converges. Since the polar decomposition of $GL(V^{\mathbb{C}})$ is a homeomorphism, it follows that $\exp X_n$ has limit $\exp X$ for some $X \in \mathfrak{p}(V^{\mathbb{C}})$. Since $\mathfrak{p}_0$ is closed in $\mathfrak{p}(V^{\mathbb{C}})$, X is in $\mathfrak{p}_0$. Therefore $g = k \exp X$ exhibits g as in G, and G is closed.

Corollary 7.10. Let G be an analytic subgroup of real or complex matrices whose Lie algebra $\mathfrak{g}_0$ is reductive, and suppose that the identity component of the center of G is compact. Then G is a closed linear group.

REMARK. In this result and some to follow, we shall work with analytic groups whose Lie algebras are direct sums. If G is an analytic group whose Lie algebra $\mathfrak{g}_0$ is a direct sum $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{b}_0$ of ideals and if A and B are the analytic subgroups corresponding to $\mathfrak{a}_0$ and $\mathfrak{b}_0$, then G is a commuting product $G = AB$. This fact follows from Proposition 1.99 or may be derived directly, as in the proof of Theorem 4.29.

PROOF. Write $\mathfrak{g}_0 = Z_{\mathfrak{g}_0} \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$ by Corollary 1.53. The analytic subgroup of G corresponding to $Z_{\mathfrak{g}_0}$ is $(Z_G)_0$, and we let G_{ss} be the analytic subgroup corresponding to $[\mathfrak{g}_0, \mathfrak{g}_0]$. By the remarks before the proof, G is the commuting product $(Z_G)_0 G_{ss}$. The group G_{ss} is closed as a group of matrices by Proposition 7.9, and $(Z_G)_0$ is compact by assumption. Hence the set of products, which is G, is closed.

Corollary 7.11. Let G be a connected closed linear group whose Lie algebra $\mathfrak{g}_0$ is reductive. Then the analytic subgroup G_{ss} of G with Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ is closed, and G is the commuting product $G = (Z_G)_0 G_{ss}$.

PROOF. The subgroup G_{ss} is closed by Proposition 7.9, and G is the commuting product $(Z_G)_0 G_{ss}$ by the remarks with Corollary 7.10.

Proposition 7.12. Let G be a compact connected linear Lie group, and let $\mathfrak{g}_0$ be its linear Lie algebra. Then the complex analytic group $G^{\mathbb{C}}$ of matrices with linear Lie algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ is a closed linear group.

REMARKS. If G is a compact connected Lie group, then Corollary 4.22 implies that G is isomorphic to a closed linear group. If G is realized as a closed linear group in two different ways, then this proposition in principle produces two different groups $G^{\mathbb{C}}$. However, Proposition 7.5 shows that the two groups $G^{\mathbb{C}}$ are isomorphic. Therefore with no reference to linear groups, we can speak of the complexification $G^{\mathbb{C}}$ of a compact connected Lie group G, and $G^{\mathbb{C}}$ is unique up to isomorphism. Proposition 7.5 shows that a homomorphism between two such groups G and G' induces a holomorphic homomorphism between their complexifications.

of the unitary group with the closed group of matrices G. Properties (iv) and (vi) follow from Propositions 1.122 and 7.9, respectively. The closed linear group of real matrices of determinant ± 1 satisfies property (v) since

$$\operatorname{Ad}(\operatorname{diag}(-1, 1, \ldots, 1)) = \operatorname{Ad}(\operatorname{diag}(e^{i\pi(n-1)/n}, e^{-i\pi/n}, \ldots, e^{-i\pi/n})).$$

But as noted in Example 3, the orthogonal group $O(n)$ does not satisfy property (v) if n is even.

5) G is the centralizer in a reductive group $\tilde{G}$ of a θ stable abelian subalgebra of the Lie algebra of $\tilde{G}$. Here K is obtained by intersection, and θ and B are obtained by restriction. The verification that G is a reductive Lie group will be given below in Proposition 7.25.

If G is semisimple with finite center and if K, θ, and B are specified so that G is considered as a reductive group, then θ is forced to be a Cartan involution in the sense of Chapter VI. This is the content of Proposition 7.17. Hence the new terms "Cartan involution" and "Cartan decomposition" are consistent with the terminology of Chapter VI in the case that G is semisimple.

An alternative way of saying (iii) is that the symmetric bilinear form

$$B_\theta(X, Y) = -B(X, \theta Y) \tag{7.18}$$

is positive definite on $\mathfrak{g}_0$.

We use the notation $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$, etc., to denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$, etc. Using complex linearity, we extend θ from $\mathfrak{g}_0$ to $\mathfrak{g}$ and B from $\mathfrak{g}_0 \times \mathfrak{g}_0$ to $\mathfrak{g} \times \mathfrak{g}$.

Proposition 7.19. If G is a reductive Lie group, then

(a) K is a maximal compact subgroup of G
(b) K meets every component of G, i.e., $G = KG_0$
(c) each member of $\operatorname{Ad}(K)$ leaves $\mathfrak{k}_0$ and $\mathfrak{p}_0$ stable and therefore commutes with θ
(d) $(\operatorname{ad} X)^* = -\operatorname{ad} \theta X$ relative to B_θ if X is in $\mathfrak{g}_0$
(e) θ leaves $Z_{\mathfrak{g}_0}$ and $[\mathfrak{g}_0, \mathfrak{g}_0]$ stable, and the restriction of θ to $[\mathfrak{g}_0, \mathfrak{g}_0]$ is a Cartan involution
(f) the identity component G_0 is a reductive Lie group (with maximal compact subgroup obtained by intersection and with Cartan involution and invariant form unchanged).

where $\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ is a maximal abelian subspace of $\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$. Theorem 6.51 shows that any two maximal abelian subspaces of $\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ are conjugate via $\operatorname{Ad}(K)$, and it follows from (7.28) that this result extends to our reductive $\mathfrak{g}_0$.

Proposition 7.29. Let G be a reductive Lie group. If $\mathfrak{a}_0$ and $\mathfrak{a}_0'$ are two maximal abelian subspaces of $\mathfrak{p}_0$, then there is a member k of K with $\operatorname{Ad}(k)\mathfrak{a}_0' = \mathfrak{a}_0$. The member k of K can be taken to be in $K \cap G_{ss}$. Hence $\mathfrak{p}_0 = \bigcup_{k \in K_{ss}} \operatorname{Ad}(k)\mathfrak{a}_0$.

Relative to $\mathfrak{a}_0$, we can form restricted roots just as in §VI.4. A **restricted root** of $\mathfrak{g}_0$, also called a **root** of $(\mathfrak{g}_0, \mathfrak{a}_0)$, is a nonzero $\lambda \in \mathfrak{a}_0^*$ such that the space

$$(\mathfrak{g}_0)_\lambda = \{X \in \mathfrak{g}_0 \mid (\operatorname{ad} H)X = \lambda(H)X \text{ for all } H \in \mathfrak{a}_0\}$$

is nonzero. It is apparent that such a restricted root is obtained by taking a restricted root for $[\mathfrak{g}_0, \mathfrak{g}_0]$ and extending it from $\mathfrak{a}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0]$ to $\mathfrak{a}_0$ by making it be 0 on $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$. The restricted-root space decomposition for $[\mathfrak{g}_0, \mathfrak{g}_0]$ gives us a restricted-root space decomposition for $\mathfrak{g}_0$. We define $\mathfrak{m}_0 = Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$, so that the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{m}_0 \oplus \mathfrak{a}_0$.

The set of restricted roots is denoted Σ. Choose a notion of positivity for $\mathfrak{a}_0^*$ in the manner of §II.5, as for example by using a lexicographic ordering. Let Σ^+ be the set of positive restricted roots, and define $\mathfrak{n}_0 = \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_\lambda$. Then $\mathfrak{n}_0$ is a nilpotent Lie subalgebra of $\mathfrak{g}_0$, and we have an Iwasawa decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0 \tag{7.30}$$

with all the properties in Proposition 6.43.

Proposition 7.31. Let G be a reductive Lie group, let (7.30) be an Iwasawa decomposition of the Lie algebra $\mathfrak{g}_0$ of G, and let A and N be the analytic subgroups of G with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$. Then the multiplication map $K \times A \times N \to G$ given by $(k, a, n) \mapsto kan$ is a diffeomorphism onto. The groups A and N are simply connected.

PROOF. Multiplication is certainly smooth, and it is regular by Lemma 6.44. To see that it is one-one, it is enough, as in the proof of Theorem 6.46, to see that we cannot have $kan = 1$ nontrivially. The identity $kan = 1$ would force the orthogonal transformation $\operatorname{Ad}(k)$ to be upper triangular with positive diagonal entries in the matrix realization of Lemma 6.45, and consequently we may assume that $\operatorname{Ad}(k) = \operatorname{Ad}(a) = \operatorname{Ad}(n) = 1$. Thus k, a, and n are in $Z_G(\mathfrak{g}_0)$. By Lemma 7.22, a is the exponential of something in $Z_{\mathfrak{g}_0}(\mathfrak{g}_0) = Z_{\mathfrak{g}_0}$. Hence a is in Z_{vec}. By construction n is in G_{ss}, and hence k and n are in ${}^0G$. By Proposition 7.27f, $a = 1$ and $kn = 1$. But then the identity $kn = 1$ is valid in G_{ss}, and Theorem 6.46 implies that $k = n = 1$.

To see that multiplication is onto G, we observe from Theorem 6.46 that $\exp(\mathfrak{p}_0 \cap [\mathfrak{g}_0, \mathfrak{g}_0])$ is in the image. By Proposition 7.27a, the image contains ${}^0G$. Also Z_{vec} is in the image (of $1 \times A \times 1$), and Z_{vec} commutes with ${}^0G$. Hence the image contains ${}^0GZ_{vec}$. This is all of G by Proposition 7.27f.

We define $\mathfrak{n}_0^- = \bigoplus_{\lambda \in \Sigma^+} (\mathfrak{g}_0)_{-\lambda}$. Then $\mathfrak{n}_0^-$ is a nilpotent Lie subalgebra of $\mathfrak{g}_0$, and we let N^- be the corresponding analytic subgroup. Since $-\Sigma^+$ is the set of positive restricted roots for another notion of positivity on $\mathfrak{a}_0^*$, $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0^-$ is another Iwasawa decomposition of $\mathfrak{g}_0$ and $G = KAN^-$ is another Iwasawa decomposition of G. The identity $\theta(\mathfrak{g}_0)_\lambda = (\mathfrak{g}_0)_{-\lambda}$ given in Proposition 6.40c implies that $\theta\mathfrak{n}_0 = \mathfrak{n}_0^-$. By Proposition 7.21, $\Theta N = N^-$.

We write M for the group $Z_K(\mathfrak{a}_0)$. This is a compact subgroup since it is closed in K, and its Lie algebra is $Z_{\mathfrak{k}_0}(\mathfrak{a}_0)$. This subgroup normalizes each $(\mathfrak{g}_0)_\lambda$ since

$$\begin{aligned} \operatorname{ad}(H)(\operatorname{Ad}(m)X_\lambda) &= \operatorname{Ad}(m)\operatorname{ad}(\operatorname{Ad}(m)^{-1}H)X_\lambda \\ &= \operatorname{Ad}(m)\operatorname{ad}(H)X_\lambda = \lambda(H)\operatorname{Ad}(m)X_\lambda \end{aligned}$$

for $m \in M$, $H \in \mathfrak{a}_0$, and $X_\lambda \in (\mathfrak{g}_0)_\lambda$. Consequently M normalizes $\mathfrak{n}_0$. Thus M centralizes A and normalizes N. Since M is compact and AN is closed, MAN is a closed subgroup.

Reflections in the restricted roots generate a group $W(\Sigma)$, which we call the **Weyl group** of Σ. The elements of $W(\Sigma)$ are nothing more than the elements of the Weyl group for the restricted roots of $[\mathfrak{g}_0, \mathfrak{g}_0]$, with each element extended to $\mathfrak{a}_0^*$ by being defined to be the identity on $\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}$.

We define $W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0)$. By the same proof as for Lemma 6.56, the Lie algebra of $N_K(\mathfrak{a}_0)$ is $\mathfrak{m}_0$. Therefore $W(G, A)$ is a finite group.

Proposition 7.32. If G is a reductive Lie group, then the group $W(G, A)$ coincides with $W(\Sigma)$.

PROOF. Just as with the corresponding result in the semisimple case (Theorem 6.57), we know that $W(\Sigma) \subseteq W(G, A)$. Fix a simple system Σ^+ for Σ. As in the proof of Theorem 6.57, it suffices to show that if $k \in N_K(\mathfrak{a}_0)$ has $\operatorname{Ad}(k)\Sigma^+ = \Sigma^+$, then k is in $Z_K(\mathfrak{a}_0)$. By Lemma

Proposition 7.35. Let G be a reductive Lie group. If two θ stable Cartan subalgebras of $\mathfrak{g}_0$ are conjugate via G, then they are conjugate via G_{ss} and in fact by $K \cap G_{ss}$.

PROOF. Let $\mathfrak{h}_0$ and $\mathfrak{h}'_0$ be θ stable Cartan subalgebras, and suppose that $\mathrm{Ad}(g)(\mathfrak{h}_0) = \mathfrak{h}'_0$. By (7.23), $\mathrm{Ad}(\Theta g)(\mathfrak{h}_0) = \mathfrak{h}'_0$. If $g = k \exp X$ with $k \in K$ and $X \in \mathfrak{p}_0$, then it follows that Ad of $(\Theta g)^{-1} g = \exp 2X$ normalizes $\mathfrak{h}_0$. Applying Lemma 7.22 to $\exp 2X$, we see that $[X, \mathfrak{h}_0] \subseteq \mathfrak{h}_0$. Therefore $\exp X$ normalizes $\mathfrak{h}_0$, and $\mathrm{Ad}(k)$ carries $\mathfrak{h}_0$ to $\mathfrak{h}'_0$.

Since $\mathrm{Ad}(k)$ commutes with θ, $\mathrm{Ad}(k)$ carries $\mathfrak{h}_0 \cap \mathfrak{p}_0$ to $\mathfrak{h}'_0 \cap \mathfrak{p}_0$. Let $\mathfrak{a}_0$ be a maximal abelian subspace of $\mathfrak{p}_0$ containing $\mathfrak{h}_0 \cap \mathfrak{p}_0$, and choose $k_0 \in K_0$ by Proposition 7.29 so that $\mathrm{Ad}(k_0 k)(\mathfrak{a}_0) = \mathfrak{a}_0$. Comparing Proposition 7.32 and Theorem 6.57, we can find $k_1 \in K_0$ so that $k_1 k_0 k$ centralizes $\mathfrak{a}_0$. Then $\mathrm{Ad}(k)|_{\mathfrak{a}_0} = \mathrm{Ad}(k_0^{-1} k_1^{-1})|_{\mathfrak{a}_0}$, and the element $k' = k_0^{-1} k_1^{-1}$ of K_0 has the property that $\mathrm{Ad}(k')(\mathfrak{h}_0 \cap \mathfrak{p}_0) = \mathfrak{h}'_0 \cap \mathfrak{p}_0$. The θ stable Cartan subalgebras $\mathfrak{h}_0$ and $\mathrm{Ad}(k')^{-1}(\mathfrak{h}'_0)$ therefore have the same $\mathfrak{p}_0$ part, and Lemma 6.62 shows that they are conjugate via $K \cap G_{ss}$.

3. KAK Decomposition

Throughout this section we let G be a reductive Lie group, and we let other notation be as in §2.

From the global Cartan decomposition $G = K \exp \mathfrak{p}_0$ and from the equality $\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k)\mathfrak{a}_0$ of Proposition 7.29, it is immediate that $G = KAK$ in the sense that every element of G can be decomposed as a product of an element of K, an element of A, and a second element of K. In this section we shall examine the degree of nonuniqueness of this decomposition.

Lemma 7.36. If X is in $\mathfrak{p}_0$, then $Z_G(\exp X) = Z_G(\mathbb{R}X)$.

PROOF. Certainly $Z_G(\mathbb{R}X) \subseteq Z_G(\exp X)$. In the reverse direction if g is in $Z_G(\exp X)$, then $\mathrm{Ad}(g)\mathrm{Ad}(\exp X) = \mathrm{Ad}(\exp X)\mathrm{Ad}(g)$. By Proposition 7.19d, $\mathrm{Ad}(\exp X)$ is positive definite on $\mathfrak{g}_0$, thus diagonable. Consequently $\mathrm{Ad}(g)$ carries each eigenspace of $\mathrm{Ad}(\exp X)$ to itself, and it follows that $\mathrm{Ad}(g)\mathrm{ad}(X) = \mathrm{ad}(X)\mathrm{Ad}(g)$. By Lemma 1.95,

$$\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{ad}(X). \tag{7.37}$$

Write $X = Y + Z$ with $Y \in Z_{\mathfrak{g}_0}$ and $Z \in [\mathfrak{g}_0, \mathfrak{g}_0]$. By property (v) of a reductive group, $\mathrm{Ad}(g)Y = Y$. Comparing this equality with (7.37), we see that $\mathrm{ad}(\mathrm{Ad}(g)Z) = \mathrm{ad}(Z)$, hence that $\mathrm{Ad}(g)Z - Z$ is in the center of $\mathfrak{g}_0$. Since it is in $[\mathfrak{g}_0, \mathfrak{g}_0]$ also, it is 0. Therefore $\mathrm{Ad}(g)X = X$, and g is in the centralizer of $\mathbb{R}X$.

The Bruhat decomposition describes the double coset decomposition $MAN\backslash G/MAN$ of G with respect to MAN. Here is an example.

EXAMPLE. Let $G = SL(2,\mathbb{R})$. Here $MAN = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\}$. The normalizer $N_K(\mathfrak{a}_0)$ consists of the four matrices $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$ and $\pm\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$, while the centralizer $Z_K(\mathfrak{a}_0)$ consists of the two matrices $\pm\begin{pmatrix} 1 & 0 \\ 0 & 1\end{pmatrix}$. Thus $|W(G,A)| = 2$, and $\tilde{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}$ is a representative of the nontrivial element of $W(G,A)$. Let $g = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$ be given in G. If $c = 0$, then g is in MAN. If $c \neq 0$, then

$$\begin{aligned}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} c & d \\ -a & -b\end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ -ac^{-1} & 1\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix}.\end{aligned}$$

Hence

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} 1 & ac^{-1} \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix}\begin{pmatrix} c & d \\ 0 & c^{-1}\end{pmatrix}$$

exhibits $\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ as in $MAN\tilde{w}MAN$. Thus the double-coset space $MAN\backslash G/MAN$ consists of two elements, with 1 and $\tilde{w}$ as representatives.

Theorem 7.40 (Bruhat decomposition). The double cosets of $MAN\backslash G/MAN$ are parametrized in a one-one fashion by $W(G,A)$, the double coset corresponding to $w \in W(G,A)$ being $MAN\tilde{w}MAN$, where $\tilde{w}$ is any representative of w in $N_K(\mathfrak{a}_0)$.

PROOF OF UNIQUENESS. Suppose that w_1 and w_2 are in $W(G,A)$, with $\tilde{w}_1$ and $\tilde{w}_2$ as representatives, and that x_1 and x_2 in MAN have

$$x_1\tilde{w}_1 = \tilde{w}_2 x_2. \tag{7.41}$$

Now $\operatorname{Ad}(N) = \exp(\operatorname{ad}(\mathfrak{n}_0))$ by Theorem 1.104, and hence $\operatorname{Ad}(N)$ carries $\mathfrak{a}_0$ to $\mathfrak{a}_0 \oplus \mathfrak{n}_0$ while leaving the $\mathfrak{a}_0$ component unchanged. Meanwhile under Ad, $N_K(\mathfrak{a}_0)$ permutes the restricted-root spaces and thus carries $\mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma}(\mathfrak{g}_0)_\lambda$ to itself. Apply Ad of both sides of (7.41) to an element $H \in \mathfrak{a}_0$ and project to $\mathfrak{a}_0$ along $\mathfrak{m}_0 \oplus \bigoplus_{\lambda\in\Sigma}(\mathfrak{g}_0)_\lambda$. The resulting left side is in $\mathfrak{a}_0 \oplus \mathfrak{n}_0$ with $\mathfrak{a}_0$ component $\operatorname{Ad}(\tilde{w}_1)H$, while the right side is in $\operatorname{Ad}(\tilde{w}_2)H + \operatorname{Ad}(\tilde{w}_2)(\mathfrak{m}_0 \oplus \mathfrak{n}_0)$. Hence $\operatorname{Ad}(\tilde{w}_1)H = \operatorname{Ad}(\tilde{w}_2)H$. Since H is arbitrary, $\tilde{w}_2^{-1}\tilde{w}_1$ centralizes $\mathfrak{a}_0$. Therefore $w_1 = w_2$.

So $Z = \mathrm{Ad}(x^{-1}n_1)H$ is in $\mathfrak{s}_0$. Since $\mathrm{ad}_{\mathfrak{g}} Z$ and $\mathrm{ad}_{\mathfrak{g}} H$ have the same eigenvalues, Lemma 7.43b shows that Z is in $\mathfrak{a}_0 \oplus \mathfrak{n}_0 \oplus (\mathfrak{m}_0 \cap Z_{\mathfrak{g}_0})$. Since $\mathrm{Ad}(x^{-1}n_1)^{-1}$ fixes $Z_{\mathfrak{g}_0}$ (by property (v)), Z is in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. Write $Z = H' + X'$ correspondingly. Here $\mathrm{ad}\, H$ and $\mathrm{ad}\, H'$ have the same eigenvalues, so that $\lambda(H') \neq 0$ for all $\lambda \in \Sigma$. By Lemma 7.42 there exists $n_2 \in N$ with $\mathrm{Ad}(n_2)^{-1}H' - H' = X'$. Then $\mathrm{Ad}(n_2)^{-1}H' = H' + X' = Z$, and

$$H' = \mathrm{Ad}(n_2)Z = \mathrm{Ad}(n_2 x^{-1} n_1)H.$$

The centralizers of H' and H are both $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ by Lemma 6.50. Thus

$$\mathrm{Ad}(n_2 x^{-1} n_1)(\mathfrak{a}_0 \oplus \mathfrak{m}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0. \tag{7.47}$$

If X is in $\mathfrak{a}_0$, then $\mathrm{ad}_{\mathfrak{g}}(X)$ has real eigenvalues by Lemma 7.43b. Since $\mathrm{ad}_{\mathfrak{g}}(\mathrm{Ad}(n_2x^{-1}n_1)X)$ and $\mathrm{ad}_{\mathfrak{g}}(X)$ have the same eigenvalues, Lemma 7.43b shows that $\mathrm{Ad}(n_2x^{-1}n_1)X$ is in $\mathfrak{a}_0 \oplus (\mathfrak{m}_0 \cap Z_{\mathfrak{g}_0})$. Since $\mathrm{Ad}(n_2x^{-1}n_1)^{-1}$ fixes $Z_{\mathfrak{g}_0}$ (by property (v)), $\mathrm{Ad}(n_2x^{-1}n_1)X$ is in $\mathfrak{a}_0$. We conclude that $n_2x^{-1}n_1$ is in $N_G(\mathfrak{a}_0)$.

Let $n_2x^{-1}n_1 = u \exp X_0$ be the global Cartan decomposition of $n_2x^{-1}n_1$. By Lemma 7.22, u is in $N_K(\mathfrak{a}_0)$ and X_0 is in $N_{\mathfrak{g}_0}(\mathfrak{a}_0)$. By the same argument as in Lemma 6.56, $N_{\mathfrak{g}_0}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Since X_0 is in $\mathfrak{p}_0$, X_0 is in $\mathfrak{a}_0$. Therefore u is in $N_K(\mathfrak{a}_0)$ and $\exp X_0$ is in A. In other words, $n_2x^{-1}n_1$ is in uA, and x is in the same MAN double coset as the member u^{-1} of $N_K(\mathfrak{a}_0)$.

5. Structure of M

We continue to assume that G is a reductive Lie group and that other notation is as in §2. The fundamental source of disconnectedness in the structure theory of semisimple groups is the behavior of the subgroup $M = Z_K(\mathfrak{a}_0)$. We shall examine M in this section, paying particular attention to its component structure. For the first time we shall make serious use of results from Chapter V.

Proposition 7.48. *M is a reductive Lie group.*

PROOF. Proposition 7.25 shows that $Z_G(\mathfrak{a}_0)$ is a reductive Lie group, necessarily of the form $Z_K(\mathfrak{a}_0)\exp(Z_{\mathfrak{g}_0}(\mathfrak{a}_0) \cap \mathfrak{p}_0) = MA$. By Proposition 7.27, ${}^0(MA) = M$ is a reductive Lie group.

Proposition 7.33 already tells us that M meets every component of G. But M can be disconnected even when G is connected. (Recall from the examples in §VI.5 that M is disconnected when $G = SL(n, \mathbb{R})$.) Choose and fix a maximal abelian subspace $\mathfrak{t}_0$ of $\mathfrak{m}_0$. Then $\mathfrak{a}_0 \oplus \mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{g}_0$.

Proposition 7.49. Every component of M contains a member of M that centralizes $\mathfrak{t}_0$, so that $M = Z_M(\mathfrak{t}_0)M_0$.

REMARK. The proposition says that we may focus our attention on $Z_M(\mathfrak{t}_0)$. After this proof we shall study $Z_M(\mathfrak{t}_0)$ by considering it as a subgroup of $Z_K(\mathfrak{t}_0)$.

PROOF. If $m \in M$ is given, then $\mathrm{Ad}(m)\mathfrak{t}_0$ is a maximal abelian subspace of $\mathfrak{m}_0$. By Theorem 4.34 (applied to M_0), there exists $m_0 \in M_0$ such that $\mathrm{Ad}(m_0)\mathrm{Ad}(m)\mathfrak{t}_0 = \mathfrak{t}_0$. Then $m_0 m$ is in $N_M(\mathfrak{m}_0)$. Introduce a positive system Δ^+ for the root system $\Delta = \Delta(\mathfrak{m}, \mathfrak{t})$. Then $\mathrm{Ad}(m_0 m)\Delta^+$ is a positive system for Δ, and Theorems 4.54 and 2.63 together say that we can find $m_1 \in M_0$ such that $\mathrm{Ad}(m_1 m_0 m)$ maps Δ^+ to itself. By Proposition 7.48, M satisfies property (v) of reductive Lie groups. Therefore $\mathrm{Ad}_{\mathfrak{m}}(m_1 m_0 m)$ is in $\mathrm{Int}\,\mathfrak{m}$. Then $\mathrm{Ad}_{\mathfrak{m}}(m_1 m_0 m)$ must be induced by an element in $\mathrm{Int}_{\mathfrak{m}}[\mathfrak{m}, \mathfrak{m}]$, and Theorem 7.8 says that this element fixes each member of Δ^+. Therefore $m_1 m_0 m$ centralizes $\mathfrak{t}_0$, and the result follows.

Suppose that the root α in $\Delta(\mathfrak{g}, \mathfrak{a} \oplus \mathfrak{t})$ is real, i.e., α vanishes on $\mathfrak{t}$. As in the discussion following (6.66), the root space $\mathfrak{g}_\alpha$ in $\mathfrak{g}$ is invariant under the conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. Since $\dim_{\mathbb{C}} \mathfrak{g}_\alpha = 1$, $\mathfrak{g}_\alpha$ contains a nonzero root vector E_α that is in $\mathfrak{g}_0$. Also as in the discussion following (6.66), we may normalize E_α by a real constant so that $B(E_\alpha, \theta E_\alpha) = -2/|\alpha|^2$. Put $H'_\alpha = 2|\alpha|^{-2}H_\alpha$. Then $\{H'_\alpha, E_\alpha, \theta E_\alpha\}$ spans a copy of $\mathfrak{sl}(2, \mathbb{R})$ with

$$H'_\alpha \leftrightarrow h, \qquad E_\alpha \leftrightarrow e, \qquad \theta E_\alpha \leftrightarrow -f. \tag{7.50}$$

Let us write $(\mathfrak{g}_0)_\alpha$ for $\mathbb{R}E_\alpha$ and $(\mathfrak{g}_0)_{-\alpha}$ for $\mathbb{R}\theta E_\alpha$.

Proposition 7.51. The subgroup $Z_G(\mathfrak{t}_0)$ of G

(a) is reductive with global Cartan decomposition

$$Z_G(\mathfrak{t}_0) = Z_K(\mathfrak{t}_0)\exp(\mathfrak{p}_0 \cap Z_{\mathfrak{g}_0}(\mathfrak{t}_0))$$

(b) has Lie algebra

$$Z_{\mathfrak{g}_0}(\mathfrak{t}_0) = \mathfrak{t}_0 \oplus \mathfrak{a}_0 \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g}, \mathfrak{a}\oplus\mathfrak{t}), \\ \alpha \text{ real}}} (\mathfrak{g}_0)_\alpha,$$

which is the direct sum of its center with a real semisimple Lie algebra that is a split real form of its complexification

(c) is such that the component groups of G, K, $Z_G(\mathfrak{t}_0)$, and $Z_K(\mathfrak{t}_0)$ are all isomorphic.

PROOF.

(a) Every member of $K_{\text{split}} \cap \exp i\mathfrak{a}_0$ centralizes $\mathfrak{a}_0$ and lies in K_{split}, hence lies in F. For the reverse inclusion we have $F \subseteq K_{\text{split}}$ by definition. To see that $F \subseteq \exp i\mathfrak{a}_0$, let U_{split} be the analytic subgroup of $G^{\mathbb{C}}$ with Lie algebra the intersection of $\mathfrak{u}_0$ with the Lie algebra $[Z_{\mathfrak{g}}(\mathfrak{t}_0), Z_{\mathfrak{g}}(\mathfrak{t}_0)]$. Then U_{split} is compact, and $i\mathfrak{a}_0 \cap [Z_{\mathfrak{g}}(\mathfrak{t}_0), Z_{\mathfrak{g}}(\mathfrak{t}_0)]$ is a maximal abelian subspace of its Lie algebra. By Corollary 4.52 the corresponding torus is its own centralizer. Hence the centralizer of $\mathfrak{a}_0$ in U_{split} is contained in $\exp i\mathfrak{a}_0$. Since $K_{\text{split}} \subseteq U_{\text{split}}$, it follows that $F \subseteq \exp i\mathfrak{a}_0$.

(b, c) Corollary 7.52 says that $M = FM_0$. By (a), every element of F commutes with any element that centralizes $\mathfrak{a}_0$. Hence F is central in M, and (b) and (c) follow.

(d) Since G_{split} has finite center, F is compact. Its Lie algebra is 0, and thus it is finite. By (b), F is abelian. We still have to prove that every element $f \neq 1$ in F has order 2.

Since G has a complexification, so does G_{split}. Call this group $G^{\mathbb{C}}_{\text{split}}$, let $\tilde{G}^{\mathbb{C}}_{\text{split}}$ be a simply connected covering group, and let φ be the covering map. Let $\tilde{G}_{\text{split}}$ be the analytic subgroup with the same Lie algebra as for G_{split}, and form the subgroups $\tilde{K}_{\text{split}}$ and $\tilde{F}$ of $\tilde{G}_{\text{split}}$. The subgroup $\tilde{F}$ is the complete inverse image of F under φ. Let $\tilde{U}_{\text{split}}$ play the same role for $\tilde{G}^{\mathbb{C}}_{\text{split}}$ that U plays for $G^{\mathbb{C}}$. The automorphism θ of the Lie algebra of G_{split} complexifies and lifts to an automorphism $\tilde{\theta}$ of $\tilde{G}^{\mathbb{C}}_{\text{split}}$ that carries $\tilde{U}_{\text{split}}$ into itself. The automorphism $\tilde{\theta}$ acts as $x \mapsto x^{-1}$ on $\exp i\mathfrak{a}_0$ and as the identity on $\tilde{K}_{\text{split}}$. The elements of $\tilde{F}$ are the elements of the intersection, by (a), and hence $\tilde{f}^{-1} = \tilde{f}$ for every element $\tilde{f}$ of $\tilde{F}$. That is $\tilde{f}^2 = 1$. Applying φ and using the fact that φ maps $\tilde{F}$ onto F, we conclude that every element $f \neq 1$ in F has order 2.

EXAMPLE. When G does not have a complexification, the subgroup F need not be abelian. For an example we observe that the group K for $SL(3,\mathbb{R})$ is $SO(3)$, which has $SU(2)$ as a 2-sheeted simply connected covering group. Thus $SL(3,\mathbb{R})$ has a 2-sheeted simply connected covering group, and we take this covering group as G. We already noted in §VI.5 that the group M for $SL(3,\mathbb{R})$ consists of the diagonal matrices with diagonal entries ± 1 and determinant 1. Thus M is the direct sum of two 2-element groups. The subgroup F of G is the complete inverse image of M under the covering map and thus has order 8. Moreover it is a subgroup of $SU(2)$, which has only one element of order 2. Thus F is a group of order 8 with only one element of order 2 and no element of order 8. Of the five abstract groups of order 8, only the 8-element subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternions has this property. This group is nonabelian, and hence F is nonabelian.

$H \in \mathfrak{a}_0$ such that $\lambda(H) \neq 0$ for all $\lambda \in \Sigma$, then

$$\begin{aligned} \mathrm{Ad}(x)H - H &= e^{\mathrm{ad}\,X}H - H \\ &= [X, H] + \tfrac{1}{2}[X, [X, H]] + \cdots \\ &= [X_{\mu_0}, H] + \text{terms for lower restricted roots.} \end{aligned}$$

In particular, $\mathrm{Ad}(x)H - H$ is in $\mathfrak{n}_0^-$ and is not 0. On the other hand, if x is in MAN, then $\mathrm{Ad}(x)H - H$ is in $\mathfrak{n}_0$. Since $\mathfrak{n}_0^- \cap \mathfrak{n}_0 = 0$, we must have $N^- \cap MAN = \{1\}$.

Lemma 7.65. The map $K/M \to G/MAN$ induced by inclusion is a diffeomorphism.

PROOF. The given map is certainly smooth. If $\kappa(g)$ denotes the K component of g in the Iwasawa decomposition $G = KAN$ of Proposition 7.31, then $g \mapsto \kappa(g)$ is smooth, and the map $gMAN \mapsto \kappa(g)M$ is a two-sided inverse to the given map.

Theorem 7.66. Suppose that the reductive Lie group G is semisimple, is of real rank one, and has a complexification $G^{\mathbb{C}}$. Then M is connected unless $\dim \mathfrak{n}_0 = 1$.

REMARKS. Since G is semisimple, it is in the Harish-Chandra class. The above remarks about simple components are therefore applicable. The condition $\dim \mathfrak{n}_0 = 1$ is the same as the condition that the simple component of $\mathfrak{g}_0$ containing $\mathfrak{a}_0$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$. In fact, if $\dim \mathfrak{n}_0 = 1$, then $\mathfrak{n}_0$ is of the form $\mathbb{R}X$ for some X. Then X, θX, and $[X, \theta X]$ span a copy of $\mathfrak{sl}(2, \mathbb{R})$, and we obtain $\mathfrak{g}_0 \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{m}_0$. The Lie subalgebra $\mathfrak{m}_0$ must centralize X, θX, and $[X, \theta X]$ and hence must be an ideal in $\mathfrak{g}_0$. The complementary ideal is $\mathfrak{sl}(2, \mathbb{R})$, as asserted.

PROOF. The multiplication map $N^- \times M_0AN \to G$ is smooth and everywhere regular by Lemma 6.44. Hence the map $N^- \to G/M_0AN$ induced by inclusion is smooth and regular, and so is the map

$$N^- \to G/MAN, \tag{7.67}$$

which is the composition of $N^- \to G/M_0AN$ and a covering map. Also the map (7.67) is one-one by Lemma 7.64. Therefore (7.67) is a diffeomorphism onto an open set.

Since G is semisimple and has real rank 1, the Weyl group $W(\Sigma)$ has two elements. By Proposition 7.32, $W(G, A)$ has two elements. Let $\tilde{w} \in N_K(\mathfrak{a}_0)$ represent the nontrivial element of $W(G, A)$. By the Bruhat decomposition (Theorem 7.40),

$$G = MAN \cup MAN\tilde{w}MAN = MAN \cup N\tilde{w}MAN. \tag{7.68}$$

EXAMPLES.

1) Let $G = SL(n, \mathbb{K})$, where $\mathbb{K}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. When $\mathfrak{g}_0$ is realized as matrices, the Lie subalgebra of upper-triangular matrices is a minimal parabolic subalgebra $\mathfrak{q}_{\mathfrak{p},0}$. The other examples of parabolic subalgebras $\mathfrak{q}_0$ containing $\mathfrak{q}_{\mathfrak{p},0}$ and written as in (7.70) and (7.71) are the Lie subalgebras of block upper-triangular matrices, one subalgebra for each arrangement of blocks.

2) Let G have compact center and be of real rank one. The examples as in (7.70) and (7.71) are the minimal parabolic subalgebras and $\mathfrak{g}_0$ itself.

We shall work with a vector X in the restricted-root space $(\mathfrak{g}_0)_\gamma$ and with θX in $(\mathfrak{g}_0)_{-\gamma}$. (See Proposition 6.40c.) Proposition 6.52 shows that $B(X, \theta X)H_\gamma$ is a negative multiple of H_γ. Normalizing, we may assume that $B(X, \theta X) = -2/|\gamma|^2$. Put $H'_\gamma = 2|\gamma|^{-2}H_\gamma$. Then the linear span $\mathfrak{sl}_X$ of $\{X, \theta X, H'_\gamma\}$ is isomorphic to $\mathfrak{sl}(2, \mathbb{R})$ under the isomorphism

$$H'_\gamma \leftrightarrow h, \qquad X \leftrightarrow e, \qquad \theta X \leftrightarrow -f. \tag{7.72}$$

We shall make use of the copy $\mathfrak{sl}_X$ of $\mathfrak{sl}(2, \mathbb{R})$ in the same way as in the proof of Corollary 6.53. This subalgebra of $\mathfrak{g}_0$ acts by ad on $\mathfrak{g}_0$ and hence acts on $\mathfrak{g}$. We know from Theorem 1.64 that the resulting representation of $\mathfrak{sl}_X$ is completely reducible, and we know the structure of each irreducible subspace from Theorem 1.63.

Lemma 7.73. Let γ be a restricted root, and let $X \neq 0$ be in $(\mathfrak{g}_0)_\gamma$. Then

(a) $\operatorname{ad} X$ carries $(\mathfrak{g}_0)_\gamma$ onto $(\mathfrak{g}_0)_{2\gamma}$
(b) $(\operatorname{ad} \theta X)^2$ carries $(\mathfrak{g}_0)_\gamma$ onto $(\mathfrak{g}_0)_{-\gamma}$
(c) $(\operatorname{ad} \theta X)^4$ carries $(\mathfrak{g}_0)_{2\gamma}$ onto $(\mathfrak{g}_0)_{-2\gamma}$.

PROOF. Without loss of generality, we may assume that X is normalized as in (7.72). The complexification of $\bigoplus_{c\in\mathbb{Z}} (\mathfrak{g}_0)_{c\gamma}$ is an invariant subspace of $\mathfrak{g}$ under the representation ad of $\mathfrak{sl}_X$. Using Theorem 1.64, we decompose it as the direct sum of irreducible representations. Each member of $(\mathfrak{g}_0)_{c\gamma}$ is an eigenvector for $\operatorname{ad} H'_\gamma$ with eigenvalue $2c$, and H'_γ corresponds to the member h of $\mathfrak{sl}(2, \mathbb{R})$. From Theorem 1.63 we see that the only possibilities for irreducible subspaces are 5-dimensional subspaces consisting of one dimension each from

$$(\mathfrak{g}_0)_{2\gamma},\ (\mathfrak{g}_0)_\gamma,\ \mathfrak{m}_0,\ (\mathfrak{g}_0)_{-\gamma},\ (\mathfrak{g}_0)_{-2\gamma};$$

3-dimensional subspaces consisting of one dimension each from

$$(\mathfrak{g}_0)_\gamma,\ \mathfrak{m}_0,\ (\mathfrak{g}_0)_{-\gamma};$$

Proposition 7.78. A parabolic subalgebra $\mathfrak{q}_0$ containing the minimal parabolic subalgebra $\mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{n}_{\mathfrak{p},0}$ has the properties that

(a) $\mathfrak{m}_0$, $\mathfrak{a}_0$, and $\mathfrak{n}_0$ are Lie subalgebras, and $\mathfrak{n}_0$ is an ideal in $\mathfrak{q}_0$
(b) $\mathfrak{a}_0$ is abelian, and $\mathfrak{n}_0$ is nilpotent
(c) $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$
(d) $\mathfrak{q}_0 \cap \theta\mathfrak{q}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0$, and $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is reductive
(e) $\mathfrak{a}_{\mathfrak{p},0} = \mathfrak{a}_0 \oplus \mathfrak{a}_{M,0}$
(f) $\mathfrak{n}_{\mathfrak{p},0} = \mathfrak{n}_0 \oplus \mathfrak{n}_{M,0}$ as vector spaces
(g) $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0 \oplus \mathfrak{n}_0 \oplus \theta\mathfrak{n}_0$ orthogonally with respect to θ
(h) $\mathfrak{m}_0 = \mathfrak{m}_{\mathfrak{p},0} \oplus \mathfrak{a}_{M,0} \oplus \mathfrak{n}_{M,0} \oplus \theta\mathfrak{n}_{M,0}$.

PROOF.

(a, b, e, f) All parts of these are clear.

(c) The centralizer of $\mathfrak{a}_0$ is spanned by $\mathfrak{a}_{\mathfrak{p},0}$, $\mathfrak{m}_{\mathfrak{p},0}$, and all the restricted root spaces for restricted roots vanishing on $\mathfrak{a}_0$. The sum of these is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$.

(d) Since $\theta(\mathfrak{g}_0)_\beta = (\mathfrak{g}_0)_{-\beta}$ by Proposition 6.40c, $\mathfrak{q}_0 \cap \theta\mathfrak{q}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Then $\mathfrak{a}_0 \oplus \mathfrak{m}_0$ is reductive by Corollary 6.29.

(g, h) These follow from Proposition 6.40.

Proposition 7.79. Among the parabolic subalgebras containing $\mathfrak{q}_{\mathfrak{p},0}$, let $\mathfrak{q}_0$ be the one corresponding to the subset Π' of simple restricted roots. For $\eta \neq 0$ in $\mathfrak{a}_0^*$, let

$$(\mathfrak{g}_0)_{(\eta)} = \bigoplus_{\substack{\beta \in \mathfrak{a}_{\mathfrak{p},0}^*, \\ \beta|_{\mathfrak{a}_0} = \eta}} (\mathfrak{g}_0)_\beta.$$

Then $(\mathfrak{g}_0)_{(\eta)} \subseteq \mathfrak{n}_0$ or $(\mathfrak{g}_0)_{(\eta)} \subseteq \theta\mathfrak{n}_0$.

PROOF. We have

$$\mathfrak{a}_{M,0} = \mathfrak{a}_0^\perp = \Big(\bigcap_{\beta \in \Gamma \cap -\Gamma} \ker\beta\Big)^\perp = \Big(\bigcap_{\beta \in \Gamma \cap -\Gamma} H_\beta^\perp\Big)^\perp = \sum_{\beta \in \Gamma \cap -\Gamma} \mathbb{R}H_\beta = \sum_{\beta \in \Pi'} \mathbb{R}H_\beta.$$

Let β and β' be restricted roots with a common nonzero restriction η to members of $\mathfrak{a}_0$. Then $\beta - \beta'$ is 0 on $\mathfrak{a}_0$, and $H_\beta - H_{\beta'}$ is in $\mathfrak{a}_{M,0}$. From the formula for $\mathfrak{a}_{M,0}$, the expansion of $\beta - \beta'$ in terms of simple restricted roots involves only the members of Π'. Since $\eta \neq 0$, the individual expansions of β and β' involve nonzero coefficients for at least one simple restricted root other than the ones in Π'. The coefficients for this other simple restricted root must be equal and in particular of the same sign. By Proposition 2.49, β and β' are both positive or both negative, and the result follows.

PROOF.

(a, b) The subgroups $Z_G(\mathfrak{a}_0)$ and ${}^0Z_G(\mathfrak{a}_0)$ are reductive by Propositions 7.25 and 7.27. By Proposition 7.78, $Z_{\mathfrak{g}_0}(\mathfrak{a}_0) = \mathfrak{a}_0 \oplus \mathfrak{m}_0$. Thus the space Z_{vec} for the group $Z_G(\mathfrak{a}_0)$ is the analytic subgroup corresponding to the intersection of $\mathfrak{p}_0$ with the center of $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. From the definition of $\mathfrak{m}_0$, the center of $Z_{\mathfrak{g}_0}(\mathfrak{a}_0)$ has to be contained in $\mathfrak{a}_{\mathfrak{p},0} \oplus \mathfrak{m}_{\mathfrak{p},0}$, and the $\mathfrak{p}_0$ part of this is $\mathfrak{a}_{\mathfrak{p},0}$. The part of $\mathfrak{a}_{\mathfrak{p},0}$ that commutes with $\mathfrak{m}_0$ is $\mathfrak{a}_0$ by definition of $\mathfrak{m}_0$. Therefore $Z_{vec} = \exp \mathfrak{a}_0 = A$, and $Z_G(\mathfrak{a}_0) = ({}^0Z_G(\mathfrak{a}_0))A$ by Proposition 7.27. Then (a) and (b) follow.

(c) By (a), M is reductive. It is clear that $\mathfrak{a}_{M,0}$ is a maximal abelian subspace of $\mathfrak{p}_0 \cap \mathfrak{m}_0$, since $\mathfrak{m}_0 \cap \mathfrak{a}_0 = 0$. The restricted roots of $\mathfrak{m}_0$ relative to $\mathfrak{a}_{M,0}$ are then the members of $\Gamma \cap -\Gamma$, and the sum of the restricted-root spaces for the positive such restricted roots is $\mathfrak{n}_{M,0}$. Therefore the minimal parabolic subgroup in question for M is $M_M A_M N_M$. The computation

$$\begin{aligned} M_M = Z_{K\cap M}(\mathfrak{a}_{M,0}) &= MA \cap Z_K(\mathfrak{a}_{M,0}) \\ &= Z_G(\mathfrak{a}_0) \cap Z_K(\mathfrak{a}_{M,0}) = Z_K(\mathfrak{a}_{\mathfrak{p},0}) = M_{\mathfrak{p}} \end{aligned}$$

identifies M_M, and $M = K_M A_M N_M$ by the Iwasawa decomposition for M (Proposition 7.31).

(d) By (a), M is reductive. Hence $M = M_M M_0$ by Proposition 7.33. But (c) shows that $M_M = M_{\mathfrak{p}}$, and Corollary 7.52 shows that $M_{\mathfrak{p}} = F(M_{\mathfrak{p}})_0$. Hence $M = FM_0$.

(e) This follows from Proposition 7.78e and the simple connectivity of $A_{\mathfrak{p}}$.

(f) This follows from Proposition 7.78f, Theorem 1.102, and the simple connectivity of $N_{\mathfrak{p}}$.

Proposition 7.83. The subgroups M, A, and N have the properties that

(a) MA normalizes N, so that $Q = MAN$ is a group
(b) $Q = N_G(\mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0)$, and hence Q is a closed subgroup
(c) Q has Lie algebra $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$
(d) multiplication $M \times A \times N \to Q$ is a diffeomorphism
(e) $N^- \cap Q = \{1\}$
(f) $G = KQ$.

PROOF.

(a) Let z be in $MA = Z_G(\mathfrak{a}_0)$, and fix $(\mathfrak{g}_0)_{(\eta)} \subseteq \mathfrak{n}_0$ as in (7.80). If X is in $(\mathfrak{g}_0)_{(\eta)}$ and H is in $\mathfrak{a}_0$, then

$$[H, \mathrm{Ad}(z)X] = [\mathrm{Ad}(z)H, \mathrm{Ad}(z)X] = \mathrm{Ad}(z)[H, X] = \eta(H)\mathrm{Ad}(z)X.$$

according to $M = K_M A_M N_M$ be $m = k_M a_M n_M$. If this element is to be in AN, then $k_M = 1$, a_M is in $A_M \cap A$, and n_M is in $N_M \cap N$, by uniqueness of the Iwasawa decomposition in G. But $A_M \cap A = \{1\}$ and $N_M \cap N = \{1\}$ by (e) and (f) of Proposition 7.82. Therefore $m = 1$, and we conclude that $M \cap AN = \{1\}$.

(e) This is proved in the same way as Lemma 7.64, which is stated for a minimal parabolic subgroup.

(f) Since $Q \supseteq A_{\mathfrak{p}} N_{\mathfrak{p}}$, $G = KQ$ by the Iwasawa decomposition for G (Proposition 7.31).

Although the set of $\mathfrak{a}_0$ roots does not necessarily form an abstract root system, it is still meaningful to define

$$W(G, A) = N_K(\mathfrak{a}_0)/Z_K(\mathfrak{a}_0), \tag{7.84a}$$

just as we did in the case that $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{p}_0$. Corollary 7.81 and Proposition 7.78c show that $N_K(\mathfrak{a}_0)$ and $Z_K(\mathfrak{a}_0)$ both have $\mathfrak{k}_0 \cap \mathfrak{m}_0$ as Lie algebra. Hence $W(G, A)$ is a compact 0-dimensional group, and we conclude that $W(G, A)$ is finite. An alternative formula for $W(G, A)$ is

$$W(G, A) = N_G(\mathfrak{a}_0)/Z_G(\mathfrak{a}_0). \tag{7.84b}$$

The equality of the right sides of (7.84a) and (7.84b) is an immediate consequence of Lemma 7.22 and Corollary 7.81. To compute $N_K(\mathfrak{a}_0)$, it is sometimes handy to use the following proposition.

Proposition 7.85. Every element of $N_K(\mathfrak{a}_0)$ decomposes as a product zn, where n is in $N_K(\mathfrak{a}_{\mathfrak{p},0})$ and z is in $Z_K(\mathfrak{a}_0)$.

PROOF. Let k be in $N_K(\mathfrak{a}_0)$ and form $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$. Since $\mathfrak{a}_{M,0}$ commutes with $\mathfrak{a}_0$, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ commutes with $\mathrm{Ad}(k)\mathfrak{a}_0 = \mathfrak{a}_0$. By Proposition 7.78c, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is contained in $\mathfrak{a}_0 \oplus \mathfrak{m}_0$. Since $\mathfrak{a}_{M,0}$ is orthogonal to $\mathfrak{a}_0$ under B_θ, $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is orthogonal to $\mathrm{Ad}(k)\mathfrak{a}_0 = \mathfrak{a}_0$. Hence $\mathrm{Ad}(k)\mathfrak{a}_{M,0}$ is contained in $\mathfrak{m}_0$ and therefore in $\mathfrak{p}_0 \cap \mathfrak{m}_0$. By Proposition 7.29 there exists z in $K \cap M$ with $\mathrm{Ad}(z)^{-1}\mathrm{Ad}(k)\mathfrak{a}_{M,0} = \mathfrak{a}_{M,0}$. Then $n = z^{-1}k$ is in $N_K(\mathfrak{a}_0)$ and in $N_K(\mathfrak{a}_{M,0})$, hence in $N_K(\mathfrak{a}_{\mathfrak{p},0})$.

EXAMPLE. Let $G = SL(3, \mathbb{R})$. Take $\mathfrak{a}_{\mathfrak{p},0}$ to be the diagonal subalgebra, and let $\Sigma^+ = \{f_1 - f_2,\ f_2 - f_3,\ f_1 - f_3\}$ in the notation of Example 1 of §VI.4. Define a parabolic subalgebra $\mathfrak{q}_0$ by using $\Pi' = \{f_1 - f_2\}$. The corresponding parabolic subgroup is the block upper-triangular group with blocks of sizes 2 and 1, respectively. The subalgebra $\mathfrak{a}_0$ equals $\{\mathrm{diag}(r, r, -2r)\}$. Suppose that w is in $W(G, A)$. Proposition 7.85 says that w extends to a member of $W(G, A_{\mathfrak{p}})$ leaving $\mathfrak{a}_0$ and $\mathfrak{a}_{M,0}$ individually stable. Here $W(G, A_{\mathfrak{p}}) = W(\Sigma)$, and the only member of $W(\Sigma)$ sending $\mathfrak{a}_0$ to itself is the identity. So $W(G, A) = \{1\}$.

Proposition 7.87. Let $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ be the decomposition of a θ stable Cartan subalgebra according to θ, and suppose that a lexicographic ordering taking $\mathfrak{a}_0$ before $i\mathfrak{t}_0$ is used to define a positive system $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Define

$$\mathfrak{m}_0 = \mathfrak{g}_0 \cap \big(\mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}),\\ \alpha|_{\mathfrak{a}}=0}} \mathfrak{g}_\alpha\big)$$

and

$$\mathfrak{n}_0 = \mathfrak{g}_0 \cap \big(\bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}),\\ \alpha|_{\mathfrak{a}}\neq 0}} \mathfrak{g}_\alpha\big).$$

Then $\mathfrak{q}_0 = \mathfrak{m}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ is the Langlands decomposition of a cuspidal parabolic subalgebra of $\mathfrak{g}_0$.

PROOF. In view of the definitions, we have to relate $\mathfrak{q}_0$ to a minimal parabolic subalgebra. Let bar denote conjugation of $\mathfrak{g}$ with respect to $\mathfrak{g}_0$. If $\alpha = \alpha_{\mathfrak{a}} + \alpha_{\mathfrak{t}}$ is a root, let $\bar\alpha = -\theta\alpha = \alpha_{\mathfrak{a}} - \alpha_{\mathfrak{t}}$. Then $\overline{\mathfrak{g}_\alpha} = \mathfrak{g}_{\bar\alpha}$, and it follows that

$$\mathfrak{m} = \mathfrak{t} \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}),\\ \alpha|_{\mathfrak{a}}=0}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}),\\ \alpha|_{\mathfrak{a}}\neq 0}} \mathfrak{g}_\alpha. \tag{7.88}$$

In particular, $\mathfrak{m}_0$ is θ stable, hence reductive. Let $\mathfrak{h}_{M,0} = \mathfrak{t}_{M,0} \oplus \mathfrak{a}_{M,0}$ be the decomposition of a maximally noncompact θ stable Cartan subalgebra of $\mathfrak{m}_0$ according to θ. Since Theorem 2.15 shows that $\mathfrak{h}_M$ is conjugate to $\mathfrak{t}$ via $\operatorname{Int}\mathfrak{m}$, $\mathfrak{h}' = \mathfrak{a} \oplus \mathfrak{h}_M$ is conjugate to $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{t}$ via a member of $\operatorname{Int}\mathfrak{g}$ that fixes $\mathfrak{a}_0$. In particular, $\mathfrak{h}'_0 = \mathfrak{a}_0 \oplus \mathfrak{h}_{M,0}$ is a Cartan subalgebra of $\mathfrak{g}_0$. Applying our constructed member of $\operatorname{Int}\mathfrak{g}$ to (7.88), we obtain

$$\mathfrak{m} = \mathfrak{h}_M \oplus \bigoplus_{\substack{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}'),\\ \alpha|_{\mathfrak{a}}=0}} \mathfrak{g}_\alpha \qquad \text{and} \qquad \mathfrak{n} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h}'),\\ \alpha|_{\mathfrak{a}}\neq 0}} \mathfrak{g}_\alpha \tag{7.89}$$

for the positive system $\Delta^+(\mathfrak{g}, \mathfrak{h}')$ obtained by transferring positivity from $\Delta^+(\mathfrak{g}, \mathfrak{h})$.

Let us note that $\mathfrak{a}_{\mathfrak{p},0} = \mathfrak{a}_0 \oplus \mathfrak{a}_{M,0}$ is a maximal abelian subspace of $\mathfrak{p}_0$. In fact, the centralizer of $\mathfrak{a}_0$ in $\mathfrak{g}_0$ is $\mathfrak{a}_0 \oplus \mathfrak{m}_0$, and $\mathfrak{a}_{M,0}$ is maximal abelian in $\mathfrak{m}_0 \cap \mathfrak{p}_0$; hence the assertion follows. We introduce a lexicographic ordering for $\mathfrak{h}'_0$ that is as before on $\mathfrak{a}_0$, takes $\mathfrak{a}_0$ before $\mathfrak{a}_{M,0}$, and takes $\mathfrak{a}_{M,0}$ before $i\mathfrak{t}_{M,0}$. Then we obtain a positive system $\Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}')$ with the property that a root α with $\alpha|_{\mathfrak{a}_0} \neq 0$ is positive if and only if $\alpha|_{\mathfrak{a}_0}$ is the restriction to $\mathfrak{a}_0$ of a member of $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Consequently we can replace $\Delta^+(\mathfrak{g}, \mathfrak{h}')$ in (7.89) by $\Delta^{+\prime}(\mathfrak{g}, \mathfrak{h}')$. Then it is apparent that

PROOF. By Proposition 7.90a, $G = G_0H$. If z is in Z_{G_0}, then $\mathrm{Ad}(z) = 1$ on $\mathfrak{h}_0$, and hence z is in $Z_G(\mathfrak{h}_0) = H$. Let $g \in G$ be given, and write $g = g_0h$ with $g \in G_0$ and $h \in H$. Then $zg_0 = g_0z$ since z commutes with members of G_0, and $zh = hz$ since z is in H and H is abelian. Hence $zg = gz$, and z is in Z_G.

If H is a Cartan subgroup of G with Lie algebra $\mathfrak{h}_0$, we define

$$W(G, H) = N_G(\mathfrak{h}_0)/Z_G(\mathfrak{h}_0). \tag{7.92a}$$

Here $Z_G(\mathfrak{h}_0)$ is nothing more than H itself, by definition. When $\mathfrak{h}_0$ is θ stable, an alternative formula for $W(G, H)$ is

$$W(G, H) = N_K(\mathfrak{h}_0)/Z_K(\mathfrak{h}_0). \tag{7.92b}$$

The equality of the right sides of (7.92a) and (7.92b) is an immediate consequence of Lemma 7.22 and Proposition 2.7. Proposition 2.7 shows that $N_K(\mathfrak{h}_0)$ and $Z_K(\mathfrak{h}_0)$ both have $\mathfrak{k}_0 \cap \mathfrak{h}_0 = \mathfrak{t}_0$ as Lie algebra. Hence $W(G, H)$ is a compact 0-dimensional group, and we conclude that $W(G, H)$ is finite.

Each member of $N_G(\mathfrak{h}_0)$ sends roots of $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ to roots, and the action of $N_G(\mathfrak{h}_0)$ on Δ descends to $W(G, H)$. It is clear that only the identity in $W(G, H)$ acts as the identity on Δ. Since $\mathrm{Ad}_{\mathfrak{g}}(G) \subseteq \mathrm{Int}\,\mathfrak{g}$, it follows from Theorem 7.8 that

$$W(G, H) \subseteq W(\Delta(\mathfrak{g}, \mathfrak{h})). \tag{7.93}$$

EXAMPLE. Let $G = SL(2, \mathbb{R})$. For any $\mathfrak{h}$, $W(\mathfrak{g}, \mathfrak{h})$ has order 2. When $\mathfrak{h}_0 = \left\{\begin{pmatrix} r & 0 \\ 0 & -r \end{pmatrix}\right\}$, $W(G, H)$ has order 2, a representative of the nontrivial coset being $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. When $\mathfrak{h}_0 = \left\{\begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix}\right\}$, $W(G, H)$ has order 1.

Now we begin to work toward the main result of this section, that the union of all Cartan subgroups of G exhausts almost all of G. We shall use the notion of a "regular element" of G. Recall that in Chapter II we introduced regular elements in the complexified Lie algebra $\mathfrak{g}$. Let $\dim \mathfrak{g} = n$. For $X \in \mathfrak{g}$, we formed the characteristic polynomial

$$\det(\lambda 1 - \mathrm{ad}\, X) = \lambda^n + \sum_{j=0}^{n-1} d_j(X)\lambda^j. \tag{7.94}$$

Here each d_j is a holomorphic polynomial function on $\mathfrak{g}$. The **rank** of $\mathfrak{g}$ is the minimum index l such that $d_l(X) \not\equiv 0$, and the **regular elements** and (7.132) follows since members of $\mathfrak{g}_0$ equal their own conjugates. The real dimension of $i\mathfrak{t}_0 \oplus \mathfrak{n}^-$ is half the real dimension of $\mathfrak{t} \oplus \mathfrak{n} \oplus \mathfrak{n}^- = \mathfrak{g}$, and hence

$$\dim_{\mathbb{R}}(\mathfrak{g}_0 \oplus (i\mathfrak{t}_0 \oplus \mathfrak{n}^-)) = \dim_{\mathbb{R}} \mathfrak{g}. \tag{7.133}$$

Combining (7.132) and (7.133), we see that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus (i\mathfrak{t}_0 \oplus \mathfrak{n}^-). \tag{7.134}$$

The subgroup $H_{\mathbb{R}}N^-$ of $G^{\mathbb{C}}$ is closed by Proposition 7.83, and hence $H_{\mathbb{R}}N^-$ is an analytic subgroup, necessarily with Lie algebra $i\mathfrak{t}_0 \oplus \mathfrak{n}^-$. By Lemma 6.44 it follows from (7.134) that multiplication $G \times H_{\mathbb{R}}N^- \to G^{\mathbb{C}}$ is everywhere regular. The dimension relation (7.133) therefore implies that $GH_{\mathbb{R}}N^-$ is open in $G^{\mathbb{C}}$. Since $B = TH_{\mathbb{R}}N^-$ and $T \subseteq G$, GB equals $GH_{\mathbb{R}}N^-$ and is open in $G^{\mathbb{C}}$.

The subgroups P^+ and P^- are the N groups of parabolic subalgebras, and their Lie algebras are abelian by Lemma 7.128. Hence P^+ and P^- are Euclidean groups. Then $\exp : \mathfrak{p}^+ \to P^+$ is biholomorphic, and P^+ is biholomorphic with $\mathbb{C}^n$ for some n. Similarly P^- is biholomorphic with $\mathbb{C}^n$.

The subgroup $K^{\mathbb{C}}$ is a reductive group, being connected and having bar as a Cartan involution for its Lie algebra. It is the product of the identity component of its center by a complex semisimple Lie group, and our above considerations show that its parabolic subgroups are connected. Then B_K is a parabolic subgroup, and

$$K^{\mathbb{C}} = KB_K \tag{7.135}$$

by Proposition 7.83f.

Let A denote a specific $A_{\mathfrak{p}}$ component for the Iwasawa decomposition of G, to be specified in Lemma 7.143 below. We shall show in Lemma 7.145 that this A satisfies

$$A \subseteq P^+K^{\mathbb{C}}P^- \tag{7.136a}$$

and

(7.136b) P^+ components of members of A are bounded.

Theorem 7.39 shows that $G = KAK$. Since $\mathfrak{b} \subseteq \mathfrak{k} \oplus \mathfrak{p}^-$, we have $B \subseteq K^{\mathbb{C}}P^-$. Since Lemma 7.128 shows that $K^{\mathbb{C}}$ normalizes P^+ and P^-, (7.136a) gives

$$\begin{aligned} GB &\subseteq GK^{\mathbb{C}}P^- \subseteq KAKK^{\mathbb{C}}P^- \\ &\subseteq KP^+K^{\mathbb{C}}P^-K^{\mathbb{C}}P^- = P^+K^{\mathbb{C}}P^-. \end{aligned} \tag{7.137}$$