PREFACE

In the kind of analysis accomplished by representation theory, unitary representations play a particularly important role because they are the most convenient to decompose. However, only in rare cases does one use a classification to identify candidates for the irreducible constituents. More often one or more particular constructions will suffice to produce enough candidates.

In 1978 lectures, Gregg Zuckerman introduced a new construction, now called cohomological induction, of representations of semisimple Lie groups that were expected often to be irreducible unitary. Philosophically, cohomological induction is based on complex analysis in the same sense that George Mackey's construction of induced representations is based on real analysis. Zuckerman's construction thus serves as a natural complement to Mackey's, and it was immediately clear that the new method might go a long way toward explaining the most mysterious features of unitary representation theory. Zuckerman used his construction to produce algebraic models of the Bott-Borel-Weil Theorem, of Harish-Chandra's discrete series representations, of some special representations arising in mathematical physics, and of many more representations that are neither induced nor familiar.

Although Zuckerman's construction is based on complex analysis, it is in fact completely algebraic. In the complex-analysis setting the representation space is supposed to be a space of Dolbeault-cohomology sections over a noncompact complex homogeneous space of the group in question. But this setting turned out to be difficult to study in detail, and Zuckerman created an algebraic analog by abstracting the notion of passing to Taylor coefficients. If K is a maximal compact subgroup of the semisimple group G, the representation of G of interest is to be replaced by its subspace of vectors that transform in finite-dimensional spaces under K. This subspace, known as a $(\mathfrak{g}, K)$ module, is compatibly a representation space for K and the complexified Lie algebra $\mathfrak{g}$ of G.

Whether the construction is complex analytic or algebraic, the goal is to produce irreducible unitary representations. Zuckerman's representations, however, carry no obvious inner products, and construction of a candidate for the inner product is a serious project. By contrast, in Mackey's real-analysis theory of induced representations, the space of a representation is always the Hilbert space of square-integrable functions on some measure space, and the inner product is immediately at hand.

This book is an exposition of five fundamental theorems about cohomological induction, all related directly or indirectly to such inner products. We call them the Duality Theorem, the Irreducibility Theorem, the Signature Theorem, the Unitarizability Theorem, and the Transfer Theorem. The Introduction explains these theorems in the context of their history and motivation.

A chapter-by-chapter list of prerequisites for the book appears on p. xv. Roughly speaking, it is assumed for the first three chapters that the reader knows about elementary Lie theory, universal enveloping algebras, the abstract representation theory of compact groups, distributions on manifolds as in Appendix B, and elementary homological algebra as in Appendix C. Later chapters assume also the Cartan-Weyl theory for semisimple Lie algebras and compact connected Lie groups, some basic facts about real forms and parabolic subalgebras, and spectral sequences as in Appendix D.

Zuckerman introduced the Duality Theorem (§III.2 below) as a conjecture, showing how it could be used to construct (possibly indefinite) Hermitian forms on cohomologically induced representations. With P. Trauber, he proposed several ideas toward proofs. Among other things, Zuckerman and Trauber showed how to write down the pairing in the Duality Theorem; what was not obvious was that the pairing was invariant under the representation. Enright-Wallach [1980]* gave a proof of this invariance, and therefore of the Duality Theorem.

Zuckerman's algebraic construction of $(\mathfrak{g}, K)$ modules via Taylor coefficients uses a functor Γ that defines away the question of convergence. The functor Γ is not exact, only left exact, and the degrees of its derived functors play the role of the degrees of the Dolbeault cohomology classes. The initial definition of an invariant Hermitian form on a cohomologically induced representation involved a mixture of the derived functors of Γ and another algebraic construction ("ind" in Chapter 6 of Vogan [1981a]) that did not fit well. Zuckerman recognized that this combination was incongruous and searched for a right-exact functor Π to replace Γ. His search was unsuccessful, and a first version of Π was not announced until Bernstein [1983]. The use of a properly defined Π is critical to the approach taken in this book, and our definition is in terms of a change of rings.

We begin, following a 1970s idea of Flath and Deligne that was developed in Knapp-Vogan [1986], by introducing a "Hecke algebra" $R(\mathfrak{g}, K)$, which may be regarded as the set of bi-K finite distributions on

*A name followed by a bracketed year is an allusion to the list of References at the end of the book.

the underlying group G with support in the compact subgroup K. The set $R(\mathfrak{g}, K)$ is a complex associative algebra with an approximate identity, and $(\mathfrak{g}, K)$ modules coincide with "approximately unital" modules for $R(\mathfrak{g}, K)$. From this point of view the Zuckerman functor Γ becomes a Hom type change-of-ring functor of the kind studied in Cartan-Eilenberg [1956]. This fact immediately suggests using the corresponding tensor product change-of-ring functor as Π.

In fact, from the Hecke algebra point of view, the functors "ind" and "pro" in Chapter 6 of Vogan [1981a] are also change-of-ring functors constructed from $\otimes$ and Hom, respectively, and the same thing is true of the functors "coinvariants" and "invariants," whose derived functors give Lie algebra homology and cohomology. Thus there are really just two master functors in the theory, having to do with changes of rings by $\otimes$ and Hom. These functors are called P and I in this book because of their effect on projectives and injectives. Many fundamental results (including versions of Frobenius reciprocity) are consequences of standard associativity formulas for $\otimes$ and Hom.

With these general results in hand, Chapter V takes up the definition and first properties of cohomological induction. The functors $\mathcal{R}$ considered in Vogan [1981a] are built from Γ and pro, thus from the master functor I mentioned above. To construct Hermitian forms, it is essential to use instead $\mathcal{L}$, constructed analogously from Π and ind, thus from the master functor P.

Once invariant Hermitian forms have been constructed with the aid of the Duality Theorem, the question arises whether the forms are definite and hence are inner products. The Signature Theorem, proved in Chapter VI, addresses this question on that part of the cohomologically induced representation that is most easily related to the inducing representation. (The subspace in question is the "bottom layer" first considered in Speh-Vogan [1980].) The theorem says that cohomological induction always preserves a part of the signature of a Hermitian form. More precisely it identifies subspaces of the inducing and cohomologically induced representations and says that the Hermitian forms on these two subspaces have the same signature. An important feature of the Signature Theorem is that it makes no positivity assumption on the parameters of the inducing representation.

By contrast, the remaining three main theorems do include some positivity hypothesis. The Irreducibility Theorem (Chapter VIII) gives conditions under which cohomological induction carries irreducible representations to irreducible representations, and the Unitarizability Theorem (Chapter IX) gives conditions under which cohomological induction carries unitary representations to unitary representations. Zuckerman visualized the Irreducibility Theorem as a consequence of the Duality Theorem and gave a number of the ideas needed for a proof; all of the ideas are in place in Vogan [1981a]. The Unitarizability Theorem is newer and was first proved in Vogan [1984]. Together the Irreducibility Theorem and Unitarizability Theorem finally give confirmation that cohomological induction is actually a construction of irreducible unitary representations.

Once cohomological induction has constructed irreducible unitary representations, the question is what these representations are and how they can be related to each other. This topic is addressed in Chapter XI. A key tool in the investigation is the last of the five main theorems, the Transfer Theorem, which permits analysis of the effect of a "change of polarization" in constructions like cohomological induction. Consequently one can compare cohomological induction with Mackey induction and locate many cohomologically induced representations in the Langlands classification.

A few words are in order about the origins of this book. David Vogan sketched a proof of the Signature Theorem as early as 1984. Anthony Knapp began to study this sketch in 1985, in order to be able to use the result in some joint work with M. W. Baldoni-Silva. This study revealed various gaps and difficulties in the proof and in the literature on which it was based, and the first fruit of the study was Knapp-Vogan [1986]. Among other things, this preprint gave a rigorous development of the functor Π. The expected publication of the Signature Theorem was delayed because of other developments in the theory, and the authors eventually decided on a more complete treatment of cohomological induction. The present work may be regarded as a revision and extension of Knapp [1988] and Vogan [1981a].

For the most part, attributions of theorems appear in the end Notes. That section also mentions related papers and tells of some further results beyond those in the text.

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