Theorem 2.9. Any finite-dimensional complex Lie algebra $\mathfrak{g}$ has a Cartan subalgebra.

Before coming to the proof, we introduce "regular" elements of $\mathfrak{g}$. In $\mathfrak{sl}(n, \mathbb{C})$ the regular elements will be the matrices with distinct eigenvalues. Let us consider matters more generally.

If π is a representation of $\mathfrak{g}$ on a finite-dimensional vector space V, we can regard each $X \in \mathfrak{g}$ as generating a 1-dimensional abelian subalgebra, and we can then form $V_{0,X}$, the generalized eigenspace for eigenvalue 0 under $\pi(X)$. Let

$$l_{\mathfrak{g}}(V) = \min_{X \in \mathfrak{g}} \dim V_{0,X}$$

$$R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} \mid \dim V_{0,X} = l_{\mathfrak{g}}(V)\}.$$

To understand $l_{\mathfrak{g}}(V)$ and $R_{\mathfrak{g}}(V)$ better, form the characteristic polynomial

$$\det(\lambda 1 - \pi(X)) = \lambda^n + \sum_{j=0}^{n-1} d_j(X)\lambda^j.$$

In any basis of $\mathfrak{g}$, the $d_j(X)$ are polynomial functions on $\mathfrak{g}$, as we see by expanding $\det(\lambda 1 - \sum \mu_i \pi(X_i))$. For given X, if j is the smallest value for which $d_j(X) \neq 0$, then $j = \dim V_{0,X}$, since the degree of the last term in the characteristic polynomial is the multiplicity of 0 as a generalized eigenvalue of $\pi(X)$. Thus $l_{\mathfrak{g}}(V)$ is the minimum j such that $d_j(X) \not\equiv 0$, and

$$R_{\mathfrak{g}}(V) = \{X \in \mathfrak{g} \mid d_{l_{\mathfrak{g}}(V)}(X) \neq 0\}.$$

Let us apply these considerations to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$. The elements of $R_{\mathfrak{g}}(\mathfrak{g})$, relative to the adjoint representation, are the **regular elements** of $\mathfrak{g}$. For any X in $\mathfrak{g}$, $\mathfrak{g}_{0,X}$ is a Lie subalgebra of $\mathfrak{g}$ by the corollary of Proposition 2.5, with $\mathfrak{h} = \mathbb{C}X$.

Theorem 2.9′. If X is a regular element of the finite-dimensional complex Lie algebra $\mathfrak{g}$, then the Lie algebra $\mathfrak{g}_{0,X}$ is a Cartan subalgebra of $\mathfrak{g}$.

PROOF. First we show that $\mathfrak{g}_{0,X}$ is nilpotent. Assuming the contrary, we construct two sets:

(i) the set of $Z \in \mathfrak{g}_{0,X}$ such that $((\operatorname{ad} Z)|_{\mathfrak{g}_{0,X}})^{\dim \mathfrak{g}_{0,X}} \neq 0$, which is nonempty by Engel's Theorem (Corollary 1.38) and is open

(ii) the set of $W \in \mathfrak{g}_{0,X}$ such that $\operatorname{ad} W|_{\mathfrak{g}/\mathfrak{g}_{0,X}}$ is nonsingular, which is nonempty since X is in it (regularity is not used here) and is the set where some polynomial is nonvanishing, hence is dense (because if a polynomial vanishes on a nonempty open set, it vanishes identically).

These two sets must have nonempty intersection, and so we can find $Z \in \mathfrak{g}_{0,X}$ such that

$$((\operatorname{ad} Z)|_{\mathfrak{g}_{0,X}})^{\dim \mathfrak{g}_{0,X}} \neq 0 \qquad \text{and} \qquad \operatorname{ad} Z|_{\mathfrak{g}/\mathfrak{g}_{0,X}} \text{ is nonsingular}.$$

Then the generalized multiplicity of the eigenvalue 0 for $\operatorname{ad} Z$ is less than $\dim \mathfrak{g}_{0,X}$, and hence $\dim \mathfrak{g}_{0,Z} < \dim \mathfrak{g}_{0,X}$, in contradiction with the regularity of X. We conclude that $\mathfrak{g}_{0,X}$ is nilpotent.

Since $\mathfrak{g}_{0,X}$ is nilpotent, we can use $\mathfrak{g}_{0,X}$ to decompose $\mathfrak{g}$ as in Proposition 2.4. Let $\mathfrak{g}_0$ be the 0 generalized weight space. Then we have

$$\mathfrak{g}_{0,X} \subseteq \mathfrak{g}_0 = \bigcap_{Y \in \mathfrak{g}_{0,X}} \mathfrak{g}_{0,Y} \subseteq \mathfrak{g}_{0,X}.$$

So $\mathfrak{g}_{0,X} = \mathfrak{g}_0$, and $\mathfrak{g}_{0,X}$ is a Cartan subalgebra.

In this book we shall be interested in Cartan subalgebras $\mathfrak{h}$ only when $\mathfrak{g}$ is semisimple. In this case $\mathfrak{h}$ has special properties, as follows.

Proposition 2.10. If $\mathfrak{g}$ is a complex semisimple Lie algebra and $\mathfrak{h}$ is a Cartan subalgebra, then $\mathfrak{h}$ is abelian.

PROOF. Since $\mathfrak{h}$ is nilpotent and therefore solvable, $\operatorname{ad}\mathfrak{h}$ is solvable as a Lie algebra of transformations of $\mathfrak{g}$. By Lie's Theorem (Corollary 1.29) it is simultaneously triangular in some basis. For any three triangular matrices A, B, C, we have $\operatorname{Tr}(ABC) = \operatorname{Tr}(BAC)$. Therefore

$$\operatorname{Tr}(\operatorname{ad}[H_1, H_2]\operatorname{ad} H) = 0 \qquad \text{for } H_1, H_2, H \in \mathfrak{h}. \tag{2.11}$$

Next let α be any nonzero generalized weight, let X be in $\mathfrak{g}_\alpha$, and let H be in $\mathfrak{h}$. By Proposition 2.5c, $\operatorname{ad} H \operatorname{ad} X$ carries $\mathfrak{g}_\beta$ to $\mathfrak{g}_{\alpha+\beta}$. Thus Proposition 2.5a shows that

$$\operatorname{Tr}(\operatorname{ad} H \operatorname{ad} X) = 0. \tag{2.12}$$

Specializing (2.12) to $H = [H_1, H_2]$ and using (2.11) and Proposition 2.5a, we see that the Killing form B of $\mathfrak{g}$ satisfies

$$B([H_1, H_2], X) = 0 \qquad \text{for all } X \in \mathfrak{g}.$$

By Cartan's Criterion for Semisimplicity (Theorem 1.42), B is nondegenerate. Therefore $[H_1, H_2] = 0$, and $\mathfrak{h}$ is abelian.

Corollary 2.13. In a complex semisimple Lie algebra $\mathfrak{g}$, a Lie subalgebra $\mathfrak{h}$ is a Cartan subalgebra if $\mathfrak{h}$ is maximal abelian and $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonable.

REMARKS.

1) It is immediate from this corollary that the subalgebras $\mathfrak{h}$ in the examples of §1 are Cartan subalgebras.

2) In the direction converse to the corollary, Proposition 2.10 shows that a Cartan subalgebra $\mathfrak{h}$ is abelian, and it is maximal abelian since $\mathfrak{h} = \mathfrak{g}_0$. Corollary 2.23 will show for a Cartan subalgebra $\mathfrak{h}$ that $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonable.

PROOF. Since $\mathfrak{h}$ is abelian and hence nilpotent, Proposition 2.4 shows that $\mathfrak{g}$ has a weight-space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\beta \neq 0} \mathfrak{g}_\beta$. Since $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonable, $\mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{r}$ with $[\mathfrak{h}, \mathfrak{r}] = 0$. In view of Proposition 2.7, we are to prove that $\mathfrak{h} = N_{\mathfrak{g}}(\mathfrak{h})$. Here $\mathfrak{h} \subseteq N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{g}_0$ by (2.8), and it is enough to show that $\mathfrak{r} = 0$. If $X \neq 0$ is in $\mathfrak{r}$, then $\mathfrak{h} \oplus \mathbb{C}X$ is an abelian subalgebra properly containing $\mathfrak{h}$, in contradiction with $\mathfrak{h}$ maximal abelian. The result follows.

3. Uniqueness of Cartan Subalgebras

We turn to the question of uniqueness of Cartan subalgebras. We begin with a lemma about polynomial mappings.

Lemma 2.14. Let $P : \mathbb{C}^m \to \mathbb{C}^n$ be a holomorphic polynomial function not identically 0. Then the set of vectors z in $\mathbb{C}^m$ for which $P(z)$ is not the 0 vector is connected in $\mathbb{C}^m$.

PROOF. Suppose that z_0 and w_0 in $\mathbb{C}^m$ have $P(z_0) \neq 0$ and $P(w_0) \neq 0$. As a function of $z \in \mathbb{C}$, $P(z_0 + z(w_0 - z_0))$ is a vector-valued holomorphic polynomial nonvanishing at $z = 0$ and $z = 1$. The subset of $z \in \mathbb{C}$ where it vanishes is finite, and the complement in $\mathbb{C}$ is connected. Thus z_0 and w_0 lie in a connected set in $\mathbb{C}^m$ where P is nonvanishing. Taking the union of these connected sets with z_0 fixed and w_0 varying, we see that the set where $P(w_0) \neq 0$ is connected.

Theorem 2.15. If $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are Cartan subalgebras of a finite-dimensional complex Lie algebra $\mathfrak{g}$, then there exists $a \in \operatorname{Int} \mathfrak{g}$ with $a(\mathfrak{h}_1) = \mathfrak{h}_2$.

REMARKS.

1) In particular any two Cartan subalgebras are conjugate by an automorphism of $\mathfrak{g}$. As was explained after the introduction of $\operatorname{Int} \mathfrak{g}$ in §I.11,

Proposition 2.78. The abstract Dynkin diagram associated to the l-by-l abstract Cartan matrix A has the following properties:

(a) there are at most l pairs of vertices $i < j$ with at least one edge connecting them
(b) there are no loops
(c) at most three edges issue from any point of the diagram.

PROOF.
(a) With α_i as in (2.75), put $\alpha = \sum_{i=1}^{l} \dfrac{\alpha_i}{|\alpha_i|}$. Then

$$
\begin{aligned}
0 < |\alpha|^2 &= \sum_{i,j} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle \\
&= \sum_{i} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_i}{|\alpha_i|} \right\rangle + 2 \sum_{i<j} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle \\
&= l + \sum_{i<j} \frac{2\langle \alpha_i, \alpha_j \rangle}{|\alpha_i||\alpha_j|} \\
&= l - \sum_{i<j} \sqrt{A_{ij} A_{ji}}. \qquad (2.79)
\end{aligned}
$$

By Proposition 2.74, $\sqrt{A_{ij} A_{ji}}$ is 0 or 1 or $\sqrt{2}$ or $\sqrt{3}$. When nonzero, it is therefore ≥ 1. Therefore the right side of (2.79) is

$$\leq l - \sum_{\substack{i<j,\\ \text{connected}}} 1.$$

Hence the number of connected pairs of vertices is $< l$.

(b) If there were a loop, we could use Operation #1 to remove all vertices except those in a loop. Then (a) would be violated for the loop.

(c) Fix $\alpha = \alpha_i$ as in (2.75). Consider the vertices that are connected by edges to the i^{th} vertex. Write $\beta_1, \ldots, \beta_r$ for the α_j's associated to these vertices, and let there be $l_1, \ldots, l_r$ edges to the i^{th} vertex. Let U be the $(r+1)$-dimensional vector subspace of $\mathbb{R}^l$ spanned by $\beta_1, \ldots, \beta_r, \alpha$. Then $\langle \beta_i, \beta_j \rangle = 0$ for $i \neq j$ by (b), and hence $\{\beta_k / |\beta_k|\}_{k=1}^{r}$ is an orthonormal set. Adjoin $\delta \in U$ to this set to make an orthonormal basis of U. Then $\langle \alpha, \delta \rangle \neq 0$ since $\{\beta_1, \ldots, \beta_r, \alpha\}$ is linearly independent. By Parseval's equality, have an isomorphism, and then compute what the map is in terms of a basis. Let $T_n(\mathfrak{g}) = \bigoplus_{k=0}^n T^k(\mathfrak{g})$ be the n^{th} member of the usual filtration of $T(\mathfrak{g})$. We have defined $U_n(\mathfrak{g})$ to be the image in $U(\mathfrak{g})$ of $T_n(\mathfrak{g})$ under the passage $T(\mathfrak{g}) \to T(\mathfrak{g})/J$. Thus we can form the composition

$$T_n(\mathfrak{g}) \to (T_n(\mathfrak{g}) + J)/J = U_n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}).$$

This composition is onto and carries $T_{n-1}(\mathfrak{g})$ to 0. Since $T^n(\mathfrak{g})$ is a vector-space complement to $T_{n-1}(\mathfrak{g})$ in $T_n(\mathfrak{g})$, we obtain an onto linear map

$$T^n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}).$$

Taking the direct sum over n gives an onto linear map

$$\tilde{\psi} : T(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$$

that respects the grading.

Appendix A uses the notation I for the two-sided ideal in $T(\mathfrak{g})$ such that $S(\mathfrak{g}) = T(\mathfrak{g})/I$:

$$I = \begin{pmatrix} \text{two-sided ideal generated by all} \\ X \otimes Y - Y \otimes X \text{ with } X \text{ and } Y \\ \text{in } T^1(\mathfrak{g}) \end{pmatrix}. \tag{3.15}$$

Proposition 3.16. The linear map $\tilde{\psi} : T(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$ respects multiplication and annihilates the defining ideal I for $S(\mathfrak{g})$. Therefore ψ descends to an algebra homomorphism

$$\psi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g}) \tag{3.17}$$

that respects the grading. This homomorphism is an isomorphism.

PROOF. Let x be in $T^r(\mathfrak{g})$ and let y be in $T^s(\mathfrak{g})$. Then $x + J$ is in $U_r(\mathfrak{g})$, and we may regard $\tilde{\psi}(x)$ as the coset $x + T_{r-1}(\mathfrak{g}) + J$ in $U_r(\mathfrak{g})/U_{r-1}(\mathfrak{g})$, with 0 in all other coordinates of $\operatorname{gr} U(\mathfrak{g})$ since x is homogeneous. Arguing in a similar fashion with y and xy, we obtain

$$\tilde{\psi}(x) = x + T_{r-1}(\mathfrak{g}) + J, \qquad \tilde{\psi}(y) = y + T_{s-1}(\mathfrak{g}) + J,$$
$$\text{and} \qquad \tilde{\psi}(xy) = xy + T_{r+s-1}(\mathfrak{g}) + J.$$

Since J is an ideal, $\tilde{\psi}(x)\tilde{\psi}(y) = \tilde{\psi}(xy)$. General members x and y of $T(\mathfrak{g})$ are sums of homogeneous elements, and hence $\tilde{\psi}$ respects multiplication.

The direct sum of the maps σ_n for $n \geq 0$ (with $\sigma_0(1) = 1$) is a linear map $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ such that

$$\sigma(X_1 \cdots X_n) = \frac{1}{n!} \sum_{\tau \in \mathfrak{S}_n} X_{\tau(1)} \cdots X_{\tau(n)}.$$

The map σ is called **symmetrization**.

Lemma 3.22. The symmetrization map $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$ has associated graded map $\psi : S(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})$, with ψ as in (3.17).

REMARK. See §A.4 for "associated graded map."

PROOF. Let $\{X_i\}$ be a basis of $\mathfrak{g}$, and let $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, with $\sum_m j_m = n$, be a basis vector of $S^n(\mathfrak{g})$. Under σ, this vector is sent to a symmetrized sum, but each term of the sum is congruent mod $U_{n-1}(\mathfrak{g})$ to $(n!)^{-1} X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, by Lemma 3.9. Hence the image of $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$ under the associated graded map is

$$= X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(\mathfrak{g}) = \psi(X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}),$$

as asserted.

Proposition 3.23. Symmetrization σ is a vector-space isomorphism of $S(\mathfrak{g})$ onto $U(\mathfrak{g})$ satisfying

$$U_n(\mathfrak{g}) = \sigma(S^n(\mathfrak{g})) \oplus U_{n-1}(\mathfrak{g}). \tag{3.24}$$

PROOF. Formula (3.24) is a restatement of (3.21), and the other conclusion follows by combining Lemma 3.22 and Proposition A.37.

The canonical decomposition of $U(\mathfrak{g})$ from $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ when $\mathfrak{a}$ and $\mathfrak{b}$ are merely vector spaces is given in the following proposition.

Proposition 3.25. Suppose $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ and suppose $\mathfrak{a}$ and $\mathfrak{b}$ are subspaces of $\mathfrak{g}$. Then the mapping $a \otimes b \mapsto \sigma(a)\sigma(b)$ of $S(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{b})$ into $U(\mathfrak{g})$ is a vector-space isomorphism onto.

PROOF. The vector space $S(\mathfrak{a}) \otimes_{\mathbb{C}} S(\mathfrak{b})$ is graded consistently for the given mapping, the n^{th} space of the grading being $\bigoplus_{p=0}^n S^p(\mathfrak{a}) \otimes_{\mathbb{C}} S^{n-p}(\mathfrak{b})$. The given mapping operates on an element of this space by

$$\sum_{p=0}^n a_p \otimes b_{n-p} \mapsto \sum_{p=0}^n \sigma(a_p)\sigma(b_{n-p}),$$

for all $x \in G$ whenever $y_1^{-1}y_2$ is in U. Then

$$\begin{aligned}
\|h(y_1^{-1}x) - h(y_2^{-1}x)\|_{2,x} &\le \|h(y_1^{-1}x) - c(y_1^{-1}x)\|_{2,x} \\
&\quad + \|c(y_1^{-1}x) - c(y_2^{-1}x)\|_{2,x} \\
&\quad + \|c(y_2^{-1}x) - h(y_2^{-1}x)\|_{2,x} \\
&= 2\|h-c\|_2 + \|c(y_1^{-1}x) - c(y_2^{-1}x)\|_{2,x} \\
&\le 2\|h-c\|_2 + \sup_{x\in G}|c(y_1^{-1}x) - c(y_2^{-1}x)| \\
&< 2\epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Lemma 4.18. Let G be a compact group, and let h be in $L^2(G)$. For any $\epsilon > 0$, there exist finitely many $y_i \in G$ and Borel sets $E_i \subseteq G$ such that the E_i disjointly cover G and

$$\|h(y^{-1}x) - h(y_i^{-1}x)\|_{2,x} < \epsilon \qquad \text{for all } i \text{ and for all } y \in E_i.$$

PROOF. By Lemma 4.17 choose an open neighborhood U of 1 so that $\|h(gx) - h(x)\|_{2,x} < \epsilon$ whenever g is in U. For each $z_0 \in G$, $\|h(gz_0x) - h(z_0x)\|_{2,x} < \epsilon$ whenever g is in U. The set Uz_0 is an open neighborhood of z_0, and such sets cover G as z_0 varies. Find a finite subcover, say $Uz_1, \dots, Uz_n$, and let $U_i = Uz_i$. Define $F_j = U_j - \bigcup_{i=1}^{j-1} U_i$. Then the lemma follows with $y_i = z_i^{-1}$ and $E_i = F_i^{-1}$.

Lemma 4.19. Let G be a compact group, let f be in $L^1(G)$, and let h be in $L^2(G)$. Put $F(x) = \int_G f(y)h(y^{-1}x)\,dy$. Then F is the limit in $L^2(G)$ of a sequence of functions, each of which is a finite linear combination of left translates of h.

PROOF. Given $\epsilon > 0$, choose y_i and E_i as in Lemma 4.18, and put $c_i = \int_{E_i} f(y)\,dy$. Then

$$\begin{aligned}
&\Big\| \int_G f(y)h(y^{-1}x)\,dy - \sum_i c_i h(y_i^{-1}x)\Big\|_{2,x} \\
&\qquad\le \Big\| \sum_i \int_{E_i} |f(y)||h(y^{-1}x) - h(y_i^{-1}x)|\,dy\Big\|_{2,x} \\
&\qquad\le \sum_i \int_{E_i} |f(y)|\,\|h(y^{-1}x) - h(y_i^{-1}x)\|_{2,x}\,dy \\
&\qquad\le \sum_i \int_{E_i} |f(y)|\epsilon\,dy = \epsilon\|f\|_1.
\end{aligned}$$

Theorem 4.20 (Peter-Weyl Theorem). If G is a compact group, then the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of G is dense in $L^2(G)$.

11. Deduce from Problem 10 that Δ carries V_N onto V_{N-2}.
12. Deduce from Problem 10 that each $p \in V_N$ decomposes uniquely as
$$p = h_N + |x|^2 h_{N-2} + |x|^4 h_{N-4} + \cdots$$
with $h_N, h_{N-2}, h_{N-4}, \ldots$ homogeneous harmonic of the indicated degrees.
13. Compute the dimension of H_N.

Problems 14–16 concern Example 2 for $SU(n)$ in §1. Let V_N be the space of polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ that are homogeneous of degree N.

14. Show for each pair (p, q) with $p + q = N$ that the subspace $V_{p,q}$ of polynomials with p z-type factors and q $\bar{z}$-type factors is an invariant subspace under $SU(n)$.
15. The Laplacian in these coordinates is a multiple of $\sum_j \dfrac{\partial^2}{\partial z_j \partial \bar{z}_j}$. Using the result of Problem 11, prove that the Laplacian carries $V_{p,q}$ onto $V_{p-1,q-1}$.
16. Compute the dimension of the subspace of harmonic polynomials in $V_{p,q}$.

Problems 17–20 deal with integral forms. In each case the maximal torus T is understood to be as in the corresponding example of §5, and the notation for members of $\mathfrak{t}^*$ is to be as in the corresponding example of §II.1 (with $\mathfrak{h} = \mathfrak{t}$).

17. For $SU(n)$, a general member of $\mathfrak{t}^*$ may be written uniquely as $\sum_{j=1}^n c_j e_j$ with $\sum_{j=1}^n c_j = 0$.
    (a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all c_j in $\mathbb{Z}$.
    (b) Prove that the algebraically integral forms are those for which all c_j are in $\mathbb{Z} + \frac{k}{n}$ for some k.
    (c) Prove that every algebraically integral form is analytically integral.
    (d) Prove that the quotient of the lattice of algebraically integral forms by the lattice of $\mathbb{Z}$ combinations of roots is a cyclic group of order n.
18. For $SO(2n+1)$, a general member of $\mathfrak{t}^*$ is $\sum_{j=1}^n c_j e_j$.
    (a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all c_j in $\mathbb{Z}$.
    (b) Prove that the algebraically integral forms are those forms with all c_j in $\mathbb{Z}$ or all c_j in $\mathbb{Z} + \frac{1}{2}$.
    (c) Prove that every analytically integral form is a $\mathbb{Z}$ combination of roots.
19. For $Sp(n, \mathbb{C}) \cap U(2n)$, a general member of $\mathfrak{t}^*$ is $\sum_{j=1}^n c_j e_j$.
    (a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all c_j in $\mathbb{Z}$ and with $\sum_{j=1}^n c_j$ even.