CHAPTER I

Lie Algebras and Lie Groups

Abstract. The first part of this chapter treats Lie algebras, beginning with definitions and many examples. The notions of solvable, nilpotent, radical, semisimple, and simple are introduced, and these notions are followed by a discussion of the effect of a change of the underlying field.

The idea of a semidirect product begins the development of the main structural theorems for real Lie algebras—the iterated construction of all solvable Lie algebras from derivations and semidirect products, Lie's Theorem for solvable Lie algebras, Engel's Theorem in connection with nilpotent Lie algebras, and Cartan's criteria for solvability and semisimplicity in terms of the Killing form. From Cartan's Criterion for Semisimplicity, it follows that semisimple Lie algebras are direct sums of simple Lie algebras.

Cartan's Criterion for Semisimplicity is used also to provide a long list of classical examples of semisimple Lie algebras. Some of these examples are defined in terms of quaternion matrices. Quaternion matrices of size n-by-n may be related to complex matrices of size $2n$-by-$2n$.

The treatment of Lie algebras concludes with a study of the finite-dimensional complex-linear representations of $\mathfrak{sl}(2, \mathbb{C})$. There is a classification theorem for the irreducible representations of this kind, and the general representations are direct sums of irreducible ones.

Sections 10 through 13 contain a summary of the elementary theory of Lie groups and their Lie algebras. The abstract theory is presented, and the correspondence is made with the concrete theory of closed linear groups as in the Introduction. In addition these sections discuss the adjoint representation, covering groups, complex structures and holomorphic functions, and complex Lie groups.

The remainder of the chapter explores some aspects of the connection between Lie groups and Lie algebras. One aspect is the relationship between automorphisms and derivations. The derivations of a semisimple Lie algebra are inner, and consequently the identity component of the group of automorphisms of a semisimple Lie algebra consists of inner automorphisms. In addition, simply connected solvable Lie groups may be built one dimension at a time as semidirect products with $\mathbb{R}^1$, and consequently they are diffeomorphic to Euclidean space. For simply connected nilpotent groups the exponential map is itself a diffeomorphism. The earlier long list of classical semisimple Lie algebras corresponds to a list of the classical semisimple Lie groups. An issue that needs attention for these groups is their connectedness, and this is proved by using the polar decomposition of matrices.