Theorem 2.9. Any finite-dimensional complex Lie algebra $\mathfrak{g}$ has a Cartan subalgebra.

Before coming to the proof, we introduce “regular” elements of $\mathfrak{g}$. In $\mathfrak{sl}(n, \mathbb{C})$ the regular elements will be the matrices with distinct eigenvalues. Let us consider matters more generally.

If $\pi$ is a representation of $\mathfrak{g}$ on a finite-dimensional vector space $V$, we can regard each $X \in \mathfrak{g}$ as generating a 1-dimensional abelian subalgebra, and we can then form $V_{0,X}$, the generalized eigenspace for eigenvalue 0 under $\pi(X)$. Let

$$l_\mathfrak{g}(V) = \min_{X \in \mathfrak{g}} \dim V_{0,X},$$

and

$$R_\mathfrak{g}(V) = \{X \in \mathfrak{g} \mid \dim V_{0,X} = l_\mathfrak{g}(V)\}.$$

To understand $l_\mathfrak{g}(V)$ and $R_\mathfrak{g}(V)$ better, form the characteristic polynomial

$$\det(\lambda I - \pi(X)) = \lambda^n + \sum_{j=0}^{n-1} d_j(X)\lambda^j.$$

In any basis of $\mathfrak{g}$, the $d_j(X)$ are polynomial functions on $\mathfrak{g}$, as we see by expanding $\det(\lambda I - \sum \mu_i \pi(X_i))$. For given $X$, if $j$ is the smallest value for which $d_j(X) \neq 0$, then $j = \dim V_{0,X}$, since the degree of the last term in the characteristic polynomial is the multiplicity of 0 as a generalized eigenvalue of $\pi(X)$. Thus $l_\mathfrak{g}(V)$ is the minimum $j$ such that $d_j(X) \neq 0$, and

$$R_\mathfrak{g}(V) = \{X \in \mathfrak{g} \mid d_{l_\mathfrak{g}(V)}(X) \neq 0\}.$$

Let us apply these considerations to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$. The elements of $R_\mathfrak{g}(\mathfrak{g})$, relative to the adjoint representation, are the regular elements of $\mathfrak{g}$. For any $X$ in $\mathfrak{g}$, $\mathfrak{g}_{0,X}$ is a Lie subalgebra of $\mathfrak{g}$ by the corollary of Proposition 2.5, with $\mathfrak{h} = \mathbb{C}X$.

Theorem 2.9'. If $X$ is a regular element of the finite-dimensional complex Lie algebra $\mathfrak{g}$, then the Lie algebra $\mathfrak{g}_{0,X}$ is a Cartan subalgebra of $\mathfrak{g}$.

Proof. First we show that $\mathfrak{g}_{0,X}$ is nilpotent. Assuming the contrary, we construct two sets:

(i) the set of $Z \in \mathfrak{g}_{0,X}$ such that $(\text{ad } Z)|_{\mathfrak{g}_{0,X}}^{\dim \mathfrak{g}_{0,X}} \neq 0$, which is nonempty by Engel’s Theorem (Corollary 1.38) and is open

(ii) the set of $W \in \mathfrak{g}_{0,X}$ such that $\text{ad } W|_{\mathfrak{g}/\mathfrak{g}_{0,X}}$ is nonsingular, which is nonempty since $X$ is in it (regularity is not used here) and is the set where some polynomial is nonvanishing, hence is dense (because if a polynomial vanishes on a nonempty open set, it vanishes identically).
These two sets must have nonempty intersection, and so we can find \( Z \in g_{0,X} \) such that

\[
(\text{ad } Z)|_{g_{0,X}} \neq 0 \quad \text{and} \quad \text{ad } Z|_{g/g_{0,X}} \text{ is nonsingular.}
\]

Then the generalized multiplicity of the eigenvalue 0 for \( \text{ad } Z \) is less than \( \dim g_{0,X} \), and hence \( \dim g_{0,Z} < \dim g_{0,X} \), in contradiction with the regularity of \( X \). We conclude that \( g_{0,X} \) is nilpotent.

Since \( g_{0,X} \) is nilpotent, we can use \( g_{0,X} \) to decompose \( g \) as in Proposition 2.4. Let \( g_0 \) be the 0 generalized weight space. Then we have

\[
g_{0,X} \subseteq g_0 = \bigcap_{Y \in g_{0,X}} g_{0,Y} \subseteq g_{0,X}.
\]

So \( g_{0,X} = g_0 \), and \( g_{0,X} \) is a Cartan subalgebra.

In this book we shall be interested in Cartan subalgebras \( \mathfrak{h} \) only when \( g \) is semisimple. In this case \( \mathfrak{h} \) has special properties, as follows.

**Proposition 2.10.** If \( g \) is a complex semisimple Lie algebra and \( \mathfrak{h} \) is a Cartan subalgebra, then \( \mathfrak{h} \) is abelian.

**Proof.** Since \( \mathfrak{h} \) is nilpotent and therefore solvable, \( \text{ad } \mathfrak{h} \) is solvable as a Lie algebra of transformations of \( g \). By Lie’s Theorem (Corollary 1.29) it is simultaneously triangular in some basis. For any three triangular matrices \( A, B, C \), we have \( \text{Tr}(ABC) = \text{Tr}(BAC) \). Therefore

\[
(2.11) \quad \text{Tr}(\text{ad}[H_1, H_2]\text{ad } H) = 0 \quad \text{for } H_1, H_2, H \in \mathfrak{h}.
\]

Next let \( \alpha \) be any nonzero generalized weight, let \( X \) be in \( g_{\alpha} \), and let \( H \) be in \( \mathfrak{h} \). By Proposition 2.5c, \( \text{ad } H \) ad \( X \) carries \( g_{\beta} \) to \( g_{\alpha + \beta} \). Thus Proposition 2.5a shows that

\[
(2.12) \quad \text{Tr}(\text{ad } H \text{ ad } X) = 0.
\]

Specializing (2.12) to \( H = [H_1, H_2] \) and using (2.11) and Proposition 2.5a, we see that the Killing form \( B \) of \( g \) satisfies

\[
B([H_1, H_2], X) = 0 \quad \text{for all } X \in g.
\]

By Cartan’s Criterion for Semisimplicity (Theorem 1.42), \( B \) is nondegenerate. Therefore \( [H_1, H_2] = 0 \), and \( \mathfrak{h} \) is abelian.
Corollary 2.13. In a complex semisimple Lie algebra \( g \), a Lie subalgebra \( h \) is a Cartan subalgebra if \( h \) is maximal abelian and \( \text{ad}_g h \) is simultaneously diagonable.

Remarks.
1) It is immediate from this corollary that the subalgebras \( h \) in the examples of §1 are Cartan subalgebras.
2) In the direction converse to the corollary, Proposition 2.10 shows that a Cartan subalgebra \( h \) is abelian, and it is maximal abelian since \( h = g_0 \). Corollary 2.23 will show for a Cartan subalgebra \( h \) that \( \text{ad}_g h \) is simultaneously diagonable.

Proof. Since \( h \) is abelian and hence nilpotent, Proposition 2.4 shows that \( g \) has a weight-space decomposition \( g = g_0 \oplus \bigoplus_{\beta \neq 0} g_\beta \). Since \( \text{ad}_g h \) is simultaneously diagonable, \( g_0 = h \oplus r \) with \( [h, r] = 0 \). In view of Proposition 2.7, we are to prove that \( h = N_g(h) \subseteq g_0 \) by (2.8), and it is enough to show that \( r = 0 \). If \( X \neq 0 \) is in \( r \), then \( h \oplus \mathbb{C}X \) is an abelian subalgebra properly containing \( h \), in contradiction with \( h \) maximal abelian. The result follows.

3. Uniqueness of Cartan Subalgebras

We turn to the question of uniqueness of Cartan subalgebras. We begin with a lemma about polynomial mappings.

Lemma 2.14. Let \( P : \mathbb{C}^m \rightarrow \mathbb{C}^n \) be a holomorphic polynomial function not identically 0. Then the set of vectors \( z \) in \( \mathbb{C}^m \) for which \( P(z) \) is not the 0 vector is connected in \( \mathbb{C}^m \).

Proof. Suppose that \( z_0 \) and \( w_0 \) in \( \mathbb{C}^m \) have \( P(z_0) \neq 0 \) and \( P(w_0) \neq 0 \). As a function of \( z \in \mathbb{C} \), \( P(z_0 + z(w_0 - z_0)) \) is a vector-valued holomorphic polynomial nonvanishing at \( z = 0 \) and \( z = 1 \). The subset of \( z \in \mathbb{C} \) where it vanishes is finite, and the complement in \( \mathbb{C} \) is connected. Thus \( z_0 \) and \( w_0 \) lie in a connected set in \( \mathbb{C}^m \) where \( P \) is nonvanishing. Taking the union of these connected sets with \( z_0 \) fixed and \( w_0 \) varying, we see that the set where \( P(w_0) \neq 0 \) is connected.

Theorem 2.15. If \( h_1 \) and \( h_2 \) are Cartan subalgebras of a finite-dimensional complex Lie algebra \( g \), then there exists \( a \in \text{Int} g \) with \( a(h_1) = h_2 \).

Remarks.
1) In particular any two Cartan subalgebras are conjugate by an automorphism of \( g \). As was explained after the introduction of \( \text{Int} g \) in §1.11,
Proposition 2.78. The abstract Dynkin diagram associated to the $l$-by-$l$ abstract Cartan matrix $A$ has the following properties:

(a) there are at most $l$ pairs of vertices $i < j$ with at least one edge connecting them
(b) there are no loops
(c) at most three edges issue from any point of the diagram.

Proof.
(a) With $\alpha_i$ as in (2.75), put $\alpha = \sum_{i=1}^{l} \frac{\alpha_i}{|\alpha_i|}$. Then

$$0 < |\alpha|^2 = \sum_{i,j} \left( \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right)$$
$$= \sum_i \left( \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_i}{|\alpha_i|} \right) + 2 \sum_{i<j} \left( \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right)$$
$$= l + \sum_{i<j} \frac{2(\alpha_i, \alpha_j)}{|\alpha_i||\alpha_j|}$$

(2.79)
$$= l - \sum_{i<j} \sqrt{A_{ij}A_{ji}}.$$

By Proposition 2.74, $\sqrt{A_{ij}A_{ji}}$ is 0 or 1 or $\sqrt{2}$ or $\sqrt{3}$. When nonzero, it is therefore $\geq 1$. Therefore the right side of (2.79) is

$$\leq l - \sum_{i<j, \text{ connected}} 1.$$

Hence the number of connected pairs of vertices is $< l$.

(b) If there were a loop, we could use Operation #1 to remove all vertices except those in a loop. Then (a) would be violated for the loop.

(c) Fix $\alpha = \alpha_i$ as in (2.75). Consider the vertices that are connected by edges to the $i$th vertex. Write $\beta_1, \ldots, \beta_r$ for the $\alpha_j$’s associated to these vertices, and let there be $l_1, \ldots, l_r$ edges to the $i$th vertex. Let $U$ be the $(r+1)$-dimensional vector subspace of $\mathbb{R}^l$ spanned by $\beta_1, \ldots, \beta_r, \alpha$. Then $\langle \beta_i, \beta_j \rangle = 0$ for $i \neq j$ by (b), and hence $\{\beta_k/|\beta_k|\}_{k=1}^r$ is an orthonormal set. Adjoin $\delta \in U$ to this set to make an orthonormal basis of $U$. Then $\langle \alpha, \delta \rangle \neq 0$ since $\{\beta_1, \ldots, \beta_r, \alpha\}$ is linearly independent. By Parseval’s equality,
have an isomorphism, and then compute what the map is in terms of a basis. Let \( T_n(\mathfrak{g}) = \bigoplus_{k=0}^{n} T^k(\mathfrak{g}) \) be the \( n \)th member of the usual filtration of \( T(\mathfrak{g}) \). We have defined \( U_n(\mathfrak{g}) \) to be the image in \( U(\mathfrak{g}) \) of \( T_n(\mathfrak{g}) \) under the passage \( T(\mathfrak{g}) \to T(\mathfrak{g})/J \). Thus we can form the composition

\[
T_n(\mathfrak{g}) \to (T_n(\mathfrak{g}) + J)/J = U_n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})
\]

This composition is onto and carries \( T_{n-1}(\mathfrak{g}) \) to 0. Since \( T_n(\mathfrak{g}) \) is a vector-space complement to \( T_{n-1}(\mathfrak{g}) \) in \( T_n(\mathfrak{g}) \), we obtain an onto linear map

\[
T_n(\mathfrak{g}) \to U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})
\]

Taking the direct sum over \( n \) gives an onto linear map

\[
\tilde{\psi} : T(\mathfrak{g}) \to \text{gr} U(\mathfrak{g})
\]

that respects the grading.

Appendix A uses the notation \( I \) for the two-sided ideal in \( T(\mathfrak{g}) \) such that \( S(\mathfrak{g}) = T(\mathfrak{g})/I \):

\[
I = \left( \text{two-sided ideal generated by all } X \otimes Y - Y \otimes X \text{ with } X \text{ and } Y \right) \text{ in } T^1(\mathfrak{g}).
\]

**Proposition 3.16.** The linear map \( \tilde{\psi} : T(\mathfrak{g}) \to \text{gr} U(\mathfrak{g}) \) respects multiplication and annihilates the defining ideal \( I \) for \( S(\mathfrak{g}) \). Therefore \( \psi \) descends to an algebra homomorphism

\[
(3.17) \quad \psi : S(\mathfrak{g}) \to \text{gr} U(\mathfrak{g})
\]

that respects the grading. This homomorphism is an isomorphism.

**Proof.** Let \( x \) be in \( T'(\mathfrak{g}) \) and let \( y \) be in \( T''(\mathfrak{g}) \). Then \( x + J \) is in \( U_r(\mathfrak{g}) \), and we may regard \( \tilde{\psi}(x) \) as the coset \( x + T_{r-1}(\mathfrak{g}) + J \) in \( U_r(\mathfrak{g})/U_{r-1}(\mathfrak{g}) \), with 0 in all other coordinates of \( \text{gr} U(\mathfrak{g}) \) since \( x \) is homogeneous. Arguing in a similar fashion with \( y \) and \( xy \), we obtain

\[
\tilde{\psi}(x) = x + T_{r-1}(\mathfrak{g}) + J, \quad \tilde{\psi}(y) = y + T_{r-1}(\mathfrak{g}) + J,
\]

and

\[
\tilde{\psi}(xy) = xy + T_{r+s-1}(\mathfrak{g}) + J.
\]

Since \( J \) is an ideal, \( \tilde{\psi}(x) \tilde{\psi}(y) = \tilde{\psi}(xy) \). General members \( x \) and \( y \) of \( T(\mathfrak{g}) \) are sums of homogeneous elements, and hence \( \tilde{\psi} \) respects multiplication.
The direct sum of the maps $\sigma_n$ for $n \geq 0$ (with $\sigma_0(1) = 1$) is a linear map $\sigma : S(g) \to U(g)$ such that

$$\sigma(X_1 \cdots X_n) = \frac{1}{n!} \sum_{\tau \in S_n} X_{\tau(1)} \cdots X_{\tau(n)}.$$ 

The map $\sigma$ is called **symmetrization**.

**Lemma 3.22.** The symmetrization map $\sigma : S(g) \to U(g)$ has associated graded map $\psi : S(g) \to \text{gr } U(g)$, with $\psi$ as in (3.17).

**Remark.** See §A.4 for “associated graded map.”

**Proof.** Let $\{X_i\}$ be a basis of $g$, and let $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, with $\sum_{m} j_m = n$, be a basis vector of $S^n(g)$. Under $\sigma$, this vector is sent to a symmetrized sum, but each term of the sum is congruent mod $U_{n-1}(g)$ to $(n!)^{-1} X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$, by Lemma 3.9. Hence the image of $X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}$ under the associated graded map is

$$= X_{i_1}^{j_1} \cdots X_{i_k}^{j_k} + U_{n-1}(g) = \psi(X_{i_1}^{j_1} \cdots X_{i_k}^{j_k}),$$

as asserted.

**Proposition 3.23.** Symmetrization $\sigma$ is a vector-space isomorphism of $S(g)$ onto $U(g)$ satisfying

(3.24) \hspace{1cm} U_n(g) = \sigma(S^n(g)) \oplus U_{n-1}(g).

**Proof.** Formula (3.24) is a restatement of (3.21), and the other conclusion follows by combining Lemma 3.22 and Proposition A.37.

The canonical decomposition of $U(g)$ from $g = a \oplus b$ when $a$ and $b$ are merely vector spaces is given in the following proposition.

**Proposition 3.25.** Suppose $g = a \oplus b$ and suppose $a$ and $b$ are subspaces of $g$. Then the mapping $a \otimes b \mapsto \sigma(a) \sigma(b)$ of $S(a) \otimes S(b)$ into $U(g)$ is a vector-space isomorphism onto.

**Proof.** The vector space $S(a) \otimes S(b)$ is graded consistently for the given mapping, the $n^{th}$ space of the grading being $\bigoplus_{p=0}^{n} S^{p}(a) \otimes S^{n-p}(b)$. The given mapping operates on an element of this space by

$$\sum_{p=0}^{n} a_p \otimes b_{n-p} \mapsto \sum_{p=0}^{n} \sigma(a_p) \sigma(b_{n-p}),$$
for all $x \in G$ whenever $y_1^{-1}y_2$ is in $U$. Then
\[
\|h(y_1^{-1}x) - h(y_2^{-1}x)\|_{2,x} \leq \|h(y_1^{-1}x) - c(y_1^{-1}x)\|_{2,x} \\
+ \|c(y_1^{-1}x) - c(y_2^{-1}x)\|_{2,x} \\
+ \|c(y_2^{-1}x) - h(y_2^{-1}x)\|_{2,x}
\]
\[
= 2\|h - c\|_2 + \|c(y_1^{-1}x) - c(y_2^{-1}x)\|_{2,x} \\
\leq 2\|h - c\|_2 + \sup_{x \in G} |c(y_1^{-1}x) - c(y_2^{-1}x)|
\]
\[
< 2\varepsilon/3 + \varepsilon/3 = \varepsilon.
\]

**Lemma 4.18.** Let $G$ be a compact group, and let $h$ be in $L^2(G)$. For any $\varepsilon > 0$, there exist finitely many $y_i \in G$ and Borel sets $E_i \subseteq G$ such that the $E_i$ disjointly cover $G$ and
\[
\|h(y_1^{-1}x) - h(y_1^{-1}x)\|_{2,x} < \varepsilon \quad \text{for all } i \text{ and for all } y \in E_i.
\]

**Proof.** By Lemma 4.17 choose an open neighborhood $U$ of 1 so that $\|h(gx) - h(x)\|_{2,x} < \varepsilon$ whenever $g$ is in $U$. For each $z_0 \in G$, $\|h(z_0x) - h(z_0)\|_{2,x} < \varepsilon$ whenever $y$ is in $U$. The set $Uz_0$ is an open neighborhood of $z_0$, and such sets cover $G$ as $z_0$ varies. Find a finite subcover, say $Uz_1, \ldots, Uz_n$, and let $U_i = Uz_i$. Define $F_j = U_j - \bigcup_{i=1}^{j-1} U_i$. Then the lemma follows with $y_i = z_i^{-1}$ and $E_i = F_i^{-1}$.

**Lemma 4.19.** Let $G$ be a compact group, let $f$ be in $L^1(G)$, and let $h$ be in $L^2(G)$. Put $F(x) = \int_G f(y)h(y^{-1}x)dy$. Then $F$ is the limit in $L^2(G)$ of a sequence of functions, each of which is a finite linear combination of left translates of $h$.

**Proof.** Given $\varepsilon > 0$, choose $y_i$ and $E_i$ as in Lemma 4.18, and put $c_i = \int_{E_i} f(y)dy$. Then
\[
\left\| \int_G f(y)h(y^{-1}x)dy - \sum_i c_i h(y_i^{-1}x) \right\|_{2,x}
\]
\[
\leq \left\| \sum_i \int_{E_i} |f(y)||h(y_i^{-1}x) - h(y_i^{-1}x)|dy \right\|_{2,x}
\]
\[
\leq \sum_i \int_{E_i} |f(y)||h(y_i^{-1}x) - h(y_i^{-1}x)|_{2,x}dy
\]
\[
\leq \sum_i \int_{E_i} |f(y)|\varepsilon \ dy = \varepsilon \|f\|_1.
\]

**Theorem 4.20** (Peter-Weyl Theorem). If $G$ is a compact group, then the linear span of all matrix coefficients for all finite-dimensional irreducible unitary representations of $G$ is dense in $L^2(G)$. 

IV. Compact Lie Groups
11. Deduce from Problem 10 that $\Delta$ carries $V_N$ onto $V_{N-2}$.

12. Deduce from Problem 10 that each $p \in V_N$ decomposes uniquely as

$$p = h_N + |x|^2h_{N-2} + |x|^4h_{N-4} + \cdots$$

with $h_N, h_{N-2}, h_{N-4}, \ldots$ homogeneous harmonic of the indicated degrees.

13. Compute the dimension of $H_N$.

Problems 14–16 concern Example 2 for $SU(n)$ in §1. Let $V_N$ be the space of polynomials in $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ that are homogeneous of degree $N$.

14. Show for each pair $(p, q)$ with $p + q = N$ that the subspace $V_{p,q}$ of polynomials with $p$ $z$-type factors and $q$ $\bar{z}$-type factors is an invariant subspace under $SU(n)$.

15. The Laplacian in these coordinates is a multiple of $\sum_j \frac{\partial^2}{\partial z_j \partial \bar{z}_j}$. Using the result of Problem 11, prove that the Laplacian carries $V_{p,q}$ onto $V_{p-1,q-1}$.

16. Compute the dimension of the subspace of harmonic polynomials in $V_{p,q}$.

Problems 17–20 deal with integral forms. In each case the maximal torus $T$ is understood to be as in the corresponding example of §5, and the notation for members of $t^*$ is to be as in the corresponding example of §II.1 (with $h = 0$).

17. For $SU(n)$, a general member of $t^*$ may be written uniquely as $\sum_{j=1}^n c_j e_j$ with $\sum_{j=1}^n c_j = 0$.

(a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all $c_j$ in $\mathbb{Z}$.

(b) Prove that the algebraically integral forms are those for which all $c_j$ are in $\mathbb{Z} + \frac{k}{n}$ for some $k$.

(c) Prove that every algebraically integral form is analytically integral.

(d) Prove that the quotient of the lattice of algebraically integral forms by the lattice of $\mathbb{Z}$ combinations of roots is a cyclic group of order $n$.

18. For $SO(2n + 1)$, a general member of $t^*$ is $\sum_{j=1}^n c_j e_j$.

(a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all $c_j$ in $\mathbb{Z}$.

(b) Prove that the algebraically integral forms are those forms with all $c_j$ in $\mathbb{Z}$ or all $c_j$ in $\mathbb{Z} + \frac{1}{2}$.

(c) Prove that every analytically integral form is a $\mathbb{Z}$ combination of roots.

19. For $Sp(n, \mathbb{C}) \cap U(2n)$, a general member of $t^*$ is $\sum_{j=1}^n c_j e_j$.

(a) Prove that the $\mathbb{Z}$ combinations of roots are those forms with all $c_j$ in $\mathbb{Z}$ and with $\sum_{j=1}^n c_j$ even.