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CHAPTER II

Wedderburn–Artin Ring Theory

Abstract. This chapter studies finite-dimensional associative division algebras, as well as other finite-dimensional associative algebras and closely related rings. The chapter is in two parts that overlap slightly in Section 6. The first part gives the structure theory of the rings in question, and the second part aims at understanding limitations imposed by the structure of a division ring.

Section 1 briefly summarizes the structure theory for finite-dimensional (nonassociative) Lie algebras that was the primary historical motivation for structure theory in the associative case. All the algebras in this chapter except those explicitly called Lie algebras are understood to be associative.

Section 2 introduces left semisimple rings, defined as rings R with identity such that the left R module R is semisimple. Wedderburn's Theorem says that such a ring is the finite product of full matrix rings over division rings. The number of factors, the size of each matrix ring, and the isomorphism class of each division ring are uniquely determined. It follows that left semisimple and right semisimple are the same. If the ring is a finite-dimensional algebra over a field F, then the various division rings are finite-dimensional division algebras over F. The factors of semisimple rings are simple, i.e., are nonzero and have no nontrivial two-sided ideals, but an example is given to show that a simple ring need not be semisimple. Every finite-dimensional simple algebra is semisimple.

Section 3 introduces chain conditions into the discussion as a useful generalization of finite dimensionality. A ring R with identity is left Artinian if the left ideals of the ring satisfy the descending chain condition. Artin's Theorem for simple rings is that left Artinian is equivalent to semisimplicity, hence to the condition that the given ring be a full matrix ring over a division ring.

Sections 4–6 concern what happens when the assumption of semisimplicity is dropped but some finiteness condition is maintained. Section 4 introduces the Wedderburn–Artin radical $\operatorname{rad} R$ of a left Artinian ring R as the sum of all nilpotent left ideals. The radical is a two-sided nilpotent ideal. It is 0 if and only if the ring is semisimple. More generally $R/\operatorname{rad} R$ is always semisimple if R is left Artinian. Sections 5–6 state and prove Wedderburn's Main Theorem—that a finite-dimensional algebra R with identity over a field F of characteristic 0 has a semisimple subalgebra S such that R is isomorphic as a vector space to $S \oplus \operatorname{rad} R$. The semisimple algebra S is isomorphic to $R/\operatorname{rad} R$. Section 5 gives the hard part of the proof, which handles the special case that $R/\operatorname{rad} R$ is isomorphic to a product of full matrix algebras over F. The remainder of the proof, which appears in Section 6, follows relatively quickly from the special case in Section 5 and an investigation of circumstances under which the tensor product over F of two semisimple algebras is semisimple. Such a tensor product is not always semisimple, but it is semisimple in characteristic 0.

The results about tensor products in Section 6, but with other hypotheses in place of the condition of characteristic 0, play a role in the remainder of the chapter, which is aimed at identifying certain division rings. Sections 7–8 provide general tools. Section 7 begins with further results about tensor products. Then the Skolem–Noether Theorem gives a relationship between any two homomorphisms of a simple subalgebra into a simple algebra whose center coincides with the underlying field of

scalars. Section 8 proves the Double Centralizer Theorem, which says for this situation that the centralizer of the simple subalgebra in the whole algebra is simple and that the product of the dimensions of the subalgebra and the centralizer is the dimension of the whole algebra.

Sections 9–10 apply the results of Sections 6–8 to obtain two celebrated theorems—Wedderburn's Theorem about finite division rings and Frobenius's Theorem classifying the finite-dimensional associative division algebras over the reals.

1. Historical Motivation

Elementary ring theory came from several sources historically and was already in place by 1880. Some of the sources are field theory (studied by Galois and others), rings of algebraic integers (studied by Gauss, Dirichlet, Kummer, Kronecker, Dedekind, and others), and matrices (studied by Cayley, Hamilton, and others). More advanced general ring theory arose initially not on its own but as an effort to imitate the theory of "Lie algebras," which began about 1880.

A brief summary of some early theorems about Lie algebras will put matters in perspective. The term "algebra" in connection with a field F refers at least to an F vector space with a multiplication that is F bilinear. This chapter will deal only with two kinds of such algebras, the Lie algebras and those algebras whose multiplication is associative. If the modifier "Lie" is absent, the understanding is that the algebra is associative.

Lie algebras arose originally from "Lie groups"—which we can regard for current purposes as connected groups with finitely many smooth parameters—by a process of taking derivatives along curves at the identity element of the group. Precise knowledge of that process will be unnecessary in our treatment, but we describe one example: The vector space $M_n(\mathbb{R})$ of all n-by-n matrices over $\mathbb{R}$ becomes a Lie algebra with multiplication defined by the "bracket product" $[X, Y] = XY - YX$. If G is a closed subgroup of the matrix group $\mathrm{GL}(n, \mathbb{R})$ and $\mathfrak{g}$ is the set of all members of $M_n(\mathbb{R})$ of the form $X = c'(0)$, where c is a smooth curve in G with $c(0)$ equal to the identity, then it turns out that the vector space $\mathfrak{g}$ is closed under the bracket product and is a Lie algebra. Although one might expect the Lie algebra $\mathfrak{g}$ to give information about the Lie group G only infinitesimally at the identity, it turns out that $\mathfrak{g}$ determines the multiplication rule for G in a whole open neighborhood of the identity. Thus the Lie group and Lie algebra are much more closely related than one might at first expect.

We turn to the underlying definitions and early main theorems about Lie algebras. Let F be a field. A vector space A over F with an F bilinear multiplication $(X, Y) \mapsto [X, Y]$ is a **Lie algebra** if the multiplication has the two properties

(i) $[X, X] = 0$ for all $X \in A$,

(ii) **(Jacobi identity)** $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$ for all $X, Y, Z \in A$.

Multiplication is often referred to as **bracket**. It is usually not associative. The vector space $M_n(F)$ with $[X, Y] = XY - YX$ is a Lie algebra, as one easily checks by expanding out the various brackets that are involved; it is denoted by $\mathfrak{gl}(n, F)$.

The elementary structural definitions with Lie algebras run parallel to those with rings. A **Lie subalgebra** S of A is a vector subspace closed under brackets, an **ideal** I of A is a vector subspace such that $[X, Y]$ is in I for $X \in I$ and $Y \in A$, a **homomorphism** $\varphi : A_1 \to A_2$ of Lie algebras is a linear mapping respecting brackets in the sense that $\varphi[X, Y] = [\varphi(X), \varphi(Y)]$ for all $X, Y \in A_1$, and an **isomorphism** is an invertible homomorphism. Every ideal is a Lie subalgebra. In contrast to the case of rings, there is no distinction between "left ideals" and "right ideals" because the bracket product is skew symmetric. Under the passage from Lie groups to Lie algebras, abelian Lie groups yield Lie algebras with all brackets 0, and thus one says that a Lie algebra is **abelian** if all its brackets are 0.

Examples of Lie subalgebras of $\mathfrak{gl}(n, F)$ are the subalgebra $\mathfrak{sl}(n, F)$ of all matrices of trace 0, the subalgebra $\mathfrak{so}(n, F)$ of all skew-symmetric matrices, and the subalgebra of all upper-triangular matrices.

The elementary properties of subalgebras, homomorphisms, and so on for Lie algebras mimic what is true for rings: The kernel of a homomorphism is an ideal. Any ideal is the kernel of a quotient homomorphism. If I is an ideal in A, then the ideals of A/I correspond to the ideals of A containing I, just as in the First Isomorphism Theorem for rings. If I and J are ideals in A, then $(I + J)/I \cong J/(I \cap J)$, just as in the Second Isomorphism Theorem for rings.

The connection of Lie algebras to Lie groups makes one want to introduce definitions that lead toward classifying all Lie algebras that are finite-dimensional. We therefore assume for the remainder of this section that all Lie algebras under discussion are finite-dimensional over F. Some of the steps require conditions on F, and we shall assume that F has characteristic 0.

Group theory already had a notion of "solvable group" from Galois, and this leads to the notion of solvable Lie algebra. In A, let $[A, A]$ denote the linear span of all $[X, Y]$ with $X, Y \in A$; $[A, A]$ is called the **commutator ideal** of A, and $A/[A, A]$ is abelian. In fact, $[A, A]$ is the smallest ideal I in A such that A/I is abelian. Starting from A, let us form successive commutator ideals. Thus put $A_0 = A$, $A_1 = [A_0, A_0], \dots, A_n = [A_{n-1}, A_{n-1}]$, so that

$$A = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n \supseteq \cdots .$$

The terms of this sequence are all the same from some point on, by finite dimensionality, and we say that A is **solvable** if the terms are ultimately 0. One easily checks that the sum $I + J$ of two solvable ideals in A, i.e., the set of sums, is a solvable ideal. By finite dimensionality, there exists a unique largest solvable ideal. This is called the **radical** of A and is denoted by $\operatorname{rad} A$. The Lie algebra

A is said to be **semisimple** if $\operatorname{rad} A = 0$. It is easy to use the First Isomorphism Theorem to check that $A/\operatorname{rad} A$ is always semisimple.

In the direction of classifying Lie algebras, one might therefore want to see how all solvable Lie algebras can be constructed by successive extensions, identify all semisimple Lie algebras, and determine how a general Lie algebra can be constructed from a semisimple Lie algebra and a solvable Lie algebra by an extension.

The first step in this direction historically concerned identifying semisimple Lie algebras. We say that the Lie algebra A is **simple** if $\dim A > 1$ and if A contains no nonzero proper ideals.

Working with the field $\mathbb{C}$ but in a way that applies to other fields of characteristic 0, W. Killing proved in 1888 that A is semisimple if and only if A is the (internal) direct sum of simple ideals. In this case the direct summands are unique, and the only ideals in A are the partial direct sums.

This result is strikingly different from what happens for abelian Lie algebras, for which the theory reduces to the theory of vector spaces. A 2-dimensional vector space is the internal direct sum of two 1-dimensional subspaces in many ways. But Killing's theorem says that the decomposition of semisimple Lie algebras into simple ideals is unique, not just unique up to some isomorphism.

É. Cartan in his 1894 thesis classified the simple Lie algebras, up to isomorphism, for the case that the field is $\mathbb{C}$. The Lie algebras $\mathfrak{sl}(n, \mathbb{C})$ for $n \geq 2$ and $\mathfrak{so}(n, \mathbb{C})$ for $n = 3$ and $n \geq 5$ were in his list, and there were others. Killing had come close to this classification in his 1888 work, but he had made a number of errors in both his statements and his proofs.

E. E. Levi in 1905 addressed the extension problem for obtaining all finite-dimensional Lie algebras over $\mathbb{C}$ from semisimple ones and solvable ones. His theorem is that for any Lie algebra A, there exists a subalgebra S isomorphic to $A/\operatorname{rad} A$ such that $A = S \oplus \operatorname{rad} A$ as vector spaces. In essence, this result says that the extension defining A is given by a semidirect product.

The final theorem in this vein at this time in history was a 1914 result of Cartan classifying the simple Lie algebras when the field F is $\mathbb{R}$. This classification is a good bit more complicated than the classification when F is $\mathbb{C}$.

With this background in mind, we can put into context the corresponding developments for associative algebras. Although others had done some earlier work, J. H. M. Wedderburn made the first big advance for associative algebras in 1905. Wedderburn's theory in a certain sense is more complicated than the theory for Lie algebras because left ideals in the associative case are not necessarily two-sided ideals. Let us sketch this theory.

For the remainder of this section until the last paragraph, A will denote a finite-dimensional associative algebra over a field F of characteristic 0, possibly the 0 algebra. We shall always assume that A has an identity. Although we shall make some definitions here, we shall repeat them later in the chapter at the appropriate times. For many results later in the chapter, the field F will not be assumed to be of characteristic 0.

As in Chapter X of *Basic Algebra*, a unital left A module M is said to be simple if it is nonzero and it has no proper nonzero A submodules, semisimple if it is the sum (or equivalently the direct sum) of simple A submodules. The algebra A is **semisimple** if the left A module A is a semisimple module, i.e., if A is the direct sum of simple left ideals; A is **simple** if it is nonzero and has no nontrivial two-sided ideals. In contrast to the setting of Lie algebras, we make no exception for the 1-dimensional case; this distinction is necessary and is continually responsible for subtle differences between the two theories.

Wedderburn's first theorem has two parts to it, the first one modeled on Killing's theorem for Lie algebras and the second one modeled on Cartan's thesis:

(i) The algebra A is semisimple if and only if it is the (internal) direct sum of simple two-sided ideals. In this case the direct summands are unique, and the only two-sided ideals of A are the partial direct sums.

(ii) The algebra A is simple if and only if $A \cong M_n(D)$ for some integer $n \geq 1$ and some division algebra D over F. In particular, if F is algebraically closed, then $A \cong M_n(F)$ for some n.

E. Artin generalized the Wedderburn theory to a suitable kind of "semisimple ring." For part of the theory, he introduced a notion of "radical" for the associative case—the **radical** of a finite-dimensional associative algebra A being the sum of the "nilpotent" left ideals of A. Here a left ideal I is called **nilpotent** if $I^k = 0$ for some k. The radical $\operatorname{rad} A$ is a two-sided ideal, and $A/\operatorname{rad} A$ is a semisimple ring.

Wedderburn's Main Theorem, proved later in time and definitely assuming characteristic 0, is an analog for associative algebras of Levi's result about Lie algebras. The result for associative algebras is that A decomposes as a vector-space direct sum $A = S \oplus \operatorname{rad} A$, where S is a semisimple subalgebra isomorphic to $A/\operatorname{rad} A$.

The remaining structural question for finite-dimensional associative algebras is to say something about simple algebras when the field is not algebraically closed. Such a result may be regarded as an analog of the 1914 work by Cartan. In the associative case one then wants to know what the F isomorphism classes of finite-dimensional associative division algebras D are for a given field F. We now drop the assumption that the field F has characteristic 0. In asking this question, one does not want to repeat the theory of field extensions. Consequently one looks only for classes of division algebras whose center is F. If F is algebraically closed, the only such D is F itself, as we shall observe in more detail in Section 2.

If F is a finite field, one is led to another theorem of Wedderburn's, saying that D has to be commutative and hence that $D = F$; this theorem appears in Section 9. If F is $\mathbb{R}$, one is led to a theorem of Frobenius saying that there are just two such D's up to $\mathbb{R}$ isomorphism, namely $\mathbb{R}$ itself and the quaternions $\mathbb{H}$; this theorem appears in Section 10. For a general field F, it turns out that the set of classes of finite-dimensional division algebras with center F forms an abelian group. The group is called the "Brauer group" of F. Its multiplication is defined by the condition that the class of D_1 times D_2 is the class of a division algebra D_3 such that $D_1 \otimes_F D_2 \cong M_n(D_3)$ for some n; the inverse of the class of D is the class of the opposite algebra D^o, and the identity is the class of F. The study of the Brauer group is postponed to Chapter III. This group has an interpretation in terms of cohomology of groups, and it has applications to algebraic number theory.

2. Semisimple Rings and Wedderburn's Theorem

We now begin our detailed investigation of associative algebras over a field. In this section we shall address the first theorem of Wedderburn's that is mentioned in the previous section. It has two parts, one dealing with semisimple algebras and one dealing with finite-dimensional simple algebras. The first part does not need the finite dimensionality as a hypothesis, and we begin with that one.

Let R be a ring with identity. The ring R is **left semisimple** if the left R module R is a semisimple module, i.e., if R is the direct sum of minimal left ideals.[1] In this case $R = \bigoplus_{i\in S} I_i$ for some set S and suitable minimal left ideals I_i. Since R has an identity, we can decompose the identity according to the direct sum as $1 = 1_{i_1} + \cdots + 1_{i_n}$ for some finite subset $\{i_1, \ldots, i_n\}$ of S, where 1_{i_k} is the component of 1 in I_{i_k}. Multiplying by $r \in R$ on the left, we see that $R \subseteq \bigoplus_{k=1}^n I_{i_k}$. Consequently R has to be a *finite* sum of minimal left ideals. A ring R with identity is **right semisimple** if the right R module R is a semisimple module. We shall see later in this section that left semisimple and right semisimple are equivalent.

EXAMPLES OF SEMISIMPLE RINGS.

(1) If D is a division ring, then we saw in Example 4 in Section X.1 of *Basic Algebra* that the ring $R = M_n(D)$ is left semisimple in the sense of the above definition. Actually, that example showed more. It showed that R as a left R module is given by $M_n(D) \cong D^n \oplus \cdots \oplus D^n$, where each D^n is a simple left R module and the j^{th} summand D^n corresponds to the matrices whose only nonzero entries are in the j^{th} column. The left R module $M_n(D)$ has a composition series whose terms are the partial sums of the n summands D^n. If M is any simple left $M_n(D)$ module and if $x \neq 0$ is in M, then $M = M_n(D)x$. If we set $I = \{r \in M_n(D) \mid rx = 0\}$, then I is a left ideal in $M_n(D)$ and $M \cong M_n(D)/I$

[1] By convention, a "minimal left ideal" always means a "minimal nonzero left ideal."

as a left $M_n(D)$ module. In other words, M is an irreducible quotient module of the left $M_n(D)$ module $M_n(D)$. By the Jordan–Hölder Theorem (Corollary 10.7 of *Basic Algebra*), M occurs as a composition factor. Hence $M \cong D^n$ as a left $M_n(D)$ module. Hence every simple left $M_n(D)$ module is isomorphic to D^n. We shall use this style of argument repeatedly but will ordinarily include less detail.

(2) If $R_1, \dots, R_n$ are left semisimple rings, then the direct product $R = \prod_{i=1}^n R_i$ is left semisimple.[2] In fact, each minimal left ideal of R_i, when included into R, is a minimal left ideal of R. Hence R is the sum of minimal left ideals and is left semisimple. By the same kind of argument as for Example 1, every simple left R module is isomorphic to one of these minimal left ideals.

Lemma 2.1. Let D be a division ring, let $R = M_n(D)$, and let D^n be the simple left R module of column vectors. Each member of D acts on D^n by scalar multiplication on the *right* side, yielding a member of $\operatorname{End}_R(D^n)$. In turn, $\operatorname{End}_R(D^n)$ is a ring, and this identification therefore is an inclusion of the members of D into the right D module $\operatorname{End}_R(D^n)$. The inclusion is in fact an isomorphism of rings: $D^o \cong \operatorname{End}_R(D^n)$, where D^o is the opposite ring of D.

PROOF. Let $\varphi : D \to \operatorname{End}_R(D^n)$ be the function given by $\varphi(d)(v) = vd$. Then $\varphi(dd')(v) = v(dd') = (vd)d' = \varphi(d')(vd) = \varphi(d')(\varphi(d)(v))$. Since the order of multiplication in D is reversed by φ, φ is a ring homomorphism of D^o into $\operatorname{End}_R(D^n)$. It is one-one because D^o is a division ring and has no nontrivial two-sided ideals. To see that it is onto $\operatorname{End}_R(D^n)$, let f be in $\operatorname{End}_R(D^n)$. Put $f\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} = \begin{pmatrix}d\\d_2\\\vdots\\d_n\end{pmatrix}$. Since f is an R module homomorphism,

$$f\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix} = f\left(\begin{pmatrix}a_1&0&\cdots&0\\a_2&0&\cdots&0\\\vdots&&&\\a_n&0&\cdots&0\end{pmatrix}\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\right) = \begin{pmatrix}a_1&0&\cdots&0\\a_2&0&\cdots&0\\\vdots&&&\\a_n&0&\cdots&0\end{pmatrix} f\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$$

$$= \begin{pmatrix}a_1&0&\cdots&0\\a_2&0&\cdots&0\\\vdots&&&\\a_n&0&\cdots&0\end{pmatrix}\begin{pmatrix}d\\d_2\\\vdots\\d_n\end{pmatrix} = \begin{pmatrix}a_1d\\a_2d\\\vdots\\a_nd\end{pmatrix} = \varphi(d)\begin{pmatrix}a_1\\a_2\\\vdots\\a_n\end{pmatrix}.$$

Therefore $\varphi(d) = f$, and φ is onto. □

[2]Some comment is appropriate about the notation $R = \prod_{i=1}^n R_i$ and the terminology "direct product." Indeed, $\prod_{i=1}^n R_i$ is a product in the sense of category theory within the category of rings or the category of rings with identity. Sometimes one views R alternatively as built from n two-sided ideals, each corresponding to one of the n coordinates; in this case, one may say that R is the "direct sum" of these ideals. This direct sum is to be regarded as a direct sum of abelian groups, or perhaps vector spaces or R modules, but it is not a coproduct within the category of rings with identity.

Theorem 2.2 (Wedderburn). If R is any left semisimple ring, then

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for suitable division rings $D_1, \ldots, D_n$ and positive integers $n_1, \ldots, n_r$. The number r is uniquely determined by R, and the ordered pairs $(n_1, D_1), \ldots, (n_r, D_r)$ are determined up to a permutation of $\{1, \ldots, r\}$ and an isomorphism of each D_j. There are exactly r mutually nonisomorphic simple left R modules, namely $(D_1)^{n_1}, \ldots, (D_r)^{n_r}$.

PROOF. Write R as the direct sum of minimal left ideals, and then regroup the summands according to their R isomorphism type as $R \cong \bigoplus_{j=1}^r n_j V_j$, where $n_j V_j$ is the direct sum of n_j submodules R isomorphic to V_j and where $V_i \ncong V_j$ for $i \neq j$. The isomorphism is one of unital left R modules. Put $D_i^o = \operatorname{End}_R(V_i)$. This is a division ring by Schur's Lemma (Proposition 10.4b of *Basic Algebra*). Using Proposition 10.14 of *Basic Algebra*, we obtain an isomorphism of rings

$$R^o \cong \operatorname{End}_R R \cong \operatorname{Hom}_R\Big(\bigoplus_{i=1}^r n_i V_i, \bigoplus_{j=1}^r n_j V_j\Big). \qquad (*)$$

Define $p_i : \bigoplus_{j=1}^r n_j V_j \to n_i V_i$ to be the i^{th} projection and $q_i : n_i V_i \to \bigoplus_{j=1}^r n_j V_j$ to be the i^{th} inclusion. Let us see that the right side of $(*)$ is isomorphic as a ring to $\prod_i \operatorname{End}_R(n_i V_i)$ via the mapping $f \mapsto (p_1 f q_1, \ldots, p_r f q_r)$. What is to be shown is that $p_j f q_i = 0$ for $i \neq j$. Here $p_j f q_i$ is a member of $\operatorname{Hom}_R(n_i V_i, n_j V_j)$. The abelian group $\operatorname{Hom}_R(n_i V_i, n_j V_j)$ is the direct sum of abelian groups isomorphic to $\operatorname{Hom}_R(V_i, V_j)$ by Proposition 10.12, and each $\operatorname{Hom}_R(V_i, V_j)$ is 0 by Schur's Lemma (Proposition 10.4a).

Referring to $(*)$, we therefore obtain ring isomorphisms

$$\begin{aligned} R^o &\cong \prod_{i=1}^r \operatorname{Hom}_R(n_i V_i, n_i V_i) = \prod_{i=1}^r \operatorname{End}_R(n_i V_i) \\ &\cong \prod_{i=1}^r M_{n_i}(\operatorname{End}_R(V_i)) && \text{by Corollary 10.13} \\ &\cong \prod_{i=1}^r M_{n_i}(D_i^o) && \text{by definition of } D_i^o. \end{aligned}$$

Reversing the order of multiplication in R^o and using the transpose map to reverse the order of multiplication in each $M_{n_i}(D_i^o)$, we conclude that $R \cong \prod_{i=1}^r M_{n_i}(D_i)$. This proves existence of the decomposition in the theorem.

We still have to identify the simple left R modules and prove an appropriate uniqueness statement. As we recalled in Example 1, we have a decomposition

$M_{n_i}(D_i) \cong D_i^{n_i} \oplus \cdots \oplus D_i^{n_i}$ of left $M_{n_i}(D_i)$ modules, and each term $D_i^{n_i}$ is a simple left $M_{n_i}(D_i)$ module. The decomposition just proved allows us to regard each term $D_i^{n_i}$ as a simple left R module, $1 \le i \le r$. Each of these modules is acted upon by a different coordinate of R, and hence we have produced at least r nonisomorphic simple left R modules. Any simple left R module must be a quotient of R by a maximal left ideal, as we observed in Example 2, hence a composition factor as a consequence of the Jordan–Hölder Theorem. Thus it must be one of the V_j's in the previous part of the proof. There are only r nonisomorphic such V_j's, and we conclude that the number of simple left R modules, up to isomorphism, is exactly r.

For uniqueness suppose that $R \cong M_{n'_1}(D'_1) \times \cdots \times M_{n'_s}(D'_s)$ as rings. Let $V'_j = (D'_j)^{n'_j}$ be the unique simple left $M_{n'_j}(D'_j)$ module up to isomorphism, and regard V'_j as a simple left R module. Then we have $R \cong \bigoplus_{j=1}^{s} n'_j V'_j$ as left R modules. By the Jordan–Hölder Theorem we must have $r = s$ and, after a suitable renumbering, $n_i = n'_i$ and $V_i \cong V'_i$ for $1 \le i \le r$. Thus we have ring isomorphisms

$$\begin{aligned} (D'_i)^o &\cong \operatorname{End}_{M_{n'_i}(D'_i)}(V'_i) && \text{by Lemma 2.1} \\ &\cong \operatorname{End}_R(V'_i) \\ &\cong \operatorname{End}_R(V_i) && \text{since } V_i \cong V'_i \\ &\cong D_i^o. \end{aligned}$$

Reversing the order of multiplication gives $D'_i \cong D_i$, and the proof is complete. □

Corollary 2.3. For a ring R, left semisimple coincides with right semisimple.

REMARK. Therefore we can henceforth refer to left semisimple rings unambiguously as **semisimple**.

PROOF. The theorem gives the form of any left semisimple ring, and each ring of this form is certainly right semisimple. □

Wedderburn's original formulation of Theorem 2.2 was for algebras over a field F, and he assumed finite dimensionality. The theorem in this case gives

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r),$$

and the proof shows that $D_i^o \cong \operatorname{End}_R(V_i)$, where V_i is a minimal left ideal of R of the i^{th} isomorphism type. The field F lies inside $\operatorname{End}_R(V_i)$, each member of F yielding a scalar mapping, and hence each D_i is a division algebra over F. Each D_i is necessarily finite-dimensional over F, since R was assumed to be finite-dimensional.

We shall make occasional use in this chapter of the fact that if D is a finite-dimensional division algebra over an algebraically closed field F, then $D = F$. To see this equality, suppose that x is a member of D but not of F, i.e., is not an F multiple of the identity. Then x and F together generate a subfield $F(x)$ of D that is a nontrivial algebraic extension of F, contradiction. Consequently every finite-dimensional semisimple algebra R over an algebraically closed field F is of the form

$$R \cong M_{n_1}(F) \times \cdots \times M_{n_r}(F),$$

for suitable integers $n_1, \dots, n_r$.

As we saw, the finite dimensionality plays no role in decomposing semisimple rings as the finite product of rings that we shall call "simple." The place where finite dimensionality enters the discussion is in identifying simple rings as semisimple, hence in establishing a converse theorem that every finite direct product of simple rings, each equal to an ideal of the given ring, is necessarily semisimple. We say that a nonzero ring R with identity is **simple** if its only two-sided ideals are 0 and R.

EXAMPLES OF SIMPLE RINGS.

(1) If D is a division ring, then $M_n(D)$ is a simple ring. In fact, let J be a two-sided ideal in $M_n(D)$, fix an ordered pair (i, j) of indices, and let

$$I = \{x \in D \mid \text{some member } X \text{ of } J \text{ has } X_{ij} = x\}.$$

Multiplying X in this definition on each side by scalar matrices with entries in D, we see that I is a two-sided ideal in D. If $I = 0$ for all (i, j), then $J = 0$. So assume for some (i, j) that $I \neq 0$. Then $I = D$ for that (i, j), and we may suppose that some X in J has $X_{ij} = 1$. If E_{kl} denotes the matrix that is 1 in the $(k, l)^{\text{th}}$ place and is 0 elsewhere, then $E_{ii} X E_{jj} = E_{ij}$ has to be in J. Hence $E_{kl} = E_{ki} E_{ij} E_{jl}$ has to be in J, and $J = M_n(D)$.

(2) Let R be the **Weyl algebra** over $\mathbb{C}$ in one variable, namely

$$R = \Big\{ \sum_{n \geq 0} P_n(x) \Big(\frac{d}{dx}\Big)^n \ \Big|\ \text{each } P_n \text{ is in } \mathbb{C}[x], \text{ and the sum is finite} \Big\}.$$

To give a more abstract construction of R, we can view R as $\mathbb{C}\big[x, \frac{d}{dx}\big]$ subject to the relation $\frac{d}{dx}\, x = x\, \frac{d}{dx} + 1$; this is not to be a quotient of a polynomial algebra in two variables but a quotient of a tensor algebra in two variables. We omit the details. We shall now prove that the ring R is simple but not semisimple.

To see that R is a simple ring, we easily check the two identities

(i) $\frac{d}{dx}\big(x^m \frac{d^n}{dx^n}\big) = m x^{m-1} \frac{d^n}{dx^n} + x^m \frac{d^{n+1}}{dx^{n+1}}$ by the product rule,

(ii) $\frac{d^n}{dx^n}\, x = n \frac{d^{n-1}}{dx^{n-1}} + x \frac{d^n}{dx^n}$ by induction when applied to a polynomial $f(x)$.

Let I be a nonzero two-sided ideal in R, and fix an element $X \neq 0$ in I. Let x^m be the highest power of x appearing in X, and let $\frac{d^n}{dx^n}$ be the highest power of $\frac{d}{dx}$ appearing in terms of X involving x^m. Let l and r denote "left multiplication by" and "right multiplication by," and apply $\big(l\big(\frac{d}{dx}\big) - r\big(\frac{d}{dx}\big)\big)^m$ to X. Since (i) shows that

$$\big(l\big(\tfrac{d}{dx}\big) - r\big(\tfrac{d}{dx}\big)\big)x^k\big(\tfrac{d}{dx}\big)^l = kx^{k-1}\big(\tfrac{d}{dx}\big)^l,$$

the result of computing $\big(l\big(\frac{d}{dx}\big) - r\big(\frac{d}{dx}\big)\big)^m X$ is a polynomial in $\frac{d}{dx}$ of degree exactly n with no x's. Application of $(r(x) - l(x))^n$ to the result, using (ii), yields a nonzero constant. We conclude that 1 is in I and therefore that $I = R$. Hence R is simple.

To show that R is not semisimple, first note that $\mathbb{C}[x]$ is a natural unital left R module. We shall show that R has infinite length as a left R module, in the sense of the length of finite filtrations. In fact,

$$R \supseteq R\big(\tfrac{d}{dx}\big) \supseteq R\big(\tfrac{d}{dx}\big)^2 \supseteq \cdots \supseteq R\big(\tfrac{d}{dx}\big)^n \qquad (*)$$

is a finite filtration of left R submodules of R. If $R\big(\frac{d}{dx}\big)^k = R\big(\frac{d}{dx}\big)^{k+1}$, then $\big(\frac{d}{dx}\big)^k = r\big(\frac{d}{dx}\big)^{k+1}$ for some $r \in R$. Applying these two equal expressions for a member of R to the member x^k of the left R module $\mathbb{C}[x]$, we arrive at a contradiction and conclude that every inclusion in $(*)$ is strict. Therefore R has infinite length and is not semisimple.

The extra hypothesis that Wedderburn imposed so that simple rings would turn out to be semisimple is finite dimensionality. Wedderburn's result in this direction is Theorem 2.4 below. This hypothesis is quite natural to the extent that the subject was originally motivated by the theory of Lie algebras. E. Artin found a substitute for the assumption of finite dimensionality that takes the result beyond the realm of algebras, and we take up Artin's idea in the next section.

Theorem 2.4 (Wedderburn). Let R be a finite-dimensional algebra with identity over a field F. If R is a simple ring, then R is semisimple and hence is isomorphic to $M_n(D)$ for some integer $n \geq 1$ and some finite-dimensional division algebra D over F. The integer n is uniquely determined by R, and D is unique up to isomorphism.

PROOF. By finite dimensionality, R has a minimal left ideal V. For r in R, form the set Vr. This is a left ideal, and we claim that it is minimal or is 0. In fact, the function $v \mapsto vr$ is R linear from V onto Vr. Since V is simple as a left R module, Vr is simple or 0. The sum $I = \sum_{r \text{ with } Vr \neq 0} Vr$ is a two-sided ideal in R, and it is not 0 because $V1 \neq 0$. Since R is simple, $I = R$. Then the left R module R is exhibited as the sum of simple left R modules and is therefore semisimple. The isomorphism with $M_n(D)$ and the uniqueness now follow from Theorem 2.2. □

3. Rings with Chain Condition and Artin's Theorem

Parts of Chapters VIII and IX of *Basic Algebra* made considerable use of a hypothesis that certain commutative rings are "Noetherian," and we now extend this notion to noncommutative rings. A ring R with identity is **left Noetherian** if the left R module R satisfies the ascending chain condition for its left ideals. It is **left Artinian** if the left R module R satisfies the descending chain condition for its left ideals. The notions of **right Noetherian** and **right Artinian** are defined similarly.

We saw many examples of Noetherian rings in the commutative case in *Basic Algebra*. The ring of integers $\mathbb{Z}$ is Noetherian, and so is the ring of polynomials $R[X]$ in an indeterminate over a nonzero Noetherian ring R. It follows from the latter example that the ring $F[X_1, \dots, X_n]$ in finitely many indeterminates over a field is a Noetherian ring. Other examples arose in connection with extensions of Dedekind domains.

Any finite direct product of fields is Noetherian and Artinian because it has a composition series and because its ideals therefore satisfy both chain conditions. If p is any prime, the ring $\mathbb{Z}/p^2\mathbb{Z}$ is Noetherian and Artinian for the same reason, and it is not a direct product of fields.

In the noncommutative setting, any semisimple ring is necessarily left Noetherian and left Artinian because it has a composition series for its left ideals and the left ideals therefore satisfy both chain conditions.

Proposition 2.5. Let R be a ring with identity, and let M be a finitely generated unital left R module. If R is left Noetherian, then M satisfies the ascending chain condition for its R submodules; if R is left Artinian, then M satisfies the descending chain condition for its R submodules.

PROOF. We prove the first conclusion by induction on the number of generators, and the proof of the second conclusion is completely similar. The result is trivial if M has 0 generators. If $M = Rx$, then M is a quotient of the left R module R and satisfies the ascending chain condition for its R submodules, according to Proposition 10.10 of *Basic Algebra*. For the inductive step with ≥ 2 generators, write $M = Rx_1 + \cdots + Rx_n$ and $N = Rx_1 + \cdots + Rx_{n-1}$. Then N satisfies the ascending chain condition for its R submodules by the inductive hypothesis, and M/N is isomorphic to $Rx_n/(N \cap Rx_n)$, which satisfies the ascending chain condition for its R submodules by the inductive hypothesis. Therefore M satisfies the ascending chain condition for its R submodules by application of the converse direction of Proposition 10.10. □

Artin's theorem (Theorem 2.6 below) will make use of the hypothesis "left Artinian" in identifying those simple rings that are semisimple. The hypothesis left Artinian may therefore be regarded as a useful generalization of finite dimensionality. Before we come to that theorem, we give a construction that produces large numbers of nontrivial examples of such rings.

EXAMPLE (triangular rings). Let R and S be nonzero rings with identity, and let M be an (R, S) bimodule.[3] Define a set A and operations of addition and multiplication symbolically by

$$A = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix} = \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \;\middle|\; r \in R,\ m \in M,\ s \in S \right\}$$

with
$$\begin{pmatrix} r & m \\ 0 & s \end{pmatrix} \begin{pmatrix} r' & m' \\ 0 & s' \end{pmatrix} = \begin{pmatrix} rr' & rm' + ms' \\ 0 & ss' \end{pmatrix}.$$

Then A is a ring with identity, the bimodule property entering the proof of associativity of multiplication in A. We can identify R, M, and S with the additive subgroups of A given by $\left(\begin{smallmatrix} R & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & M \\ 0 & 0 \end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & 0 \\ 0 & S \end{smallmatrix}\right)$. Problems 8–11 at the end of the chapter ask one to check the following facts:

(i) The left ideals in A are of the form $I_1 \oplus I_2$, where I_2 is a left ideal in S and I_1 is a left R submodule of $R \oplus M$ containing MI_2.

(ii) The right ideals in A are of the form $J_1 \oplus J_2$, where J_1 is a right ideal in R and J_2 is a right S submodule of $M \oplus S$ containing $J_1 M$.

(iii) The ring A is left Noetherian if and only if R and S are left Noetherian and M satisfies the ascending chain condition for its left R submodules. The ring A is right Noetherian if and only if R and S are right Noetherian and M satisfies the ascending chain condition for its right S submodules.

(iv) The previous item remains valid if "Noetherian" is replaced by "Artinian" and "ascending" is replaced by "descending."

(v) If $A = \left(\begin{smallmatrix} R & R \\ 0 & S \end{smallmatrix}\right)$ is a ring such as $\left(\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right)$ in which S is a (commutative) Noetherian integral domain with field of fractions R and if $S \neq R$, then A is left Noetherian and not right Noetherian, and A is neither left nor right Artinian.

(vi) If $A = \left(\begin{smallmatrix} R & R \\ 0 & S \end{smallmatrix}\right)$ is a ring such as $\left(\begin{smallmatrix} \mathbb{Q}(x) & \mathbb{Q}(x) \\ 0 & \mathbb{Q} \end{smallmatrix}\right)$ in which R and S are fields with $S \subseteq R$ and $\dim_S R$ infinite, then A is left Noetherian and left Artinian, and A is neither right Noetherian nor right Artinian.

From these examples we see, among other things, that "left" and "right" are somewhat independent for both the Noetherian and the Artinian conditions. We

[3]This means that M is an abelian group with the structure of a unital left R module and the structure of a unital right S module in such a way that $(rm)s = r(ms)$ for all $r \in R$, $m \in M$, and $s \in S$.

already know from the commutative case that Noetherian does not imply Artinian, $\mathbb{Z}$ being a counterexample. We shall see in Theorem 2.15 later that left Artinian implies left Noetherian and that right Artinian implies right Noetherian.

Theorem 2.6 (E. Artin). If R is a simple ring, then the following conditions are equivalent:

(a) R is left Artinian,
(b) R is semisimple,
(c) R has a minimal left ideal,
(d) $R \cong M_n(D)$ for some integer $n \geq 1$ and some division ring D.

In particular, a left Artinian simple ring is right Artinian.

REMARK. Theorem 2.4 is a special case of the assertion that (a) implies (d). In fact, if R is a finite-dimensional algebra over a field F, then the finite dimensionality forces R to be left Artinian.

PROOF. It is evident from Wedderburn's Theorem (Theorem 2.2) that (b) and (d) are equivalent. For the rest we prove that (a) implies (c), that (c) implies (b), and that (b) implies (a).

Suppose that (a) holds. Applying the minimum condition for left ideals in R, we obtain a minimal left ideal. Thus (c) holds.

Suppose that (c) holds. Let V be a minimal left ideal. Then the sum $I = \sum_{r \in R} Vr$ is a two-sided ideal in R, and it is nonzero because the term for $r = 1$ is nonzero. Since R is simple, $I = R$. Then the left R module R is spanned by the simple left R modules Vr, and R is semisimple. Thus (b) holds.

Suppose that (b) holds. Since R is semisimple, the left R module R has a composition series. Then the left ideals in R satisfy both chain conditions, and it follows that R is left Artinian. Thus (a) holds. $\square$

4. Wedderburn–Artin Radical

In this section we introduce one notion of "radical" for certain rings with identity, and we show how it is related to semisimplicity. This notion, the "Wedderburn–Artin radical," is defined under the hypothesis that the ring is left Artinian. It is not the only notion of radical studied by ring theorists, however. There is a useful generalization, known as the "Jacobson radical," that is defined for arbitrary rings with identity. We shall not define and use the Jacobson radical in this text.

Fix a ring R with identity. A **nilpotent element** in R is an element a with $a^n = 0$ for some integer $n \geq 1$. A **nil left ideal** is a left ideal in which every element is nilpotent; nil right ideals and nil two-sided ideals are defined similarly.

A **nilpotent left ideal** is a left ideal I such that $I^n = 0$ for some integer $n \geq 1$, i.e., for which $a_1 \cdots a_n = 0$ for all n-fold products of elements from I; nilpotent right ideals and nilpotent two-sided ideals are defined similarly.

Lemma 2.7. If I_1 and I_2 are nilpotent left ideals in a ring R with identity, then $I_1 + I_2$ is nilpotent.

PROOF. Let $I_1^r = 0$ and $I_2^s = 0$. Expand $(I_1 + I_2)^k$ as $\sum I_{i_1} I_{i_2} \cdots I_{i_k}$ with each i_j equal to 1 or 2. Take $k = r + s$. In any term of the sum, there are $\geq r$ indices 1 or $\geq s$ indices 2. In the first case let there be t indices 2 at the right end. Since $I_2 I_1 \subseteq I_1$, we can absorb all other indices 2, and the term of the sum is contained in $I_1^r I_2^t = 0$. Similarly in the second case if there are t' indices 1 at the right end, then the term is contained in $I_2^s I_1^{t'} = 0$. $\square$

Lemma 2.8. If I is a nilpotent left ideal in a ring R with identity, then I is contained in a nilpotent two-sided ideal J.

PROOF. Put $J = \sum_{r \in R} Ir$. This is a two-sided ideal. For any integer $k \geq 0$, $J^k = \big(\sum_{r \in R} Ir\big)^k \subseteq \sum_{r_1, \dots, r_k} Ir_1 Ir_2 \cdots Ir_k \subseteq \sum_{r_k} I^k r_k$. If $I^k = 0$, then $J^k = 0$. $\square$

Lemma 2.9. If R is a ring with identity, then the sum of all nilpotent left ideals in a nil two-sided ideal.

PROOF. Let K be the sum of all nilpotent left ideals in R, and let a be a member of K. Write $a = a_1 + \cdots + a_n$ with $a_i \in I_i$ for a nilpotent left ideal I_i. Lemma 2.7 shows that $I = \sum_{i=1}^n I_i$ is a nilpotent left ideal. Since a is in I, a is a nilpotent element.

The set K is certainly a left ideal, and we need to see that aR is in K in order to see that K is a two-sided ideal. Lemma 2.8 shows that $I \subseteq J$ for some nilpotent two-sided ideal J. Then $J \subseteq K$ because J is one of the nilpotent left ideals whose sum is K. Since a is in I and therefore in J and since J is a two-sided ideal, aR is contained in J. Therefore aR is contained in K, and K is a two-sided ideal. $\square$

Theorem 2.10. If R is a left Artinian ring, then any nil left ideal in R is nilpotent.

REMARK. Readers familiar with a little structure theory for finite-dimensional Lie algebras will recognize this theorem as an analog for associative algebras of Engel's Theorem.

PROOF. Let I be a nil left ideal of R, and form the filtration

$$I \supseteq I^2 \supseteq I^3 \supseteq \cdots .$$

Since R is left Artinian, this filtration is constant from some point on, and we have $I^k = I^{k+1} = I^{k+2} = \cdots$ for some $k \geq 1$. Put $J = I^k$. We shall show that $J = 0$, and then we shall have proved that I is a nilpotent ideal.

Suppose that $J \neq 0$. Since $J^2 = I^{2k} = I^k = J$, we have $J^2 = J$. Thus the left ideal J has the property that $JJ \neq 0$. Since R is left Artinian, the set of left ideals $K \subseteq J$ with $JK \neq 0$ has a minimal element K_0. Choose $a \in K_0$ with $Ja \neq 0$. Since $Ja \subseteq JK_0 \subseteq K_0$ and $J(Ja) = J^2a = Ja \neq 0$, the minimality of K_0 implies that $Ja = K_0$. Thus there exists $x \in J$ with $xa = a$. Applying powers of x, we obtain $x^n a = a$ for every integer $n \geq 1$. But x is a nilpotent element, being in I, and thus we have a contradiction. □

Corollary 2.11. If R is a left Artinian ring, then there exists a unique largest nilpotent two-sided ideal I in R. This ideal is the sum of all nilpotent left ideals and also is the sum of all nilpotent right ideals.

REMARKS. The two-sided ideal I of the corollary is called the **Wedderburn–Artin radical** of R and will be denoted by rad R. This exists under the hypothesis that R is left Artinian.

PROOF. By Lemma 2.9 and Theorem 2.10 the sum of all nilpotent left ideals in R is a two-sided nilpotent ideal I. Lemma 2.8 shows that any nilpotent right ideal is contained in a nilpotent two-sided ideal J. Since J is in particular a nilpotent left ideal, the definition of I forces $J \subseteq I$. Hence the sum of all nilpotent right ideals is contained in I. But I itself is a nilpotent right ideal and hence equals the sum of all the nilpotent right ideals. □

Lemma 2.12 (Brauer's Lemma). If R is any ring with identity and if V is a minimal left ideal in R, then either $V^2 = 0$ or $V = Re$ for some element e of V with $e^2 = e$.

REMARK. An element e with the property that $e^2 = e$ is said to be **idempotent**.

PROOF. Being a minimal left ideal, V is a simple left R module. Schur's Lemma (Proposition 10.4b of *Basic Algebra*) shows that $\operatorname{End}_R V$ is a division ring. If a is in V, then the map $v \mapsto va$ of V into itself lies in $\operatorname{End}_R V$ and hence is the 0 map or is one-one onto. If it is the 0 map for all $a \in V$, then $V^2 = 0$. Otherwise suppose that a is an element for which $v \mapsto va$ is one-one onto. Then there exists $e \in V$ with $ea = a$. Multiplying on the left by e gives $e^2 a = ea$ and therefore $(e^2 - e)a = 0$. Since the map $v \mapsto va$ is assumed to be one-one onto, we must have $e^2 - e = 0$ and $e^2 = e$. □

Theorem 2.13. If R is a left Artinian ring and if the Wedderburn–Artin radical of R is 0, then R is a semisimple ring.

REMARKS. Conversely semisimple rings are left Artinian and have radical 0. In fact, we already know that semisimple rings have a composition series for their left ideals and hence are left Artinian. To see that the radical is 0, apply Theorem 2.2 and write the ring as $R = M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$. The two-sided ideals of R are the various subproducts, with 0 in the missing coordinates. Such a subproduct cannot be nilpotent as an ideal unless it is 0, since the identity element in any factor is not a nilpotent element in R.

PROOF. Let us see that any minimal left ideal I of R is a direct summand as a left R submodule. Since rad $R = 0$, I is not nilpotent. Thus $I^2 \neq 0$, and Lemma 2.12 shows that I contains an idempotent e. This element satisfies $I = Re$. Put $I' = \{r \in R \mid re = 0\}$. Then I' is a left ideal in R. Since $I' \cap I \subseteq I$ and e is not in I', the minimality of I forces $I' \cap I = 0$. Writing $r = re + (r - re)$ with $re \in I$ and $r - re \in I'$, we see that $R = I + I'$. Therefore $R = I \oplus I'$.

Now put $I_1 = I$. If I' is not 0, choose a minimal left ideal $I_2 \subseteq I'$ by the minimum condition for left ideals in R. Arguing as in the previous paragraph, we have $I_2 = Re_2$ for some element e_2 with $e_2^2 = e_2$. The argument in the previous paragraph shows that $R = I_2 \oplus I_2'$, where $I_2' = \{r \in R \mid re_2 = 0\}$. Define $I'' = \{r \in R \mid re_1 = re_2 = 0\} = I' \cap I_2'$. Since I_2 is contained in I', we can intersect $R = I_2 \oplus I_2'$ with I' and obtain $I' = I_2 \oplus I''$. Then $R = I_1 \oplus I' = I_1 \oplus I_2 \oplus I''$. Continuing in this way, we obtain $R = I_1 \oplus I_2 \oplus I_3 \oplus I'''$, etc. As this construction continues, we have $I' \supseteq I'' \supseteq I''' \supseteq \cdots$. Since R is left Artinian, this sequence must terminate, evidently in 0. Then R is exhibited as the sum of simple left R modules and is semisimple. □

Corollary 2.14. If R is a left Artinian ring, then $R/\operatorname{rad} R$ is a semisimple ring.

PROOF. Let $I = \operatorname{rad} R$, and let $\varphi : R \to R/I$ be the quotient homomorphism. Arguing by contradiction, let $\overline{J}$ be a nonzero nilpotent left ideal in R/I, and let $J = \varphi^{-1}(\overline{J}) \subseteq R$. Since $\overline{J}$ is nilpotent, $J^k \subseteq I$ for some integer $k \geq 1$. But I, being the radical, is nilpotent, say with $I^l = 0$, and hence $J^{k+l} \subseteq I^l = 0$. Therefore J is a nilpotent left ideal in R strictly containing I, in contradiction to the maximality of I. We conclude that no such $\overline{J}$ exists. Then $\operatorname{rad}(R/\operatorname{rad} R) = 0$. Since $R/\operatorname{rad} R$ is left Artinian as a quotient of a left Artinian ring, Theorem 2.13 shows that $R/\operatorname{rad} R$ is a semisimple ring. □

We shall use this corollary to prove that left Artinian rings are left Noetherian. We state the theorem, state and prove a lemma, and then prove the theorem.

Theorem 2.15 (Hopkins). If R is a left Artinian ring, then R is left Noetherian.

Lemma 2.16. If R is a semisimple ring, then every unital left R module M is semisimple. Consequently any unital left R module satisfying the descending chain condition has a composition series and therefore satisfies the ascending chain condition.

PROOF. For each $m \in M$, let R_m be a copy of the left R module R, and define $\widetilde{M} = \bigoplus_{m \in M} R_m$ as a left R module. Since each R_m is semisimple, $\widetilde{M}$ is semisimple. Define a function $\varphi : \widetilde{M} \to M$ as follows: if $r_{m_1} + \cdots + r_{m_k}$ is given with r_{m_j} in R_{m_j} for each j, let $\varphi(r_{m_1} + \cdots + r_{m_k}) = \sum_{j=1}^{k} r_{m_j} m_j$. Then φ is an R module map with the property that $\varphi(1_m) = m$, and consequently φ carries $\widetilde{M}$ onto M. As the image of a semisimple R module under an R module map, M is semisimple.

Now suppose that M is a unital left R module satisfying the descending chain condition. We have just seen that M is semisimple, and thus we can write $M = \bigoplus_{i \in S} M_i$ as a direct sum over a set S of simple left R modules M_i. Let us see that S is a finite set. If S were not a finite set, then we could choose an infinite sequence $i_1, i_2, \dots$ of distinct members of S, and we would obtain

$$M \underset{\neq}{\supseteq} \bigoplus_{i \neq i_1} M_i \underset{\neq}{\supseteq} \bigoplus_{i \neq i_1, i_2} M_i \underset{\neq}{\supseteq} \cdots,$$

in contradiction to the fact that the R submodules of M satisfy the descending chain condition. □

PROOF OF THEOREM 2.15. Let $I = \operatorname{rad} R$. Since I is nilpotent, $I^n = 0$ for some n. Each I^k for $k \geq 0$ is a left R submodule of R. Since R is left Artinian, its left R submodules satisfy the descending chain condition, and the same thing is true of the R submodules of each I^k. Consequently the R submodules of each I^k/I^{k+1} satisfy the descending chain condition.

In the action of R on I^k/I^{k+1} on the left, I acts as 0. Hence I^k/I^{k+1} becomes a left R/I module, and the R/I submodules of this left R/I module must satisfy the descending chain condition. Corollary 2.14 shows that $R/I = R/\operatorname{rad} R$ is a semisimple ring. Since the R/I submodules of I^k/I^{k+1} satisfy the descending chain condition, Lemma 2.16 shows that these R/I submodules satisfy the ascending chain condition. Therefore the R submodules of each left R module I^k/I^{k+1} satisfy the ascending chain condition.

We shall show inductively for $k \geq 0$ that the R submodules of R/I^{k+1} satisfy the ascending chain condition. Since $I^n = 0$, this conclusion will establish that R is left Noetherian, as required. The case $k = 0$ was shown in the previous paragraph. Assume inductively that the R submodules of R/I^k satisfy the ascending chain condition. Since $R/I^k \cong (R/I^{k+1})\big/(I^k/I^{k+1})$ and since the R submodules of R/I^k and of I^k/I^{k+1} satisfy the ascending chain condition, the same is true for R/I^{k+1}. This completes the proof. □