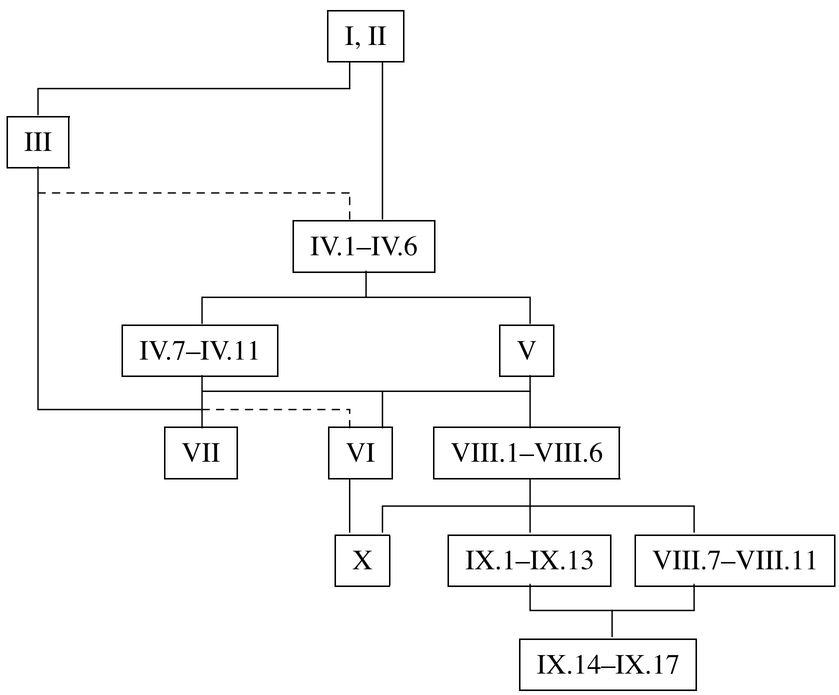
DEPENDENCE AMONG CHAPTERS

Below is a chart of the main lines of dependence of chapters on prior chapters. The dashed lines indicate helpful motivation but no logical dependence. Apart from that, particular examples may make use of information from earlier chapters that is not indicated by the chart.



STANDARD NOTATION

See the Index of Notation, pp. 717–719, for symbols defined starting on page 1.

Item	**Meaning**
$\#S$ or $\lvert S\rvert$	number of elements in S
$\varnothing$	empty set
$\{x \in E \mid P\}$	the set of x in E such that P holds
E^c	complement of the set E
$E \cup F,\ E \cap F,\ E - F$	union, intersection, difference of sets
$\bigcup_\alpha E_\alpha,\ \bigcap_\alpha E_\alpha$	union, intersection of the sets E_α
$E \subseteq F,\ E \supseteq F$	E is contained in F, E contains F
$E \subsetneqq F,\ E \supsetneqq F$	E properly contained in F, properly contains F
$E \times F,\ \bigtimes_{s\in S} X_s$	products of sets
$(a_1, \ldots, a_n),\ \{a_1, \ldots, a_n\}$	ordered n-tuple, unordered n-tuple
$f : E \to F,\ x \mapsto f(x)$	function, effect of function
$f \circ g$ or $fg,\ f\big\vert_E$	composition of g followed by f, restriction to E
$f(\,\cdot\,, y)$	the function $x \mapsto f(x, y)$
$f(E),\ f^{-1}(E)$	direct and inverse image of a set
δ_{ij}	Kronecker delta: 1 if $i = j$, 0 if $i \neq j$
$\binom{n}{k}$	binomial coefficient
n positive, n negative	$n > 0$, $n < 0$
$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$	integers, rationals, reals, complex numbers
max (and similarly min)	maximum of a finite subset of a totally ordered set
$\sum$ or $\prod$	sum or product, possibly with a limit operation
countable	finite or in one-one correspondence with $\mathbb{Z}$
$[x]$	greatest integer $\leq x$ if x is real
Re z, Im z	real and imaginary parts of complex z
$\bar{z}$	complex conjugate of z
$\lvert z\rvert$	absolute value of z
1	multiplicative identity
1 or I	identity matrix or operator
1_X	identity function on X
$\mathbb{Q}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$	spaces of column vectors
$\mathrm{diag}(a_1, \ldots, a_n)$	diagonal matrix
$\cong$	is isomorphic to, is equivalent to

GUIDE FOR THE READER

This section is intended to help the reader find out what parts of each chapter are most important and how the chapters are interrelated. Further information of this kind is contained in the abstracts that begin each of the chapters.

The book pays attention to at least three recurring themes in algebra, allowing a person to see how these themes arise in increasingly sophisticated ways. These are the analogy between integers and polynomials in one indeterminate over a field, the interplay between linear algebra and group theory, and the relationship between number theory and geometry. Keeping track of how these themes evolve will help the reader understand the mathematics better and anticipate where it is headed.

In Chapter I the analogy between integers and polynomials in one indeterminate over the rationals, reals, or complex numbers appears already in the first three sections. The main results of these sections are theorems about unique factorization in each of the two settings. The relevant parts of the underlying structures for the two settings are the same, and unique factorization can therefore be proved in both settings by the same argument. Many readers will already know this unique factorization, but it is worth examining the parallel structure and proof at least quickly before turning to the chapters that follow.

Before proceeding very far into the book, it is worth looking also at the appendix to see whether all its topics are familiar. Readers will find Section A1 useful at least for its summary of set-theoretic notation and for its emphasis on the distinction between range and image for a function. This distinction is usually unimportant in analysis but becomes increasingly important as one studies more advanced topics in algebra. Readers who have not specifically learned about equivalence relations and partial orderings can learn about them from Sections A2 and A5. Sections A3 and A4 concern the real and complex numbers; the emphasis is on notation and the Intermediate Value Theorem, which plays a role in proving the Fundamental Theorem of Algebra. Zorn's Lemma and cardinality in Sections A5 and A6 are usually unnecessary in an undergraduate course. They arise most importantly in Sections II.9 and IX.4, which are normally omitted in an undergraduate course, and in Proposition 8.8, which is invoked only in the last few sections of Chapter VIII.

The remainder of this section is an overview of individual chapters and pairs of chapters.

Chapter I is in three parts. The first part, as mentioned above, establishes unique factorization for the integers and for polynomials in one indeterminate over the rationals, reals, or complex numbers. The second part defines permutations and shows that they have signs such that the sign of any composition is the product of the signs; this result is essential for defining general determinants in Section II.7. The third part will likely be a review for all readers. It establishes notation for row reduction of matrices and for operations on matrices, and it uses row reduction to show that a one-sided inverse for a square matrix is a two-sided inverse.

Chapters II–III treat the fundamentals of linear algebra. Whereas the matrix computations in Chapter I were concrete, Chapters II–III are relatively abstract. Much of this material is likely to be a review for graduate students. The geometric interpretation of vectors spaces, subspaces, and linear mappings is not included in the chapter, being taken as known previously. The fundamental idea that a newly constructed object might be characterized by a "universal mapping property" appears for the first time in Chapter II, and it appears more and more frequently throughout the book. One aspect of this idea is that it is sometimes not so important what certain constructed objects are, but what they do. A related idea being emphasized is that the mappings associated with a newly constructed object are likely to be as important as the object, if not more so; at the least, one needs to stop and find what those mappings are. Section II.9 uses Zorn's Lemma and can be deferred until Chapter IX if one wants. Chapter III discusses special features of real and complex vector spaces endowed with inner products. The main result is the Spectral Theorem in Section 3. Many of the problems at the end of the chapter make contact with real analysis. The subject of linear algebra continues in Chapter V.

Chapter IV is the primary chapter on group theory and may be viewed as in three parts. Sections 1–6 form the first part, which is essential for all later chapters in the book. Sections 1–3 introduce groups and some associated constructions, along with a number of examples. Many of the examples will be seen to be related to specific or general vector spaces, and thus the theme of the interplay between group theory and linear algebra is appearing concretely for the first time. In practice, many examples of groups arise in the context of group actions, and abstract group actions are defined in Section 6. Of particular interest are group representations, which are group actions on a vector space by linear mappings. Sections 4–5 are a digression to define rings, fields, and ring homomorphisms, and to extend the theories concerning polynomials and vector spaces as presented in Chapters I–II. The immediate purpose of the digression is to make prime fields, their associated multiplicative groups, and the notion of characteristic available for the remainder of the chapter. The definition of vector space is extended to allow scalars from any field. The definition of polynomial is extended to allow coefficients from any commutative ring with identity, rather than just the rationals or reals or complex numbers, and to allow more than one indeterminate. Universal mapping properties for polynomial rings are proved. Sections 7–10 form the second part of the chapter and are a continuation of group theory. The main result is the Fundamental Theorem of Finitely Generated Abelian Groups, which is in Section 9. Section 11 forms the third part of the chapter. This section is a gentle introduction to categories and functors, which are useful for working with parallel structures in different settings within algebra. As S. Mac Lane says in his book, "Category theory asks of every type of Mathematical object: 'What are the morphisms?'; it suggests that these morphisms should be described at the same time as the objects. . . . This emphasis on (homo)morphisms is largely due to Emmy Noether, who emphasized the use of homomorphisms of groups and rings." The simplest parallel structure reflected in categories is that of an isomorphism. The section also discusses general notions of product and coproduct functors. Examples of products are direct products in linear algebra and in group theory. Examples of coproducts are direct sums in linear algebra and in *abelian* group theory, as well as disjoint unions in set theory. The theory in this section helps in unifying the mathematics that is to come in Chapters VI–VIII and X. The subject of group theory in continued in Chapter VII, which assumes knowledge of the material on category theory.

Chapters V and VI continue the development of linear algebra. Chapter VI uses categories, but Chapter V does not. Most of Chapter V concerns the analysis of a linear transformation carrying a finite-dimensional vector space over a field into itself. The questions are to find invariants of such transformations and to classify the transformations up to similarity. Section 2 at the start extends the theory of determinants so that the matrices are allowed to have entries in a commutative ring with identity; this extension is necessary in order to be able to work easily with characteristic polynomials. The extension of this theory is carried out by an important principle known as the "permanence of identities." Chapter VI largely concerns bilinear forms and tensor products, again in the context that the coefficients are from a field. This material is necessary in many applications to geometry and physics, but it is not needed in Chapters VII–IX. Many objects in the chapter are constructed in such a way that they are uniquely determined by a universal mapping property. Problems 18–22 at the end of the chapter discuss universal mapping properties in the general context of category theory, and they show that a uniqueness theorem is automatic in all cases.

Chapter VII continues the development of group theory, making use of category theory. It is in two parts. Sections 1–3 concern free groups and the topic of generators and relations; they are essential for abstract descriptions of groups and for work in topology involving fundamental groups. Section 3 constructs a notion of free product and shows that it is the coproduct functor for the category of groups. Sections 4–6 continue the theme of the interplay of group theory and linear algebra. Section 4 analyzes group representations of a finite group when the underlying field is the complex numbers, and Section 5 applies this theory to obtain a conclusion about the structure of finite groups. Section 6 studies extensions of groups and uses them to motivate the subject of cohomology of groups.

Chapter VIII introduces modules, giving many examples in Section 1, and then goes on to discuss questions of unique factorization in integral domains. Section 6 obtains a generalization for principal ideal domains of the Fundamental Theorem of Finitely Generated Abelian Groups, once again illustrating the first theme—similarities between the integers and certain polynomial rings. Section 7 introduces the third theme, the relationship between number theory and geometry, as a more sophisticated version of the first theme. The section compares a certain polynomial ring in two variables with a certain ring of algebraic integers that extends the ordinary integers. Unique factorization of elements fails for both, but the geometric setting has a more geometrically meaningful factorization in terms of ideals that is evidently unique. This kind of unique factorization turns out to work for the ring of algebraic integers as well. Sections 8–11 expand the examples in Section 7 into a theory of unique factorization of ideals in any integrally closed Noetherian domain whose nonzero prime ideals are all maximal.

Chapter IX analyzes algebraic extensions of fields. The first 13 sections make use only of Sections 1–6 in Chapter VIII. Sections 1–5 of Chapter IX give the foundational theory, which is sufficient to exhibit all the finite fields and to prove that certain classically proposed constructions in Euclidean geometry are impossible. Sections 6–8 introduce Galois theory, but Theorem 9.28 and its three corollaries may be skipped if Sections 14–17 are to be omitted. Sections 9–11 give a first round of applications of Galois theory: Gauss's theorem about which regular n-gons are in principle constructible with straightedge and compass, the Fundamental Theorem of Algebra, and the Abel–Galois theorem that solvability of a polynomial equation with rational coefficients in terms of radicals implies solvability of the Galois group. Sections 12–13 give a second round of applications: Gauss's method in principle for actually constructing the constructible regular n-gons and a converse to the Abel–Galois theorem. Sections 14–17 make use of Sections 7–11 of Chapter VIII, proving that π is transcendental and obtaining two methods for computing Galois groups.

Chapter X is a relatively short chapter developing further tools for dealing with modules over a ring with identity. The main construction is that of the tensor product over a ring of a unital right module and a unital left module, the result being an abelian group. The chapter makes use of material from Chapters VI and VIII, but not from Chapter IX.