31). The representation π therefore lifts from $\mathfrak{u}$ to U. By Proposition 4.6 we can introduce a Hermitian inner product on the representation space so that U acts by unitary operators. Then it follows that K acts by unitary operators and $i\mathfrak{t}_{\mathfrak{p}} \oplus \mathfrak{a}_{\mathfrak{p}}$ acts by Hermitian operators. In particular, distinct weight spaces are orthogonal, and so are distinct restricted-weight spaces.

(a) $\Rightarrow$ (b). Let ϕ_ν be a nonzero highest restricted-weight vector, and let ϕ_K be a nonzero K fixed vector. Since $\mathfrak{n}_{\mathfrak{p}}$ pushes restricted weights up and since the exponential map carries $\mathfrak{n}_{\mathfrak{p}}$ onto $N_{\mathfrak{p}}$ (Theorem 1.104), $\pi(n)\phi_\nu = \phi_\nu$ for $n \in N_{\mathfrak{p}}$. Therefore

$$(\pi(kan)\phi_\nu, \phi_K) = (\pi(a)\phi_\nu, \pi(k)^{-1}\phi_K) = e^{\nu \log a}(\phi_\nu, \phi_K).$$

By the irreducibility of π and the fact that $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$, the left side cannot be identically 0, and hence (ϕ_ν, ϕ_K) on the right side is nonzero. The inner product with ϕ_K is then an everywhere-nonzero linear functional on the highest restricted-weight space, and the highest restricted-weight space must be 1-dimensional. If ϕ_ν is a nonzero vector of norm 1 in this space, then $(\phi_K, \phi_\nu)\phi_\nu$ is the orthogonal projection of ϕ_K into this space. Since $M_{\mathfrak{p}}$ commutes with $\mathfrak{a}_{\mathfrak{p}}$, the action by $M_{\mathfrak{p}}$ commutes with this projection. But $M_{\mathfrak{p}}$ acts trivially on ϕ_K since $M_{\mathfrak{p}} \subseteq K$, and therefore $M_{\mathfrak{p}}$ acts trivially on ϕ_ν.

and $\mathfrak{t}_0$ be the respective Lie algebras. Let $m = \dim G$ and $l = \dim T$. As in §VII.8, an element g of G is **regular** if the eigenspace of $\mathrm{Ad}(g)$ for eigenvalue 1 has dimension l. Let G' and T' be the sets of regular elements in G and T; these are open subsets of G and T, respectively.

Theorem 4.36 implies that the smooth map $G \times T \to G$ given by $\psi(g,t) = gtg^{-1}$ is onto G. Fix $g \in G$ and $t \in T$. If we identify tangent spaces at g, t, and gtg^{-1} with $\mathfrak{g}_0$, $\mathfrak{t}_0$, and $\mathfrak{g}_0$ by left translation, then (4.45) computes the differential of ψ at (g,t) as

$$d\psi(X,H) = \mathrm{Ad}(g)((\mathrm{Ad}(t^{-1}) - 1)X + H) \qquad \text{for } X \in \mathfrak{g}_0,\ H \in \mathfrak{t}_0.$$

The map ψ descends to $G/T \times T \to G$, and we call the descended map ψ also. We may identify the tangent space of G/T with an orthogonal complement $\mathfrak{t}_0^\perp$ to $\mathfrak{t}_0$ in $\mathfrak{g}_0$ (relative to an invariant inner product). The space $\mathfrak{t}_0^\perp$ is invariant under $\mathrm{Ad}(t^{-1}) - 1$, and we can write

$$d\psi(X,H) = \mathrm{Ad}(g)((\mathrm{Ad}(t^{-1}) - 1)X + H) \qquad \text{for } X \in \mathfrak{t}_0^\perp,\ H \in \mathfrak{t}_0.$$

Now $d\psi$ at (g,t) is essentially a map of $\mathfrak{g}_0$ to itself, with matrix

$$(d\psi)_{(g,t)} = \mathrm{Ad}(g) \begin{array}{c} \begin{array}{cc} \mathfrak{t}_0 & \quad\mathfrak{t}_0^\perp \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & \mathrm{Ad}(t^{-1}) - 1 \end{pmatrix} \end{array}.$$

Since $\det \mathrm{Ad}(g) = 1$ by compactness and connectedness of G,

$$\det(d\psi)_{(g,t)} = \det((\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp}). \tag{8.53}$$

We can think of building a left-invariant $(m-l)$ form on G/T from the duals of the X's in $\mathfrak{t}_0^\perp$ and a left-invariant l form on T from the duals of the H's in $\mathfrak{t}_0$. We may think of a left-invariant m form on G as the wedge of these forms. Referring to Proposition 8.19 and (8.7b) and taking (8.53) into account, we at first expect an integral formula

(8.54a)
$$\int_G f(x)\,dx \overset{?}{=} \int_T \Big[\int_{G/T} f(gtg^{-1})\,d(gT)\Big] \big|\det(\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp}\big|\,dt$$

if the measures are normalized so that

$$\int_G f(x)\,dx = \int_{G/T} \Big[\int_T f(xt)\,dt\Big]\,d(xT). \tag{8.54b}$$

But Proposition 8.19 fails to be applicable in two ways. One is that the onto map $\psi : G/T \times T \to G$ has differential of determinant 0 at some points, and the other is that ψ is not one-one even if we exclude points of the domain where the differential has determinant 0.

From (8.53) we can exclude the points where the differential has determinant 0 if we restrict ψ to a map $\psi : G/T \times T' \to G'$. To understand T', consider $\mathrm{Ad}(t^{-1})-1$ as a linear map of the complexification $\mathfrak{g}$ to itself. If $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ is the set of roots, then $\mathrm{Ad}(t^{-1}) - 1$ is diagonable with eigenvalues 0 with multiplicity l and also $\xi_\alpha(t^{-1}) - 1$ with multiplicity 1 each. Hence $\big|\det(\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp}\big| = \big|\prod_{\alpha\in\Delta}(\xi_\alpha(t^{-1}) - 1)\big|$. If we fix a positive system Δ^+ and recognize that $\xi_\alpha(t^{-1}) = \overline{\xi_{-\alpha}(t^{-1})}$, then we see that

$$\big|\det(\mathrm{Ad}(t^{-1}) - 1)|_{\mathfrak{t}_0^\perp}\big| = \prod_{\alpha\in\Delta^+} |\xi_\alpha(t^{-1}) - 1|^2. \tag{8.55}$$

Putting $t = \exp iH$ with $iH \in \mathfrak{t}_0$, we have $\xi_\alpha(t^{-1}) = e^{-i\alpha(H)}$. Thus the set in the torus where (8.55) is 0 is a countable union of lower-dimensional sets and is a lower-dimensional set. By (8.25) the singular set in T has dt measure 0. The singular set in G is the smooth image of the product of G/T and the singular set in T, hence is lower dimensional and is of measure 0 for $d\mu(gT)$. Therefore we may disregard the singular set and consider ψ as a map $G/T \times T' \to G'$.

The map $\psi : G/T \times T' \to G'$ is not, however, one-one. If w is in $N_G(\mathfrak{t}_0)$, then

$$\psi(gwT, w^{-1}tw) = \psi(gT, t). \tag{8.56}$$

Since $gwT \neq gT$ when w is not in $Z_G(\mathfrak{t}_0) = T$, each member of G' has at least $|W(G, T)|$ preimages.

Lemma 8.57. Each member of G' has exactly $|W(G, T)|$ preimages under the map $\psi : G/T \times T' \to G'$.

PROOF. Let us call two members of $G/T \times T'$ equivalent, written $\sim$, if they are related by a member w of $N_G(\mathfrak{t}_0)$ as in (8.56), namely

$$(gwT, w^{-1}tw) \sim (gT, t).$$

Each equivalence class has exactly $|W(G, T)|$ members.

Now suppose that $\psi(gT, s) = \psi(hT, t)$ with s and t regular. We shall show that

$$(gT, s) \sim (hT, t), \tag{8.58}$$

and then the lemma will follow. The given equality $\psi(gT, s) = \psi(hT, t)$ means that $gsg^{-1} = hth^{-1}$. Proposition 4.53 shows that s and t are conjugate via $N_G(\mathfrak{t}_0)$. Say $s = w^{-1}tw$. Then $hth^{-1} = gw^{-1}twg^{-1}$, and $wg^{-1}h$ centralizes the element t. Since t is regular and G has a complexification, Corollary 7.106 shows that $wg^{-1}h$ is in $N_G(\mathfrak{t}_0)$, say $wg^{-1}h = w'$. Then $h = gw^{-1}w'$, and we have

$$\begin{aligned}(hT, t) &= (gw^{-1}w'T, t)\\ &= (gw^{-1}w'T, w'^{-1}tw')\\ &\sim (gw^{-1}T, t)\\ &\sim (gT, w^{-1}tw)\\ &= (gT, s).\end{aligned}$$

This proves (8.58) and the lemma.

Now we return to Proposition 8.19. Instead of assuming that $\Phi : M \to N$ is an orientation-preserving diffeomorphism, we assume for some n that Φ is an everywhere regular n-to-1 map of M onto N with $\dim M = \dim N$. Then the proof of Proposition 8.19 applies with easy modifications to give

$$n\int_N f\omega = \int_M (f \circ \Phi)\Phi^*\omega. \tag{8.59}$$

Therefore we have the following result in place of (8.54).

Theorem 8.60 (Weyl Integration Formula). Let T be a maximal torus of the compact connected Lie group G, and let invariant measures on G, T, and G/T be normalized so that

$$\int_G f(x)\,dx = \int_{G/T}\Big[\int_T f(xt)\,dt\Big]\,d(xT)$$

for all continuous f on G. Then every Borel function $F \geq 0$ on G satisfies

$$\int_G F(x)\,dx = \frac{1}{|W(G,T)|}\int_T\Big[\int_{G/T} F(gtg^{-1})\,d(gT)\Big]\,|D(t)|^2\,dt,$$

where $$|D(t)|^2 = \prod_{\alpha\in\Delta^+}|1 - \xi_\alpha(t^{-1})|^2.$$

The integration formula in Theorem 8.60 is a starting point for an analytic treatment of parts of representation theory for compact connected Lie groups. For a given such group for which δ is analytically integral, let us sketch how the theorem leads simultaneously to a construction of an irreducible representation with given dominant analytically integral highest weight and to a proof of the Weyl Character Formula.

Define

$$D(t) = \xi_\delta(t) \prod_{\alpha \in \Delta^+} (1 - \xi_{-\alpha}(t)), \tag{8.61}$$

so that Theorem 8.60 for any Borel function f constant on conjugacy classes and either nonnegative or integrable reduces to

$$\int_G f(x)\,dx = \frac{1}{|W(G,T)|} \int_T f(t)|D(t)|^2\,dt \tag{8.62}$$

if we take dx, dt, and $d(gT)$ to have total mass one. For $\lambda \in \mathfrak{t}^*$ dominant and analytically integral, define

$$\chi_\lambda(t) = \frac{\sum_{s \in W(G,T)} \varepsilon(s) \xi_{s(\lambda+\delta)}(t)}{D(t)}.$$

Then χ_λ is invariant under $W(G,T)$, and Proposition 4.53 shows that $\chi_\lambda(t)$ extends to a function χ_λ on G constant on conjugacy classes. Applying (8.62) with $f = |\chi_\lambda|^2$, we see that

$$\int_G |\chi_\lambda|^2\,dx = 1. \tag{8.63a}$$

Applying (8.62) with $f = \chi_\lambda \overline{\chi_{\lambda'}}$, we see that

$$\int_G \chi_\lambda(x)\overline{\chi_{\lambda'}(x)}\,dx = 0 \qquad \text{if } \lambda \neq \lambda'. \tag{8.63b}$$

Let χ be the character of an irreducible finite-dimensional representation of G. On T, $\chi(t)$ must be of the form $\sum_\mu \xi_\mu(t)$, where the μ's are the weights repeated according to their multiplicities. Also $\chi(t)$ is even under $W(G,T)$. Then $D(t)\chi(t)$ is odd under $W(G,T)$ and is of the form $\sum_\nu n_\nu \xi_\nu(t)$ with each n_ν in $\mathbb{Z}$. Focusing on the dominant ν's and seeing that the ν's orthogonal to a root must drop out, we find that $\chi(t) = \sum_\lambda a_\lambda \chi_\lambda(t)$ with $a_\lambda \in \mathbb{Z}$. By (8.63),

$$\int_G |\chi(x)|^2\,dx = \sum_\lambda |a_\lambda|^2.$$

For an irreducible character Corollary 4.16 shows that the left side is 1. So one a_λ is ± 1 and the others are 0. Since $\chi(t)$ is of the form $\sum_\mu \xi_\mu(t)$, we readily find that $a_\lambda = +1$ for some λ. Hence every irreducible character is of the form $\chi = \chi_\lambda$ for some λ. This proves the Weyl Character Formula. Using the Peter-Weyl Theorem (Theorem 4.20), we readily see that no L^2 function on G that is constant on conjugacy classes can be orthogonal to all irreducible characters. Then it follows from (8.63b) that every χ_λ is an irreducible character. This proves the existence of an irreducible representation corresponding to a given dominant analytically integral form as highest weight.

For reductive Lie groups that are not necessarily compact, there is a formula analogous to Theorem 8.60. This formula is a starting point for the analytic treatment of representation theory on such groups. We state the result as Theorem 8.64 but omit the proof. The proof makes use of Theorem 7.108 and of other variants of results that we applied in the compact case.

Theorem 8.64 (Harish-Chandra). Let G be a reductive Lie group, let $(\mathfrak{h}_1)_0, \dots, (\mathfrak{h}_r)_0$ be a maximal set of nonconjugate θ stable Cartan subalgebras of $\mathfrak{g}_0$, and let $H_1, \dots, H_r$ be the corresponding Cartan subgroups. Let the invariant measures on each H_j and G/H_j be normalized so that

$$\int_G f(x)\,dx = \int_{G/H_j}\Big[\int_{H_j} f(gh)\,dh\Big]\,d(gH_j) \qquad \text{for all } f \in C_{\mathrm{com}}(G).$$

Then every Borel function $F \geq 0$ on G satisfies

$$\int_G F(x)\,dx = \sum_{j=1}^{r} \frac{1}{|W(G,H_j)|}\int_{H_j}\Big[\int_{G/H_j} F(ghg^{-1})\,d(gH_j)\Big]|D_{H_j}(h)|^2\,dh,$$

where
$$|D_{H_j}(h)|^2 = \prod_{\alpha\in\Delta(\mathfrak{g},\mathfrak{h}_j)} |1-\xi_\alpha(h^{-1})|.$$

6. Problems

1. Prove that if M is an oriented m-dimensional manifold, then M admits a nowhere-vanishing smooth m form.
2. Prove that the zero locus of a nonzero real analytic function on a cube in $\mathbb{R}^n$ has Lebesgue measure 0.

3. Let G be the group of all real matrices $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$. Show that $a^{-2}\,da\,db$ is a left Haar measure and that $a^{-1}\,da\,db$ is a right Haar measure.

4. Let G be a noncompact semisimple Lie group with finite center, and let $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ be a minimal parabolic subgroup. Prove that $G/M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ has no nonzero G invariant Borel measure.

5. Prove that the complement of the set of regular points in a reductive Lie group G is a closed set of Haar measure 0.

Problems 6–8 concern Haar measure on $GL(n, \mathbb{R})$.

6. Why is Haar measure on $GL(n, \mathbb{R})$ two-sided invariant?

7. Regard $\mathfrak{gl}(n, \mathbb{R})$ as an n^2-dimensional vector space over $\mathbb{R}$. For each $x \in GL(n, \mathbb{R})$, let L_x denote left multiplication by x. Prove that $\det L_x = (\det x)^n$.

8. Let E_{ij} be the matrix that is 1 in the $(i,\ j)^{\text{th}}$ place and is 0 elsewhere. Regard $\{E_{ij}\}$ as the standard basis of $\mathfrak{gl}(n, \mathbb{R})$, and introduce Lebesgue measure accordingly.
   (a) Why is the set of $x \in \mathfrak{gl}(n, \mathbb{R})$ with $\det x = 0$ a set of Lebesgue measure 0?
   (b) Deduce from Problem 7 that $|\det y|^{-n}\,dy$ is a Haar measure for $GL(n, \mathbb{R})$.

Problems 9–12 concern the function $e^{\nu H_{\mathfrak{p}}(x)}$ for a semisimple Lie group G with a complexification $G^{\mathbb{C}}$. Here it is assumed that $G = KA_{\mathfrak{p}}N_{\mathfrak{p}}$ is an Iwasawa decomposition of G and that elements decompose as $x = \kappa(g)\exp H_{\mathfrak{p}}(x)\,n$. Let $\mathfrak{a}_{\mathfrak{p}}$ be the Lie algebra of $A_{\mathfrak{p}}$, and let ν be in $\mathfrak{a}_{\mathfrak{p}}^*$.

9. Let π be an irreducible finite-dimensional representation of G on V, and introduce a Hermitian inner product in V as in the proof of Theorem 8.49. If π has highest restricted weight ν and if v is in the restricted-weight space for ν, prove that $\|\pi(x)v\|^2 = e^{2\nu H_{\mathfrak{p}}(x)}\|v\|^2$.

10. In $G = SL(3, \mathbb{R})$, let $K = SO(3)$ and let $M_{\mathfrak{p}}A_{\mathfrak{p}}N_{\mathfrak{p}}$ be upper-triangular. Introduce parameters for $N_{\mathfrak{p}}^-$ by writing $N_{\mathfrak{p}}^- = \left\{ \bar{n} = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \right\}$. Let $f_1 - f_2$, $f_2 - f_3$, and $f_1 - f_3$ be the positive restricted roots as usual, and let $\rho_{\mathfrak{p}}$ denote half their sum (namely $f_1 - f_3$).
   (a) Show that $e^{2f_1 H_{\mathfrak{p}}(\bar{n})} = 1 + x^2 + z^2$ and $e^{2(f_1+f_2)H_{\mathfrak{p}}(\bar{n})} = 1 + y^2 + (z - xy)^2$ for $\bar{n} \in N_{\mathfrak{p}}^-$.
   (b) Deduce that $e^{2\rho_{\mathfrak{p}} H_{\mathfrak{p}}(\bar{n})} = (1 + x^2 + z^2)(1 + y^2 + (z - xy)^2)$ for $\bar{n} \in N_{\mathfrak{p}}^-$.

11. In $G = SO(n,1)_0$, let $K = SO(n) \times \{1\}$ and $\mathfrak{a}_\mathfrak{p} = \mathbb{R}(E_{1,n+1} + E_{n+1,1})$, with E_{ij} as in Problem 8. If $\lambda(E_{1,n+1} + E_{n+1,1}) > 0$, say that $\lambda \in \mathfrak{a}_\mathfrak{p}^*$ is positive, and obtain $G = KA_\mathfrak{p}N_\mathfrak{p}$ accordingly.
    (a) Using the standard representation of $SO(n,1)_0$, compute $e^{2\lambda H_\mathfrak{p}(x)}$ for a suitable λ and all $x \in G$.
    (b) Deduce a formula for $e^{2\rho_\mathfrak{p} H_\mathfrak{p}(x)}$ from the result of (a). Here $\rho_\mathfrak{p}$ is half the sum of the positive restricted roots repeated according to their multiplicities.

12. In $G = SU(n,1)$, let $K = S(U(n) \times U(1))$, and let $\mathfrak{a}_\mathfrak{p}$ and positivity be as in Problem 11. Repeat the two parts of Problem 11 for this group.

and let $\sigma : T^n(E) \to T^n(E)$ be its linear extension. We call σ the **symmetrizer** operator. The image of σ is denoted $\tilde{S}^n(E)$, and the members of this subspace are called **symmetrized** tensors.

Corollary A.23. Let $\Bbbk$ have characteristic 0, and let E be a vector space over $\Bbbk$. Then the symmetrizer operator σ satisfies $\sigma^2 = \sigma$. The kernel of σ is exactly $T^n(E) \cap I$, and therefore

$$T^n(E) = \tilde{S}^n(E) \oplus (T^n(E) \cap I).$$

REMARK. In view of this corollary, the quotient map $T^n(E) \to S^n(E)$ carries $\tilde{S}^n(E)$ one-one onto $S^n(E)$. Thus $\tilde{S}^n(E)$ can be viewed as a copy of $S^n(E)$ embedded as a direct summand of $T^n(E)$.

PROOF. We have

$$\begin{aligned}
\sigma^2(v_1 \otimes \cdots \otimes v_n) &= \frac{1}{(n!)^2} \sum_{\rho,\tau \in \mathfrak{S}_n} v_{\rho\tau(1)} \otimes \cdots \otimes v_{\rho\tau(n)} \\
&= \frac{1}{(n!)^2} \sum_{\rho \in \mathfrak{S}_n} \sum_{\substack{\omega \in \mathfrak{S}_n, \\ (\omega = \rho\tau)}} v_{\omega(1)} \otimes \cdots \otimes v_{\omega(n)} \\
&= \frac{1}{n!} \sum_{\rho \in \mathfrak{S}_n} \sigma(v_1 \otimes \cdots \otimes v_n) \\
&= \sigma(v_1 \otimes \cdots \otimes v_n).
\end{aligned}$$

Hence $\sigma^2 = \sigma$. Consequently $T^n(E)$ is the direct sum of image σ and $\ker \sigma$. We thus are left with identifying $\ker \sigma$ as $T^n(E) \cap I$.

The subspace $T^n(E) \cap I$ is spanned by elements

$$x_1 \otimes \cdots \otimes x_r \otimes u \otimes v \otimes y_1 \otimes \cdots \otimes y_s - x_1 \otimes \cdots \otimes x_r \otimes v \otimes u \otimes y_1 \otimes \cdots \otimes y_s$$

with $r+2+s = n$, and it is clear that σ vanishes on such elements. Hence $T^n(E) \cap I \subseteq \ker \sigma$. Suppose that the inclusion is strict, say with t in $\ker \sigma$ but t not in $T^n(E) \cap I$. Let q be the quotient map $T^n(E) \to S^n(E)$. The kernel of q is $T^n(E) \cap I$, and thus $q(t) \neq 0$. From Proposition A.21 it is clear that q carries $\tilde{S}^n(E) = \text{image}\, \sigma$ onto $S^n(E)$. Thus choose $t' \in \tilde{S}^n(E)$ with $q(t') = q(t)$. Then $t' - t$ is in $\ker q = T^n(E) \cap I \subseteq \ker \sigma$. Since $\sigma(t) = 0$, we see that $\sigma(t') = 0$. Consequently t' is in $\ker \sigma \cap \text{image}\, \sigma = 0$, and we obtain $t' = 0$ and $q(t) = q(t') = 0$, contradiction.

PROOF. Since multiplication in $\bigwedge(E)$ satisfies (A.26) and since monomials span $T^n(E)$, the indicated set spans $\bigwedge^n(E)$. Let us see independence. For $i \in A$, let u_i^* be the member of E^* with $u_i^*(u_j)$ equal to 1 for $j = i$ and equal to 0 for $j \neq i$. Fix $r_1 < \cdots < r_n$, and define

$$l(w_1, \ldots, w_n) = \det\{u_{r_i}^*(w_j)\} \qquad \text{for } w_1, \ldots, w_n \text{ in } E.$$

Then l is alternating n-multilinear from $E \times \cdots \times E$ into $\Bbbk$ and extends by Proposition A.27a to $L : \bigwedge^n(E) \to \Bbbk$. If $k_1 < \cdots < k_n$, then

$$L(u_{k_1} \wedge \cdots \wedge u_{k_n}) = l(u_{k_1}, \ldots, u_{k_n}) = \det\{u_{r_i}^*(u_{k_j})\},$$

and the right side is 0 unless $r_1 = k_1, \ldots, r_n = k_n$, in which case it is 1. This proves that the $u_{r_1} \wedge \cdots \wedge u_{r_n}$ are linearly independent in $\bigwedge^n(E)$.

Corollary A.30. Let E be a finite-dimensional vector space over $\Bbbk$ of dimension N. Then

(a) $\dim \bigwedge^n(E) = \binom{N}{n}$ for $0 \le n \le N$ and $= 0$ for $n > N$.

(b) $\bigwedge^n(E^*)$ is canonically isomorphic to $\bigwedge^n(E)^*$ by

$$(f_1 \wedge \cdots \wedge f_n)(w_1, \ldots, w_n) = \det\{f_i(w_j)\}.$$

PROOF. Part (a) is an immediate consequence of Proposition A.29, and (b) is proved in the same way as Corollary A.22b, using Proposition A.27a as a tool.

Now let us suppose that $\Bbbk$ has characteristic 0. We define an n-multilinear function from $E \times \cdots \times E$ into $T^n(E)$ by

$$(v_1, \ldots, v_n) \mapsto \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} (\operatorname{sgn} \tau) v_{\tau(1)} \otimes \cdots \otimes v_{\tau(n)},$$

and let $\sigma' : T^n(E) \to T^n(E)$ be its linear extension. We call σ' the **antisymmetrizer** operator. The image of σ' is denoted $\tilde{\bigwedge}^n(E)$, and the members of this subspace are called **antisymmetrized** tensors.

Corollary A.31. Let $\Bbbk$ have characteristic 0, and let E be a vector space over $\Bbbk$. Then the antisymmetrizer operator σ' satisfies $\sigma'^2 = \sigma'$. The kernel of σ' is exactly $T^n(E) \cap I'$, and therefore

$$T^n(E) = \tilde{\bigwedge}^n(E) \oplus (T^n(E) \cap I').$$

REMARK. In view of this corollary, the quotient map $T^n(E) \to \bigwedge^n(E)$ carries $\tilde{\bigwedge}^n(E)$ one-one onto $\bigwedge^n(E)$. Thus $\tilde{\bigwedge}^n(E)$ can be viewed as a copy of $\bigwedge^n(E)$ embedded as a direct summand of $T^n(E)$.

E_6

$V = \{v \in \mathbb{R}^8 \mid \langle v, e_6 - e_7\rangle = \langle v, e_7 + e_8\rangle = 0)\}$
$\Delta = \{\pm e_i \pm e_j \mid i < j \le 5\} \cup \left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}e_i \in V \,\middle|\, \sum_{i=1}^{8} n(i) \text{ even}\right\}$

$|\Delta| = 72$
$\dim \mathfrak{g} = 78$
$|W| = 2^7 \cdot 3^4 \cdot 5$
$\det(A_{ij}) = 3$

$\Delta^+ = \{e_i \pm e_j \mid i > j\}$
$\quad \cup \left\{\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{i=1}^{5}(-1)^{n(i)}e_i) \,\middle|\, \sum_{i=1}^{5} n(i) \text{ even}\right\}$
$\Pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6\}$
$\quad = \{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1),$
$\quad\quad e_2 + e_1,\ e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,\ e_5 - e_4\}$
Numbering of simple roots in Dynkin diagram $= \begin{pmatrix} 2 \\ 65431 \end{pmatrix}$

Fundamental weights in terms of simple roots:

$$\begin{aligned}
\varpi_1 &= \tfrac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6)\\
\varpi_2 &= 1\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 1\alpha_6\\
\varpi_3 &= \tfrac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6)\\
\varpi_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6\\
\varpi_5 &= \tfrac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6)\\
\varpi_6 &= \tfrac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)
\end{aligned}$$

Positive roots having a coefficient ≥ 2:

$$\begin{pmatrix} 1 \\ 01210 \end{pmatrix}, \begin{pmatrix} 1 \\ 11210 \end{pmatrix}, \begin{pmatrix} 1 \\ 01211 \end{pmatrix}, \begin{pmatrix} 1 \\ 12210 \end{pmatrix}, \begin{pmatrix} 1 \\ 11211 \end{pmatrix}, \begin{pmatrix} 1 \\ 01221 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 12211 \end{pmatrix}, \begin{pmatrix} 1 \\ 11221 \end{pmatrix}, \begin{pmatrix} 1 \\ 12221 \end{pmatrix}, \begin{pmatrix} 1 \\ 12321 \end{pmatrix}, \begin{pmatrix} 2 \\ 12321 \end{pmatrix}$$

$\delta = e_2 + 2e_3 + 3e_4 + 4e_5 - 4e_6 - 4e_7 + 4e_8$

E_7

$V = \{v \in \mathbb{R}^8 \mid \langle v, e_7 + e_8\rangle = 0\}$
$\Delta = \{\pm e_i \pm e_j \mid i < j \le 6\} \cup \{\pm(e_7 - e_8)\}$
$\quad \cup \left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}e_i \in V \,\middle|\, \sum_{i=1}^{8} n(i) \text{ even}\right\}$

$|\Delta| = 126$
$\dim \mathfrak{g} = 133$
$|W| = 2^{10}\cdot 3^4 \cdot 5 \cdot 7$
$\det(A_{ij}) = 2$

$\Delta^+ = \{e_i \pm e_j \mid i > j\} \cup \{e_8 - e_7\}$
$\quad \cup \left\{\frac{1}{2}(e_8 - e_7 + \sum_{i=1}^{6}(-1)^{n(i)}e_i) \,\middle|\, \sum_{i=1}^{6} n(i) \text{ odd}\right\}$
$\Pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4,\ \alpha_5,\ \alpha_6,\ \alpha_7\}$
$\quad = \{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1),$
$\quad\quad e_2 + e_1,\ e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,\ e_5 - e_4,\ e_6 - e_5\}$
Numbering of simple roots in Dynkin diagram $= \begin{pmatrix} 2 \\ 765431 \end{pmatrix}$

Fundamental weights in terms of simple roots:

$$\begin{aligned}
\varpi_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + 1\alpha_7\\
\varpi_2 &= \tfrac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7)\\
\varpi_3 &= 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7\\
\varpi_4 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7\\
\varpi_5 &= \tfrac{1}{2}(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7)\\
\varpi_6 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7\\
\varpi_7 &= \tfrac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)
\end{aligned}$$

Positive roots having a coefficient ≥ 2 and involving α_7:

$$\begin{pmatrix} 1 \\ 111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 111221 \end{pmatrix}, \begin{pmatrix} 1 \\ 112211 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 112221 \end{pmatrix}, \begin{pmatrix} 1 \\ 122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 112321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 122321 \end{pmatrix}, \begin{pmatrix} 2 \\ 123321 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 123431 \end{pmatrix}, \begin{pmatrix} 2 \\ 123432 \end{pmatrix}$$

$\delta = \frac{1}{2}(2e_2 + 4e_3 + 6e_4 + 8e_5 + 10e_6 - 17e_7 + 17e_8)$

E_8

$V = \mathbb{R}^8$
$\Delta = \{\pm e_i \pm e_j \mid i < j\}$
$\quad \cup \left\{\frac{1}{2}\sum_{i=1}^{8}(-1)^{n(i)}e_i \,\middle|\, \sum_{i=1}^{8} n(i) \text{ even}\right\}$

$|\Delta| = 240$
$\dim \mathfrak{g} = 248$
$|W| = 2^{14}\cdot 3^5 \cdot 5^2 \cdot 7$
$\det(A_{ij}) = 1$

$\Delta^+ = \{e_i \pm e_j \mid i > j\}$
$\quad \cup \left\{\frac{1}{2}(e_8 + \sum_{i=1}^{7}(-1)^{n(i)}e_i) \,\middle|\, \sum_{i=1}^{7} n(i) \text{ even}\right\}$
$\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$
$\quad = \{\frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1),$
$\quad\quad e_2 + e_1,\ e_2 - e_1,\ e_3 - e_2,\ e_4 - e_3,\ e_5 - e_4,\ e_6 - e_5,\ e_7 - e_6\}$
Numbering of simple roots in Dynkin diagram $= \begin{pmatrix} 2 \\ 8765431 \end{pmatrix}$

Fundamental weights in terms of simple roots:

$$\begin{aligned}
\varpi_1 &= 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8\\
\varpi_2 &= 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8\\
\varpi_3 &= 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8\\
\varpi_4 &= 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8\\
\varpi_5 &= 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8\\
\varpi_6 &= 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8\\
\varpi_7 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8\\
\varpi_8 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8
\end{aligned}$$

Positive roots having a coefficient ≥ 2 and involving α_8:

$$\begin{pmatrix} 1 \\ 1111210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1111221 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1112211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122211 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222210 \end{pmatrix}, \begin{pmatrix} 1 \\ 1112321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1122221 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222211 \end{pmatrix}, \begin{pmatrix} 2 \\ 1112321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222221 \end{pmatrix},$$
$$\begin{pmatrix} 2 \\ 1122321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1222321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1222321 \end{pmatrix},$$
$$\begin{pmatrix} 1 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123421 \end{pmatrix}, \begin{pmatrix} 2 \\ 1223321 \end{pmatrix}, \begin{pmatrix} 1 \\ 1233321 \end{pmatrix}, \begin{pmatrix} 2 \\ 1123431 \end{pmatrix},$$

***F*$_4$**

$V = \mathbb{R}^4$
$\Delta = \{\pm e_i \pm e_j \mid i < j\} \cup \{\pm e_i\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$

$|\Delta| = 48$
$\dim \mathfrak{g} = 52$
$|W| = 2^7 \cdot 3^2$
$\det(A_{ij}) = 1$

$\Delta^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i\} \cup \{\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$
$\Pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$
$\quad = \{\frac{1}{2}(e_1 - e_2 - e_3 - e_4),\ e_4,\ e_3 - e_4,\ e_2 - e_3\}$
Numbering of simple roots in Dynkin diagram = (1234)

Fundamental weights in terms of simple roots:
$\quad \varpi_1 = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + 1\alpha_4$
$\quad \varpi_2 = 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4$
$\quad \varpi_3 = 4\alpha_1 + 8\alpha_2 + 6\alpha_3 + 3\alpha_4$
$\quad \varpi_4 = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$

Positive roots having a coefficient ≥ 2:
(0210), (0211), (1210), (0221), (1211), (2210), (1221),
(2211), (1321), (2221), (2321), (2421), (2431), (2432)

$\delta = 11e_1 + 5e_2 + 3e_3 + e_4$

G_2

$V = \{v \in \mathbb{R}^3 \mid \langle v, e_1 + e_2 + e_3 \rangle = 0\}$
$\Delta = \{\pm(e_1 - e_2),\ \pm(e_2 - e_3),\ \pm(e_1 - e_3)\}$
$\quad \cup \{\pm(2e_1 - e_2 - e_3),\ \pm(2e_2 - e_1 - e_3),\ \pm(2e_3 - e_1 - e_2)\}$

$|\Delta| = 12$
$\dim \mathfrak{g} = 14$
$|W| = 2^2 \cdot 3$
$\det(A_{ij}) = 1$

$\Pi = \{\alpha_1,\ \alpha_2\}$
$\quad = \{e_1 - e_2,\ -2e_1 + e_2 + e_3\}$
Numbering of simple roots in Dynkin diagram $= (12)$
$\Delta^+ = \{(10), (01), (11), (21), (31), (32)\}$

Fundamental weights in terms of simple roots:
$\varpi_1 = 2\alpha_1 + 1\alpha_2$
$\varpi_2 = 3\alpha_1 + 2\alpha_2$

$\delta = 5\alpha_1 + 3\alpha_2$

$\mathfrak{so}(2p, 2q+1)$, $1 \le p \le q$

Vogan diagram:
B_{p+q}, trivial automorphism,
p^{th} simple root $e_p - e_{p+1}$ painted
$\mathfrak{k}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2q+1)$
Simple roots for $\mathfrak{k}_0$: compact simple roots and
$\begin{Bmatrix} e_{p-1} + e_p \text{ when } p > 1 \\ \text{no other when } p = 1 \end{Bmatrix}$

Real rank $= 2p$
Cayley transform list:
all $e_i \pm e_{2p+1-i}$ for $1 \le i \le p$

$\Sigma = B_{2p}$
Real-rank-one subalgebras:
$\mathfrak{sl}(2, \mathbb{R})$ for all long restricted roots
$\mathfrak{so}(2q-2p+2, 1)$ for all short restricted roots
$\mathfrak{m}_{\mathfrak{p},0} = \mathfrak{so}(2q-2p+1)$,
simple roots when $p < q$ by Cayley transform from
e_{p+q} and all $e_{2p+i} - e_{2p+i+1}$ for $1 \le i \le q-p-1$

$G = SO(2p, 2q+1)_0$
$K = SO(2p) \times SO(2q+1)$
$|M_{\mathfrak{p}}/(M_{\mathfrak{p}})_0| = 2^{2p-1}$

Special features:
G/K is Hermitian when $p = 1$,
$\mathfrak{g}_0$ is a split real form when $p = q$

Further information:
For $M_{\mathfrak{p}}$, see Example 3 in §VI.5.

$\mathfrak{so}(2p, 2q+1)$, $p > q \geq 0$

Vogan diagram:
  B_{p+q}, trivial automorphism,
  p^{th} simple root $e_p - e_{p+1}$ painted
$\mathfrak{k}_0 = \mathfrak{so}(2p) \oplus \mathfrak{so}(2q+1)$
Simple roots for $\mathfrak{k}_0$: compact simple roots and
$$\begin{Bmatrix} e_{p-1} + e_p \text{ when } p > 1 \\ \text{no other when } p = 1 \text{ and } q = 0 \end{Bmatrix}$$

Real rank $= 2q+1$
Cayley transform list:
  e_{p-q} and all $e_i \pm e_{2p+1-i}$ for $p - q + 1 \leq i \leq p$

$\Sigma = B_{2q+1}$
Real-rank-one subalgebras:
  $\mathfrak{sl}(2, \mathbb{R})$ for all long restricted roots
  $\mathfrak{so}(2p - 2q, 1)$ for all short restricted roots
$\mathfrak{m}_{\mathfrak{p},0} = \mathfrak{so}(2p - 2q - 1)$,
  simple roots when $p > q + 1$ by Cayley transform from
  e_{p-q-1} and all $e_i - e_{i+1}$ for $1 \leq i \leq p - q - 2$

$G = SO(2p, 2q+1)_0$
$K = SO(2p) \times SO(2q+1)$
$|M_{\mathfrak{p}}/(M_{\mathfrak{p}})_0| = 2^{2q}$

Special feature:
  G/K is Hermitian when $p = 1$ and $q = 0$,
  $\mathfrak{g}_0$ is a split real form when $p = q + 1$

Further information:
  For $M_{\mathfrak{p}}$, see Example 3 in §VI.5.

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16. Write the λ'' highest weight vector as $v = \sum_{\mu+\mu'=\lambda''}(v_\mu \otimes v_{\mu'})$, allowing more than one term per choice of μ and taking the $v_{\mu'}$'s to be linearly independent. Choose $\mu = \mu_0$ as large as possible so that there is a nonzero term $v_\mu \otimes v_{\mu'}$. Apply root vectors for positive roots and see that v_μ is highest for φ_λ.

17. Changing notation, suppose that the weights of $\varphi_{\lambda'}$ have multiplicity one. Let $\varphi_{\lambda''}$ occur more than once. By Problem 16 write $\lambda'' = \lambda + \mu'$ for a weight μ' of $\varphi_{\lambda'}$. The solution to Problem 16 shows that a highest weight vector for each occurrence of $\varphi_{\lambda''}$ contains a term equal to a nonzero multiple of $v_\lambda \otimes v_{\mu'}$. A suitable linear combination of these vectors does not contain such a term, in contradiction with Problem 16.

18. By Chevalley's Lemma, $\langle\lambda, \alpha\rangle = 0$ for some root α. Rewrite the sum as an iterated sum, the inner sum over $\{1, s_\alpha\}$ and the outer sum over cosets of this subgroup.

19. Putting $\mu'' = w\lambda''$ and using that $m_\lambda(w\lambda'') = m_\lambda(\lambda'')$, we have

$$\begin{aligned}
\chi_\lambda \chi_{\lambda'} &= d^{-1} \sum_{w \in W} \sum_{\mu''=\text{weight of } \varphi_\lambda} m_\lambda(\mu'')\varepsilon(w)\xi_{\mu''}\xi_{w(\lambda'+\delta)} \\
&= d^{-1} \sum_{w \in W} \sum_{\lambda''=\text{weight of } \varphi_\lambda} m_\lambda(\lambda'')\varepsilon(w)\xi_{w(\lambda''+\lambda'+\delta)} \\
&= d^{-1} \sum_{\lambda''=\text{weight of } \varphi_\lambda} m_\lambda(\lambda'')\operatorname{sgn}(\lambda''+\lambda'+\delta) \sum_{w \in W} \varepsilon(w)\xi_{w(\lambda''+\lambda'+\delta)^\vee} \\
&= d^{-1} \sum_{\lambda''=\text{weight of } \varphi_\lambda} m_\lambda(\lambda'')\operatorname{sgn}(\lambda''+\lambda'+\delta)\chi_{(\lambda''+\lambda'+\delta)^\vee-\delta}.
\end{aligned}$$

20. The lowest weight $-\mu$ has $m_\lambda(-\mu) = 1$ by Theorem 5.5e. If $\lambda' - \mu$ is dominant, then $\operatorname{sgn}(-\mu + \lambda' + \delta) = 1$. So $\lambda'' = -\mu$ contributes $+1$ to the coefficient of $\chi_{\lambda'-\mu}$. Suppose some other λ'' contributes. Then $(\lambda''+\lambda'+\delta)^\vee - \delta = \lambda' - \mu$. So $(\lambda''+\lambda'+\delta)^\vee = \lambda' - \mu + \delta$, $\lambda''+\lambda'+\delta = s(\lambda'-\mu+\delta) = \lambda' - \mu + \delta - \sum_{\alpha>0} n_\alpha \alpha$, and $\lambda'' = -\mu - \sum_{\alpha>0} n_\alpha\alpha$. This says that λ'' is lower than the lowest weight unless $\lambda'' = -\mu$.

22. Write $(\lambda'+\delta+\lambda'')^\vee = \lambda' + w\lambda + \delta$, $\lambda'+\delta+\lambda'' = s(\lambda'+w\lambda+\delta)$. Subtract $\lambda'+\delta$ from both sides and compute the length squared, taking into account that $\lambda'+\delta$ is strictly dominant and $\lambda'+w\lambda+\delta$ is dominant:

$$\begin{aligned}
|\lambda''|^2 &= |s(\lambda'+w\lambda+\delta) - (\lambda'+\delta)|^2 \\
&= |\lambda'+w\lambda+\delta|^2 - 2\langle s(\lambda'+w\lambda+\delta), \lambda'+\delta\rangle + |\lambda'+\delta|^2 \\
&\geq |\lambda'+w\lambda+\delta|^2 - 2\langle \lambda'+w\lambda+\delta, \lambda'+\delta\rangle + |\lambda'+\delta|^2 \\
&= |(\lambda'+w\lambda+\delta) - (\lambda'+\delta)|^2 \\
&= |\lambda|^2.
\end{aligned}$$

$(\widetilde{SL}(2,\mathbb{R})\times\{0\})/(D\cap(\widetilde{SL}(2,\mathbb{R})\times\{0\}))$, and the intersection on the bottom is trivial. Thus G_{ss} has infinite center. If G_{ss} were closed in G, K_{ss} would be closed in G, hence in K. Then K_{ss} would be compact, contradiction.

5. Let MAN be block upper-triangular with respective blocks of sizes 2 and 1. Then M is isomorphic to the group of 2-by-2 real matrices of determinant ± 1 and has a compact Cartan subalgebra. The group M is disconnected, and its center $Z_M = \{\pm 1\}$ is contained in M_0. Therefore $M \neq M_0 Z_M$.

6. Refer to the diagram of the root system G_2 in Figure 2.2. Take this to be the diagram of the restricted roots. Arrange for $\mathfrak{a}_0$ to correspond to the vertical axis and for $\mathfrak{t}_0$ to correspond to the horizontal axis. The nonzero projections of the roots on the $\mathfrak{a}_0$ axis are of the required form.

7. In (b) one MA is $\cong GL^+(2,\mathbb{R})\times\mathbb{Z}/2\mathbb{Z}$ (the plus referring to positive determinant), and the other is $\cong GL(2,\mathbb{R})$. If the two Cartan subalgebras were conjugate, the two MA's would be conjugate.

8. It is easier to work with $SO(2,n)_0$. For (a), conjugate the Lie algebra by $\operatorname{diag}(i,i,1,\dots,1)$. In (b), $\mathfrak{c}_0$ comes from the upper left 2-by-2 block. For (c) the Cartan subalgebra $\mathfrak{h}$ given in §II.1 is fixed by the conjugation in (a) and intersects with $\mathfrak{g}_0$ in a compact Cartan subalgebra of $\mathfrak{g}_0$. The noncompact roots are those that involve $\pm e_1$, and all others are compact. For (d) the usual ordering makes $e_1 \pm e_j$ and e_1 larger than all compact roots; hence it is good.

9. It is one-one since $N_K(\mathfrak{a}_0)\cap Z_G(\mathfrak{a}_0) = Z_K(\mathfrak{a}_0)$. To see that it is onto, let $g \in N_G(\mathfrak{a}_0)$ be given, and write $g = k\exp X$. By Lemma 7.22, k and X normalize $\mathfrak{a}_0$. Then X centralizes $\mathfrak{a}_0$. Hence g can be adjusted by the member $\exp X$ of $Z_G(\mathfrak{a}_0)$ so as to be in $N_K(\mathfrak{a}_0)$.

10. Imitate the proof of Proposition 7.85.

11. For (a) when α is real, form the associated Lie subalgebra $\mathfrak{sl}(2,\mathbb{R})$ and argue as in Proposition 6.52c. When α is compact imaginary, reduce matters to $SU(2)$. For (b), fix a positive system $\Delta^+(\mathfrak{k},\mathfrak{h})$ of compact roots. If s_α is in $W(G,H)$, choose $w \in W(\Delta(\mathfrak{k},\mathfrak{h}))$ with $ws_\alpha\Delta^+(\mathfrak{k},\mathfrak{h}) = \Delta^+(\mathfrak{k},\mathfrak{h})$. Let $\tilde{w}$ and $\tilde{s}_\alpha$ be representatives. By Theorem 7.8, $\operatorname{Ad}(\tilde{w}\tilde{s}_\alpha) = 1$ on $\mathfrak{h}$. Hence s_α is in $W(\Delta(\mathfrak{k},\mathfrak{h}))$. By Chevalley's Lemma some multiple of α is in $\Delta(\mathfrak{k},\mathfrak{h})$, contradiction. For (c) use the group of 2-by-2 real matrices of determinant ± 1.

12. Parts (a) and (c) are trivial. In (b) put $M = {}^0Z_G(\mathfrak{a}_0)$. If k is in $N_K(\mathfrak{a}_0)$, then $\operatorname{Ad}(k)$ carries $\mathfrak{t}_0$ to a compact Cartan subalgebra of $\mathfrak{m}_0$ and can be carried back to $\mathfrak{t}_0$ by Ad of a member of $K\cap M$, essentially by Proposition 6.61.

13. The given ordering on roots is compatible with an ordering on restricted roots. Any real or complex root whose restriction to $\mathfrak{a}_0$ is positive contributes to both $\mathfrak{b}$ and $\bar{\mathfrak{b}}$. Any imaginary root contributes either to $\mathfrak{b}$ or to $\bar{\mathfrak{b}}$. Therefore $\mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n} = \mathfrak{b}+\bar{\mathfrak{b}}$.

14. Otherwise $N_{\mathfrak{g}_0}(\mathfrak{k}_0)$ would contain a nonzero member X of $\mathfrak{p}_0$. Then $\operatorname{ad}X$ carries $\mathfrak{k}_0$ to $\mathfrak{k}_0$ because X is in the normalizer, and $\operatorname{ad}X$ carries $\mathfrak{k}_0$ to $\mathfrak{p}_0$

One result about structure theory that we have omitted in §3, having no Lie-theoretic proof, is the theorem of Cartan [1929a] that any compact subgroup of G is conjugate to a subgroup of K.

Cartan [1927b] shows that there is a Euclidean subgroup A of G such that any element of G/K can be reached from the identity coset by applying a member of A and then a member of K. This is the subgroup A of §4, and the geometric result establishes Theorem 6.51 in §5 and the KAK decomposition in Theorem 7.39. Cartan [1927b] introduces restricted roots. The introduction of N in §4 is due to Iwasawa [1949], and the decomposition given as Theorem 6.46 appears in the same paper. Lemma 6.44 came after Iwasawa's original proof and appears as Lemma 26 of Harish-Chandra [1953]. Cartan [1927b] uses the group $W(G, A)$ of §5, and Theorem 6.57 is implicit in that paper.

It was apparent from the work of Harish-Chandra and Gelfand-Graev in the early 1950s that Cartan subalgebras would play an important role in harmonic analysis on semisimple Lie groups. The results of §6 appear in Kostant [1955] and Harish-Chandra [1956a]. Kostant [1955] announces the existence of a classification of Cartan subalgebras up to conjugacy, but the appearance of Harish-Chandra [1956a] blocked the appearance of proofs for the results of that paper. Sugiura [1959] states and proves the classification.

In effect Cayley transforms as in §7 appear in Harish-Chandra [1957], §2. For further information, see the Notes for §VII.9.

In §8 the name "Vogan diagram" is new. In the case that $\mathfrak{a}_0 = 0$, the idea of adapting a system of positive roots to given data was present in the late 1960s and early 1970s in the work of Schmid on discrete series representations (see Schmid [1975], for example), and a Vogan diagram could capture this idea in a picture. Vogan used the same idea in the mid 1970s for general maximally compact Cartan subalgebras. He introduced the notion of a θ stable parabolic subalgebra of $\mathfrak{g}$ to handle representation-theoretic data and used the diagrams to help in understanding these subalgebras. The paper [1979] contains initial results from this investigation but no diagrams.

Because of Theorem 6.74 Vogan diagrams provide control in the problem of classifying simple real Lie algebras. This theorem was perhaps understood for a long time to be true, but Knapp [1996] gives a proof. Theorem 6.88 is due to Vogan.

The results of §9 were already recognized in Cartan [1914]. The classification in §10, as was said earlier, is in Cartan [1914]; it is the result of a remarkable computation made before the discovery of the Cartan involution. Lie algebras with a given complexification are to be classified in that paper, and the signature of the Killing form is the key invariant. The classification over $\mathbb{R}$ is recalled in Cartan [1927a], and $\mathfrak{k}_0$ is identified in each case. In this paper Cartan provided a numbering for the noncomplex noncompact simple real Lie algebras. This numbering has been retained by Helgason [1978], and we use the same numbering for the exceptional cases in Figures 6.2 and 6.3.

Cartan [1927b] improves the classification by relating Lie algebras and geometry. This paper contains tables giving more extensive information about the exceptional Lie algebras. Gantmacher [1939a] and [1939b] approached classification as a problem in classifying automorphisms and then succeeded in simplifying the proof of classification. This method was further simplified by Murakami [1965] and Wallach [1966] and [1968] independently. Murakami and Wallach made use of the Borel and de Siebenthal Theorem (Borel and de Siebenthal [1949]), which is similar to Theorem 6.96 but slightly different. The original purpose of the theorem was to find a standard form for automorphisms, and Murakami and Wallach both use the theorem that way. Helgason [1978] gives a proof of classification that is based on classifying automorphisms in a different way. The paper Knapp [1996] gives the quick proof of Theorem 6.96 and then deduces the classification as a consequence of Theorem 6.74; no additional consideration of automorphisms is needed.

The above approaches to classification make use of a maximally compact Cartan subalgebra. An alternative line of attack starts from a maximally noncompact Cartan subalgebra and is the subject of Araki [1962]. The classification is stated in terms of "Satake diagrams," which are described by Helgason [1978], 531. Problem 7 at the end of Chapter VI establishes the facts due to Satake [1960] needed to justify the definition of a Satake diagram.

The information in (6.107) and (6.108) appears in Cartan [1927b]. Appendix C shows how this information can be obtained from Vogan diagrams.

Chapter VII

§1. The essence of Theorem 7.8 is already in Cartan [1925b]. Goto [1948] and Mostow [1950] investigated conditions that ensure that an analytic subgroup is closed. The circle of ideas in this direction in §1 is based ultimately on Goto's work. The unitary trick is due to Weyl [1925–26] and consists of two parts—the existence of compact real forms and the comparison of $\mathfrak{g}$ and $\mathfrak{u}_0$.

§2. The necessity for considering reductive groups emerged from the work of Harish-Chandra, who for a semisimple group G was led to form a series of infinite-dimensional representations constructed from the M of each cuspidal parabolic subgroup. The subgroup M is not necessarily semisimple, however, and it was helpful to have a class of groups that would include a rich supply of semisimple groups G and would have the property that the M of each cuspidal parabolic subgroup of G is again in the class. Various classes have been proposed for this purpose. The Harish-Chandra class is the class defined by axioms in §3 of Harish-Chandra [1975], and its properties are developed in the first part of that paper. We have used axioms from Knapp-Vogan [1995], based on Vogan [1981]. These axioms, though more complicated to state than Harish-Chandra's axioms, have the advantage of being easier to check. The