PREFACE

This book and its companion volume *Advanced Real Analysis* systematically develop concepts and tools in real analysis that are vital to every mathematician, whether pure or applied, aspiring or established. The two books together contain what the young mathematician needs to know about real analysis in order to communicate well with colleagues in all branches of mathematics.

The books are written as textbooks, and their primary audience is students who are learning the material for the first time and who are planning a career in which they will use advanced mathematics professionally. Much of the material in the books corresponds to normal course work. Nevertheless, it is often the case that core mathematics curricula, time-limited as they are, do not include all the topics that one might like. Thus the book includes important topics that may be skipped in required courses but that the professional mathematician will ultimately want to learn by self-study.

The content of the required courses at each university reflects expectations of what students need before beginning specialized study and work on a thesis. These expectations vary from country to country and from university to university. Even so, there seems to be a rough consensus about what mathematics a plenary lecturer at a broad international or national meeting may take as known by the audience. The tables of contents of the two books represent my own understanding of what that degree of knowledge is for real analysis today.

Key topics and features of *Basic Real Analysis* are as follows:

- Early chapters treat the fundamentals of real variables, sequences and series of functions, the theory of Fourier series for the Riemann integral, metric spaces, and the theoretical underpinnings of multivariable calculus and ordinary differential equations.
- Subsequent chapters develop the Lebesgue theory in Euclidean and abstract spaces, Fourier series and the Fourier transform for the Lebesgue integral, point-set topology, measure theory in locally compact Hausdorff spaces, and the basics of Hilbert and Banach spaces.
- The subjects of Fourier series and harmonic functions are used as recurring motivation for a number of theoretical developments.
- The development proceeds from the particular to the general, often introducing examples well before a theory that incorporates them.

- More than 300 problems at the ends of chapters illuminate aspects of the text, develop related topics, and point to additional applications. A separate 55-page section "Hints for Solutions of Problems" at the end of the book gives detailed hints for most of the problems, together with complete solutions for many.

Beyond a standard calculus sequence in one and several variables, the most important prerequisite for using *Basic Real Analysis* is that the reader already know what a proof is, how to read a proof, and how to write a proof. This knowledge typically is obtained from honors calculus courses, or from a course in linear algebra, or from a first junior-senior course in real variables. In addition, it is assumed that the reader is comfortable with a modest amount of linear algebra, including row reduction of matrices, vector spaces and bases, and the associated geometry. A passing acquaintance with the notions of group, subgroup, and quotient is helpful as well.

Chapters I–IV are appropriate for a single rigorous real-variables course and may be used in either of two ways. For students who have learned about proofs from honors calculus or linear algebra, these chapters offer a full treatment of real variables, leaving out only the more familiar parts near the beginning—such as elementary manipulations with limits, convergence tests for infinite series with positive scalar terms, and routine facts about continuity and differentiability. For students who have learned about proofs from a first junior-senior course in real variables, these chapters are appropriate for a second such course that begins with Riemann integration and sequences and series of functions; in this case the first section of Chapter I will be a review of some of the more difficult foundational theorems, and the course can conclude with an introduction to the Lebesgue integral from Chapter V if time permits.

Chapters V through XII treat Lebesgue integration in various settings, as well as introductions to the Euclidean Fourier transform and to functional analysis. Typically this material is taught at the graduate level in the United States, frequently in one of three ways: The first way does Lebesgue integration in Euclidean and abstract settings and goes on to consider the Euclidean Fourier transform in some detail; this corresponds to Chapters V–VIII. A second way does Lebesgue integration in Euclidean and abstract settings, treats L^p spaces and integration on locally compact Hausdorff spaces, and concludes with an introduction to Hilbert and Banach spaces; this corresponds to Chapters V–VII, part of IX, and XI–XII. A third way combines an introduction to the Lebesgue integral and the Euclidean Fourier transform with some of the subject of partial differential equations; this corresponds to some portion of Chapters V–VI and VIII, followed by chapters from the companion volume *Advanced Real Analysis*.

In my own teaching, I have most often built one course around Chapters I–IV and another around Chapters V–VII, part of IX, and XI–XII. I have normally assigned the easier sections of Chapters II and X as outside reading, indicating the date when the lectures would begin to use that material.

More detailed information about how the book may be used with courses may be deduced from the chart "Dependence among Chapters" on page xiv and the section "Guide to the Reader" on pages xv–xvii.

The problems at the ends of chapters are an important part of the book. Some of them are really theorems, some are examples showing the degree to which hypotheses can be stretched, and a few are just exercises. The reader gets no indication which problems are of which type, nor of which ones are relatively easy. Each problem can be solved with tools developed up to that point in the book, plus any additional prerequisites that are noted.

Two omissions from the pair of books are of note. One is any treatment of Stokes's Theorem and differential forms. Although there is some advantage, when studying these topics, in having the Lebesgue integral available and in having developed an attitude that integration can be defined by means of suitable linear functionals, the topic of Stokes's Theorem seems to fit better in a book about geometry and topology, rather than in a book about real analysis.

The other omission concerns the use of complex analysis. It is tempting to try to combine real analysis and complex analysis into a single subject, but my own experience is that this combination does not work well at the level of *Basic Real Analysis*, only at the level of *Advanced Real Analysis*.

Almost all of the mathematics in the two books is at least forty years old, and I make no claim that any result is new. The books are a distillation of lecture notes from a 35-year period of my own learning and teaching. Sometimes a problem at the end of a chapter or an approach to the exposition may not be a standard one, but no attempt has been made to identify such problems and approaches. In the reverse direction it is possible that my early lecture notes have directly quoted some source without proper attribution. As an attempt to rectify any difficulties of this kind, I have included a section of "Acknowledgements" on pages xix–xx of this volume to identify the main sources, as far as I can reconstruct them, for those original lecture notes.

I am grateful to Ann Kostant and Steven Krantz for encouraging this project and for making many suggestions about pursuing it, and to Susan Knapp and David Kramer for helping with the readability. The typesetting was by $\mathcal{A}\mathcal{M}\mathcal{S}$-TeX, and the figures were drawn with Mathematica.

I invite corrections and other comments from readers. I plan to maintain a list of known corrections on my own Web page.

A. W. KNAPP
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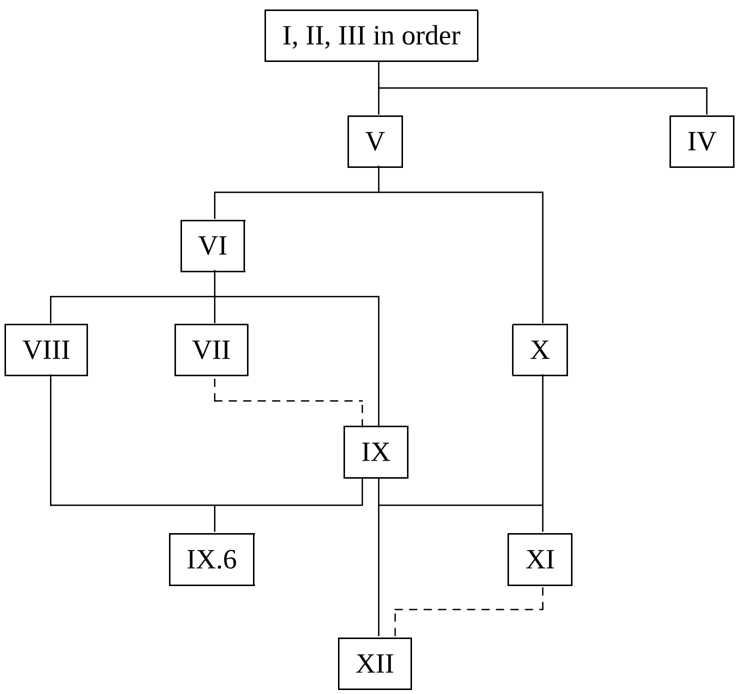
DEPENDENCE AMONG CHAPTERS

Below is a chart of the main lines of dependence of chapters on prior chapters. The dashed lines indicate helpful motivation but no logical dependence. Apart from that, particular examples may make use of information from earlier chapters that is not indicated by the chart.



GUIDE FOR THE READER

This section is intended to help the reader find out what parts of each chapter are most important and how the chapters are interrelated. Further information of this kind is contained in the abstracts that begin each of the chapters.

The book pays attention to certain recurring themes in real analysis, allowing a person to see how these themes arise in increasingly sophisticated ways. Examples are the role of interchanges of limits in theorems, the need for certain explicit formulas in the foundations of subject areas, the role of compactness and completeness in existence theorems, and the approach of handling nice functions first and then passing to general functions.

All of these themes are introduced in Chapter I, and already at that stage they interact in subtle ways. For example, a natural investigation of interchanges of limits in Sections 2–3 leads to the discovery of Ascoli's Theorem, which is a fundamental compactness tool for proving existence results. Ascoli's Theorem is proved by the "Cantor diagonal process," which has other applications to compactness questions and does not get fully explained until Chapter X. The consequence is that, no matter where in the book a reader plans to start, everyone will be helped by at least leafing through Chapter I.

The remainder of this section is an overview of individual chapters and groups of chapters.

Chapter I. Every section of this chapter plays a role in setting up matters for later chapters. No knowledge of metric spaces is assumed anywhere in the chapter. Section 1 will be a review for anyone who has already had a course in real-variable theory; the section shows how compactness and completeness address all the difficult theorems whose proofs are often skipped in calculus. Section 2 begins the development of real-variable theory at the point of sequences and series of functions. It contains interchange results that turn out to be special cases of the main theorems of Chapter V. Sections 8–9 introduce the approach of handling nice functions before general functions, and Section 10 introduces Fourier series, which provided a great deal of motivation historically for the development of real analysis and are used in this book in that same way. Fourier series are somewhat limited in the setting of Chapter I because one encounters no class of functions, other than infinitely differentiable ones, that corresponds exactly to some class of Fourier coefficients; as a result Fourier series, with Riemann integration in use, are not particularly useful for constructing new functions from old ones. This defect will be fixed with the aid of the Lebesgue integral in Chapter VI.

Chapter II. Now that continuity and convergence have been addressed on the line, this chapter establishes a framework for these questions in higher-dimensional Euclidean space and other settings. There is no point in ad hoc definitions for each setting, and metric spaces handle many such settings at once. Chapter X later will enlarge the framework from metric spaces to "topological spaces." Sections 1–6 of Chapter II are routine. Section 7, on compactness and completeness, is the core. The Baire Category Theorem in Section 9 is not used outside of Chapter II until Chapter XII, and it may therefore be skipped temporarily. Section 10 contains the Stone–Weierstrass Theorem, which is a fundamental approximation tool. Section 11 is used in some of the problems but is not otherwise used in the book.

Chapter III. This chapter does for the several-variable theory what Chapter I has done for the one-variable theory. The main results are the Inverse and Implicit Function Theorems in Section 6 and the change-of-variables formula for multiple integrals in Section 10. The change-of-variables formula has to be regarded as only a preliminary version, since what it directly accomplishes for the change to polar coordinates still needs supplementing; this difficulty will be repaired in Chapter VI with the aid of the Lebesgue integral. Section 4, on exponentials of matrices, may be skipped if linear systems of ordinary differential equations are going to be skipped in Chapter IV. Some of the problems at the end of the chapter introduce harmonic functions; harmonic functions will be combined with Fourier series in problems in later chapters to motivate and illustrate some of the development.

Chapter IV provides theoretical underpinnings for the material in a traditional undergraduate course in ordinary differential equations. Nothing later in the book is logically dependent on Chapter IV; however, Chapter XII includes a discussion of orthogonal systems of functions, and the examples of these that arise in Chapter IV are helpful as motivation. Some people shy away from differential equations and might wish to treat Chapter IV only lightly, or perhaps not at all. The subject is nevertheless of great importance, and Chapter IV is the beginning of it. A minimal treatment of Chapter IV might involve Sections 1–2 and Section 8, all of which visibly continue the themes begun in Chapter I.

Chapters V–VI treat the core of measure theory—including the basic convergence theorems for integrals, the development of Lebesgue measure in one and several variables, Fubini's Theorem, the metric spaces L^1 and L^2 and L^∞, and the use of maximal theorems for getting at differentiation of integrals and other theorems concerning almost-everywhere convergence. In Chapter V Lebesgue measure in one dimension is introduced right away, so that one immediately has the most important example at hand. The fundamental Extension Theorem for getting measures to be defined on σ-rings and σ-algebras is stated when needed but is proved only after the basic convergence theorems for integrals have been proved; the proof in Sections 5–6 may be skipped on first reading. Section 7, on Fubini's Theorem, is a powerful result about interchange of integrals. At the same time that it justifies interchange, it also constructs a "double integral"; consequently the section prepares the way for the construction in Chapter VI of n-dimensional Lebesgue measure from 1-dimensional Lebesgue measure. Section 10 introduces normed linear spaces along with the examples of L^1 and L^2 and L^∞, and it goes on to establish some properties of all normed linear spaces. Chapter VI fleshes out measure theory as it applies to Euclidean space in more than one dimension. Of special note is the Lebesgue-integration version in Section 5 of the change-of-variables formula for multiple integrals and the Riesz–Fischer Theorem in Section 7. The latter characterizes square-integrable periodic functions by their Fourier coefficients and makes the subject of Fourier series useful in constructing functions. Differentiation of integrals in approached in Section 6 of Chapter VI as a problem of estimating finiteness of a quantity, rather than its smallness; the device is the Hardy–Littlewood Maximal Theorem, and the approach becomes a routine way of approaching almost-everywhere convergence theorems. Sections 8–10 are of somewhat less importance and may be omitted if time is short; Section 10 is applied only in Section IX.6.

Chapters VII–IX are continuations of measure theory that are largely independent of each other. Chapter VII contains the traditional proof of the differentiation of integrals on the line via differentiation of monotone functions. No later chapter is logically dependent on Chapter VII; the material is included only because of its historical importance and its usefulness as motivation for the Radon–Nikodym Theorem in Chapter IX. Chapter VIII is an introduction to the Fourier transform in Euclidean space. Its core consists of the first four sections, and the rest may be considered as optional if Section IX.6 is to be omitted. Chapter IX concerns L^p spaces for $1 \le p \le \infty$; only Section 6 makes use of material from Chapter VIII.

Chapter X develops, at the latest possible time in the book, the necessary part of point-set topology that goes beyond metric spaces. Emphasis is on product and quotient spaces, and on Urysohn's Lemma concerning the construction of real-valued functions on normal spaces.

Chapter XI contains one more continuation of measure theory, namely special features of measures on locally compact Hausdorff spaces. It provides an example beyond L^p spaces in which one can usefully identify the dual of a particular normed linear space. These chapters depend on Chapter X and on the first five sections of Chapter IX but do not depend on Chapters VII–VIII.

Chapter XII is a brief introduction to functional analysis, particularly to Hilbert spaces, Banach spaces, and linear operators on them. The main topics are the geometry of Hilbert space and the three main theorems about Banach spaces.