

# RESEARCH SUMMARY (SHORT VERSION)

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My research interest is in *differential geometry*, an area of mathematics which uses calculus to study geometric shapes called *manifolds*. Examples of manifolds include curves, surfaces, and the 4-dimensional spacetime that we inhabit. A major question in differential geometry is how to tell apart and classify manifolds and their various geometric structures. One way to tackle this question is by means of *gauge theory*, which originates from particle physics and is concerned with fields on manifolds, e.g. the electromagnetic field permeating our spacetime. The dynamics of these fields is governed by differential equations, such as Maxwell's equations of electromagnetism or the *Yang–Mills equations* describing nuclear forces. In the 1980's geometers realized that fields governed by the Yang–Mills equations capture subtle geometric properties of manifolds of dimension 3 and 4. This insight led to spectacular progress on the problem of distinguishing and classifying such manifolds.

My work fits into two new, rapidly developing lines of research whose goal is to extend the scope of applications of gauge theory to geometry:

- (1) *Beyond Yang–Mills*. The focus of the previous research has been on Yang–Mills fields and monopoles, closely related objects originating from a variant of Yang–Mills theory. Modern physics, especially string theory, considers a variety of other fields. From a geometric viewpoint, especially promising are the ones which obey the Kapustin–Witten equations or the Vafa–Witten equations. Such fields, according to physical predictions, should give us new insights into the geometry of 3- and 4-dimensional manifolds.
- (2) *Beyond dimensions 3 and 4*. Of particular interest in differential geometry and string theory are higher-dimensional manifolds with a metric of special holonomy: a rare geometric structure whose defining property is the presence of rich symmetry at very small scales. There are currently many examples of such manifolds—including the famous Calabi–Yau three-folds—of which, however, there is still no systematic understanding. It is expected that Yang–Mills theory will shed new light on the classification of special holonomy manifolds.

The main difficulty in developing either of these theories stems from possible degenerations of fields, which occur when their energy concentrates in very small regions. This phenomenon is well-known from Yang–Mills theory in dimension 4 but it is still poorly understood in the cases described above. It was conjectured that whenever such a degeneration happens, it is necessarily accompanied by the emergence of two new types of fields. The first type, called *harmonic  $\mathbb{Z}/2$  spinors*, are fields which satisfy the Dirac equation, a fundamental equation of quantum physics. The symbol  $\mathbb{Z}/2$  describes the peculiar property that these fields come in pairs which differ by a sign—in the same way the two square roots  $\pm\sqrt{x}$  of a positive number  $x$  differ by a sign. The second type of fields, called *multi-monopoles*, are fields consisting of many monopoles interacting with one another.

In my thesis, I studied the question of existence of these fields. Even though their appearance is predicted by general theory, there were so far no non-trivial examples known. In papers [1] and [2], Walpuski and I remedied this situation by establishing a general existence result for harmonic  $\mathbb{Z}/2$  spinors on 3-dimensional manifolds. In particular, we produced infinitely many new examples, and confirmed a conjecture that, in an appropriate sense, these fields should appear frequently. Similarly, in papers [3] and [4], I constructed and studied the first known examples of multi-monopoles. To do this, I considered 3-dimensional manifolds which can be sliced using 2-dimensional surfaces; in the same way our 3-dimensional space with coordinates  $x, y, z$  can be sliced using the planes  $z = \text{constant}$ . I then related multi-monopoles to better-understood objects known from *algebraic geometry*, a field of mathematics which uses algebraic properties of polynomials to study manifolds. These are, at present, the only existence results for harmonic  $\mathbb{Z}/2$  spinors and multi-monopoles.

In my future research, I plan to study Yang–Mills theory on Calabi–Yau three-folds. This is a particularly attractive class of manifolds which are amenable to methods of both differential and algebraic geometry. On the algebraic side, there has been in recent years enormous progress on invariants of Calabi–Yau three-folds. A celebrated conjecture known as the MNOP Conjecture predicts deep relations between different types of these invariants. On the differential-geometric side, much less is known. In article [5], Walpuski and I, building on ideas of Donaldson–Thomas–Segal, outlined a program of using gauge theory to construct a new, differential-geometric invariant of Calabi–Yau three-folds. We are currently working towards making this proposal rigorous. Part of this project involves a deeper study of special surfaces, called *pseudo-holomorphic curves*, inside Calabi–Yau three-folds. We already made progress in this direction in papers [6] and [7]. This previously little explored, analytic approach should shed new light on the MNOP Conjecture and the geometry of Calabi–Yau three-folds.

## REFERENCES

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